Topology of 3-manifolds

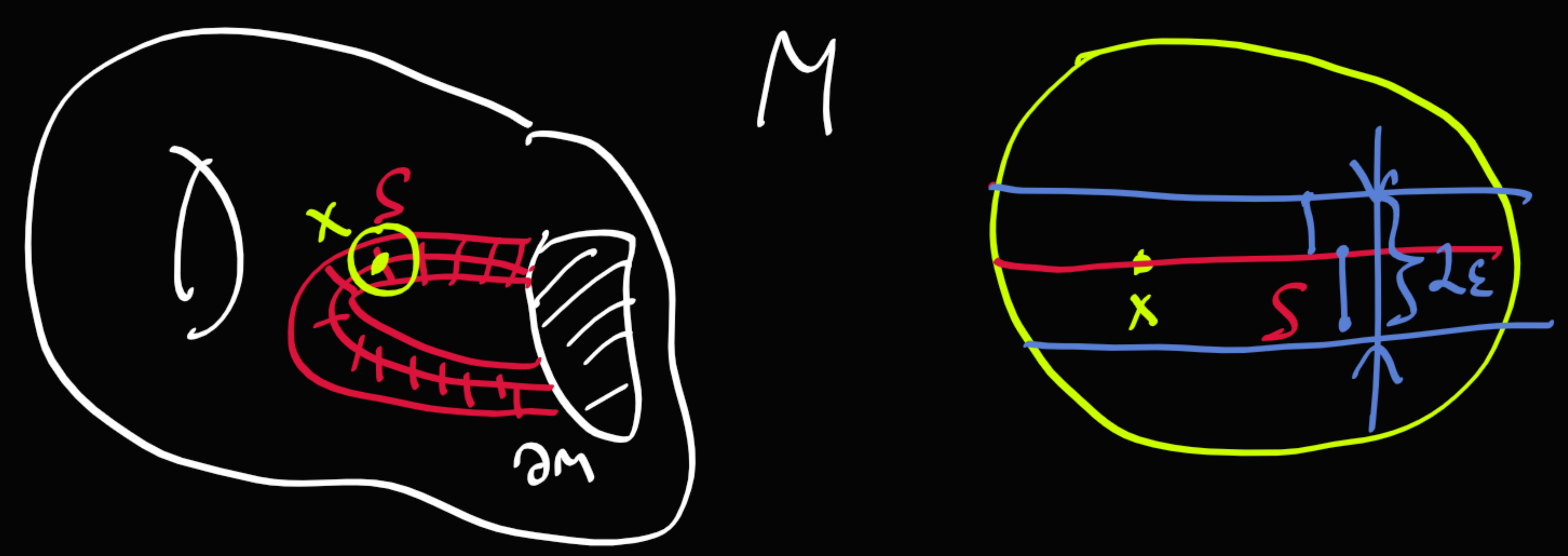
topological 3-manifold $\longrightarrow$ smooth 3-manifold

In the following we will always assume the given 3-manifold is oriented, compact and connected.

Connected sum operation # $M_1 \# M_2 = M$

Assume that M is a 3-manifold with a properly embedded surface S (S is 2-dim. mani.) with $\partial S \subseteq \partial M$ if $\partial S \neq 0$.

Tubular neighborhood (or also collar neigh.) of S in M is an open subset $N(S)$ of M which contains S which locally around S looks as $S \times (-\varepsilon, \varepsilon)$, $\varepsilon > 0$.



locally S is given as the zero locus of a smooth function on M

$S \subseteq M$ looks as $\mathbb{R}^2 \subseteq \mathbb{R}^3$

$S(M)$ looks as $\mathbb{R}^2 \times (-\varepsilon, \varepsilon)$

Construction of $N(S)$:

Fix a Riemannian metric g_M on M. Then one can restrict this metric (pointwise) to S so that for every $x \in S \setminus \partial S$

$T_x M = T_x S \oplus N_x$ where N_x is the orthogonal complement to $T_x S$ inside $T_x M$ w.r. to $(g_M)_x$.

Now for $v \in N_x$ we can consider the geodesic γ_v satisfying:

$\gamma_v(0) = x$, $\gamma_v'(0) = v$.

Consider the map

$$\exp: S \times N \longrightarrow M, \quad (x, v) \longmapsto \gamma_v(1)$$

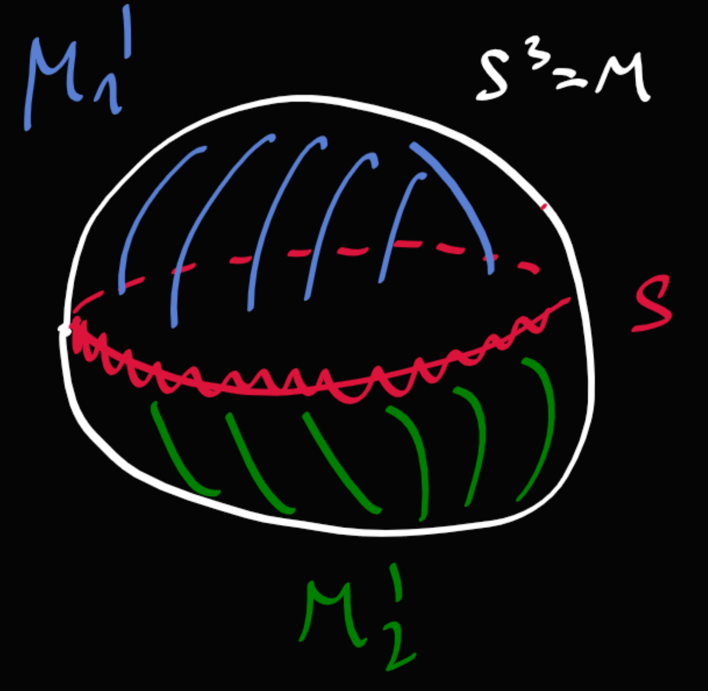
where $N = \dot{\bigcup}_{x \in S} N_x$. The space N_x is called the normal vector space to S (in M) at x. N is called a normal vector bundle.

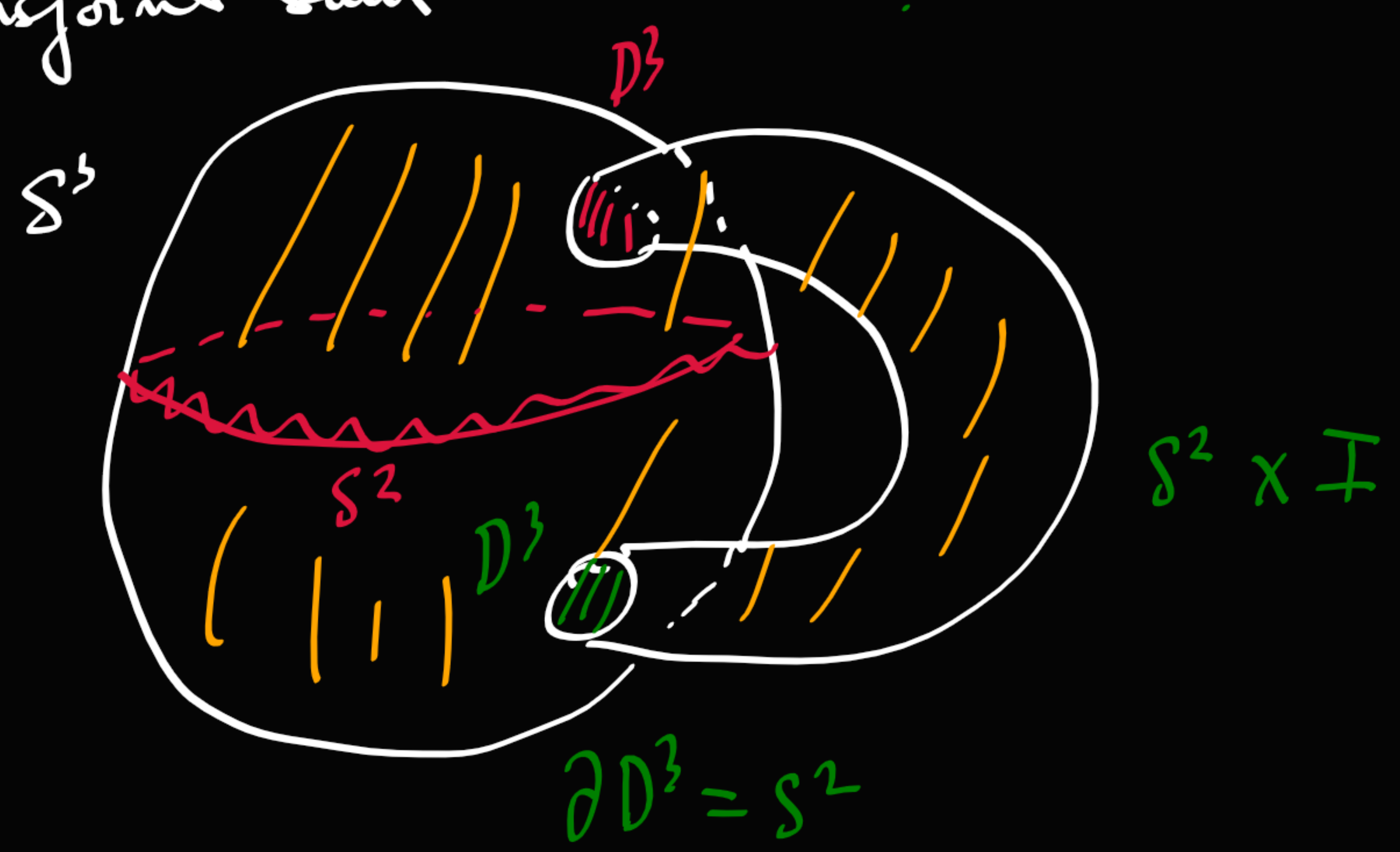
By implicit function theorem, there exists $\varepsilon > 0$ such that the map exp is a diffeomorphism onto its image when we restrict it to $N_\varepsilon = \{(v, x) : \sqrt{g_x(v,v)} = \|v\|_x < \varepsilon\}$. Then the image of exp to N_ε is called an tubular (ε-) neigh. of S in M.

Let us now assume that $S = S^2$ is the 2-dimensional sphere and that

$M|S := M \setminus N(S) = M_1' \cup M_2'$ ← two connected 3-manifolds

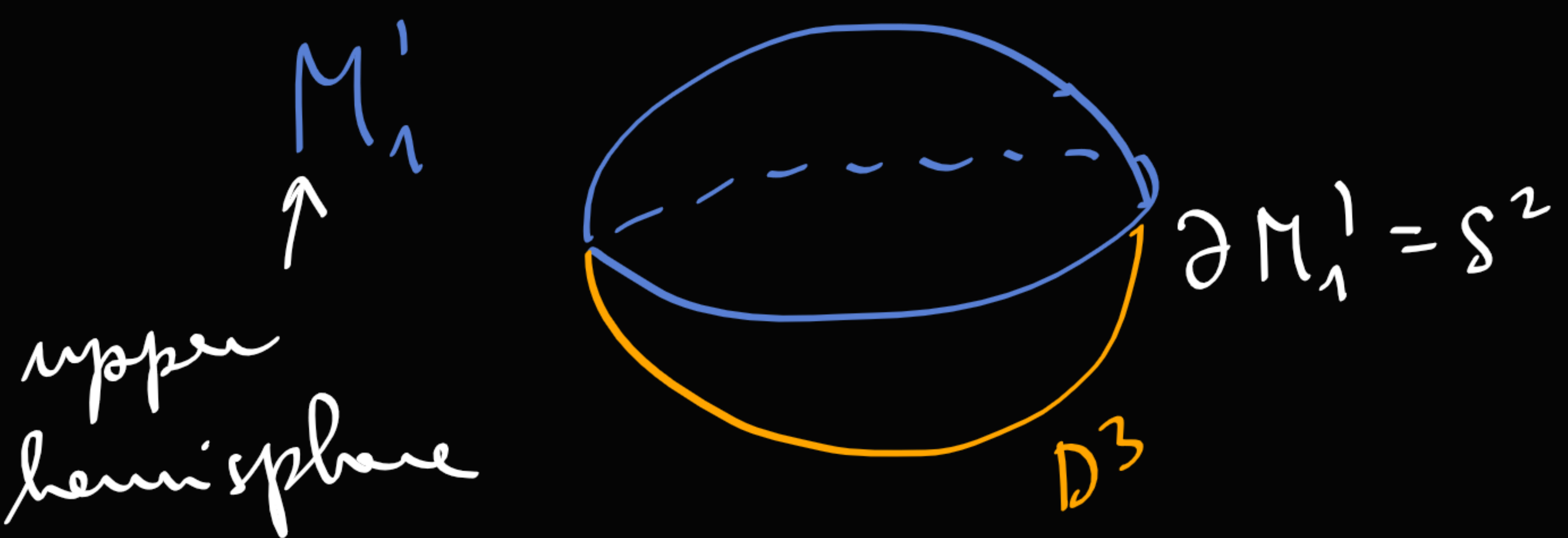
disjoint sum





$M_1 = M_1'$ with D^3 attached to $S = \partial D^3$

$M_2 = M_2'$ with D^3 attached to $S = \partial D^3$

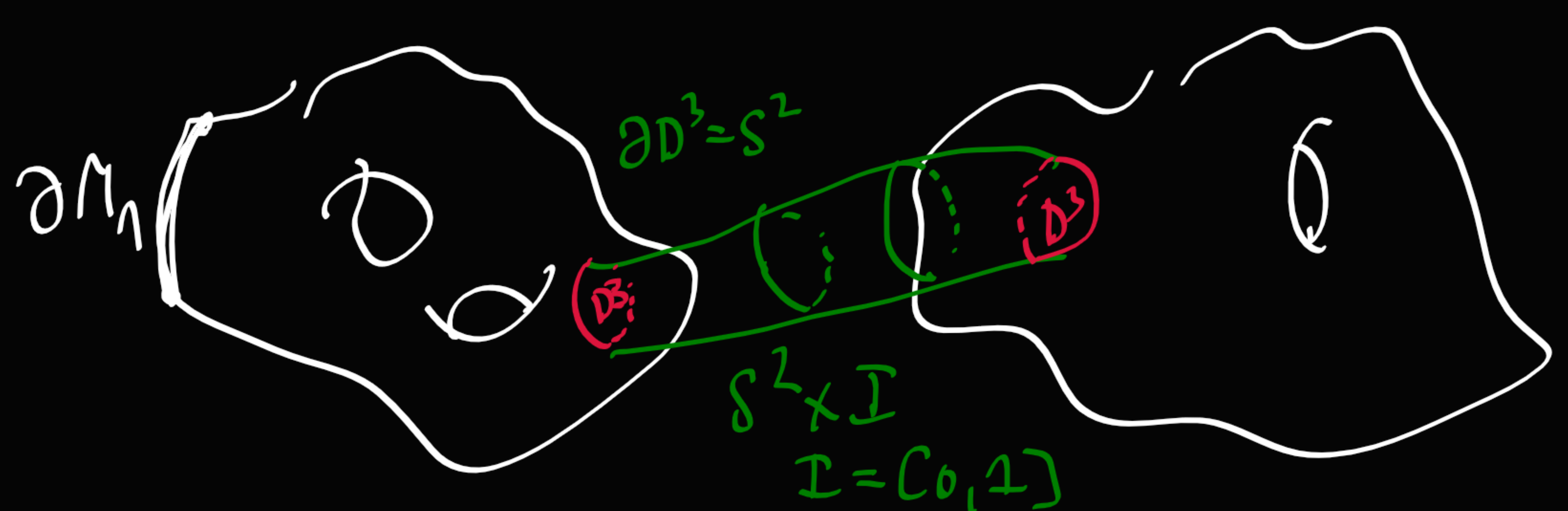


$M_1 = S^3$

In this situation, $M = M_1 \# M_2$ is the connected sum of M_1 and M_2.

M_1 M_2 3-manifolds

if M_1 and M_2 are connected, then M (up to diffeom.) does not depend on the choices of $D^3 \subset M_i$



$M = M_1 \# M_2$ is a smooth 3-manifold

smooth structure of M depends only smooth structures of M_1 and M_2

Theorem The connected sum operation # is associative, commutative with unit ($= S^3$).

Definition M is called prime if the following holds:

$$M = M_1 \# M_2 \Rightarrow M \text{ is diffeo. to } M_1 \text{ or } M_2 \text{ and } M_2 \text{ or } M_1 \text{ is diffeo. to } S^3.$$

Definition M is called irreducible if every embedded 2-sphere in M bounds an embedded ball.
(Every $S^2 \subseteq M$ is the boundary of $D^3 \subseteq M$.)

Theorem The only orientable 3-manifold which is prime but not irreducible is $S^1 \times S^2$.

Theorem Let M be compact, connected and orientable 3-man. Then there exists a decomposition

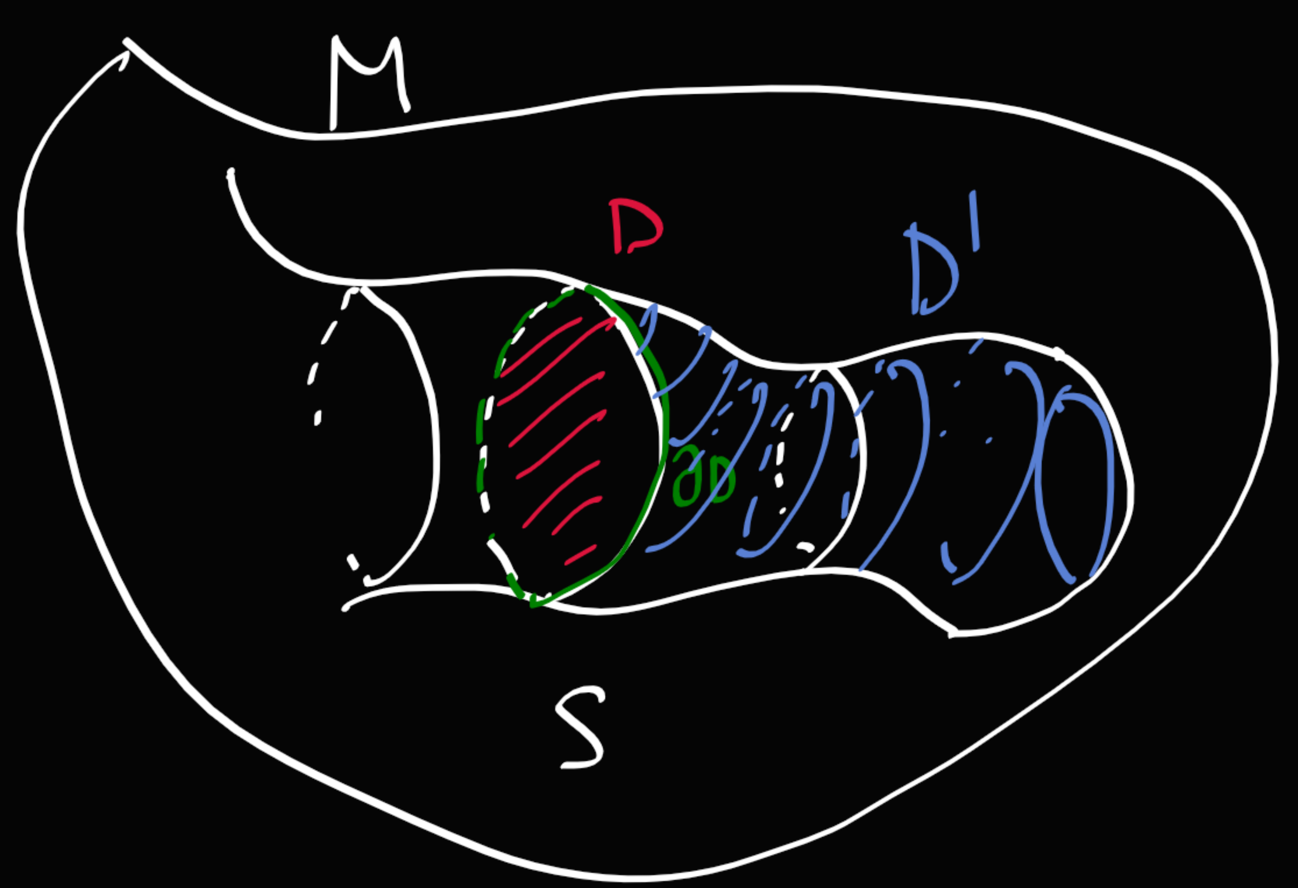
$$M = P_1 \# \cdots \# P_n$$

with each P_i prime. This decomposition is unique up to insertion or deletion of S^3's.

Torus decomposition

Surface $S \subset M$ properly embedded is called 2-sided or 1-sided if the normal bundle N is trivial or not (if $N = S \times I$ then S is called 2-sided and 1-sided otherwise).

2-sided surface S without S^2 or D^2 components is called incompressible if each disc $D \subset M$ with $D \cap S = \partial D$, then $\exists$ a disc $D' \subset S$ with $\partial D = \partial D'$.



A properly embedded surface $S \subseteq M$ is called ∂-parallel if it is isotopic, fixing ∂S, to a surface in ∂M. An irreducible manifold is called atoroidal if every incompressible torus in M is ∂-parallel.

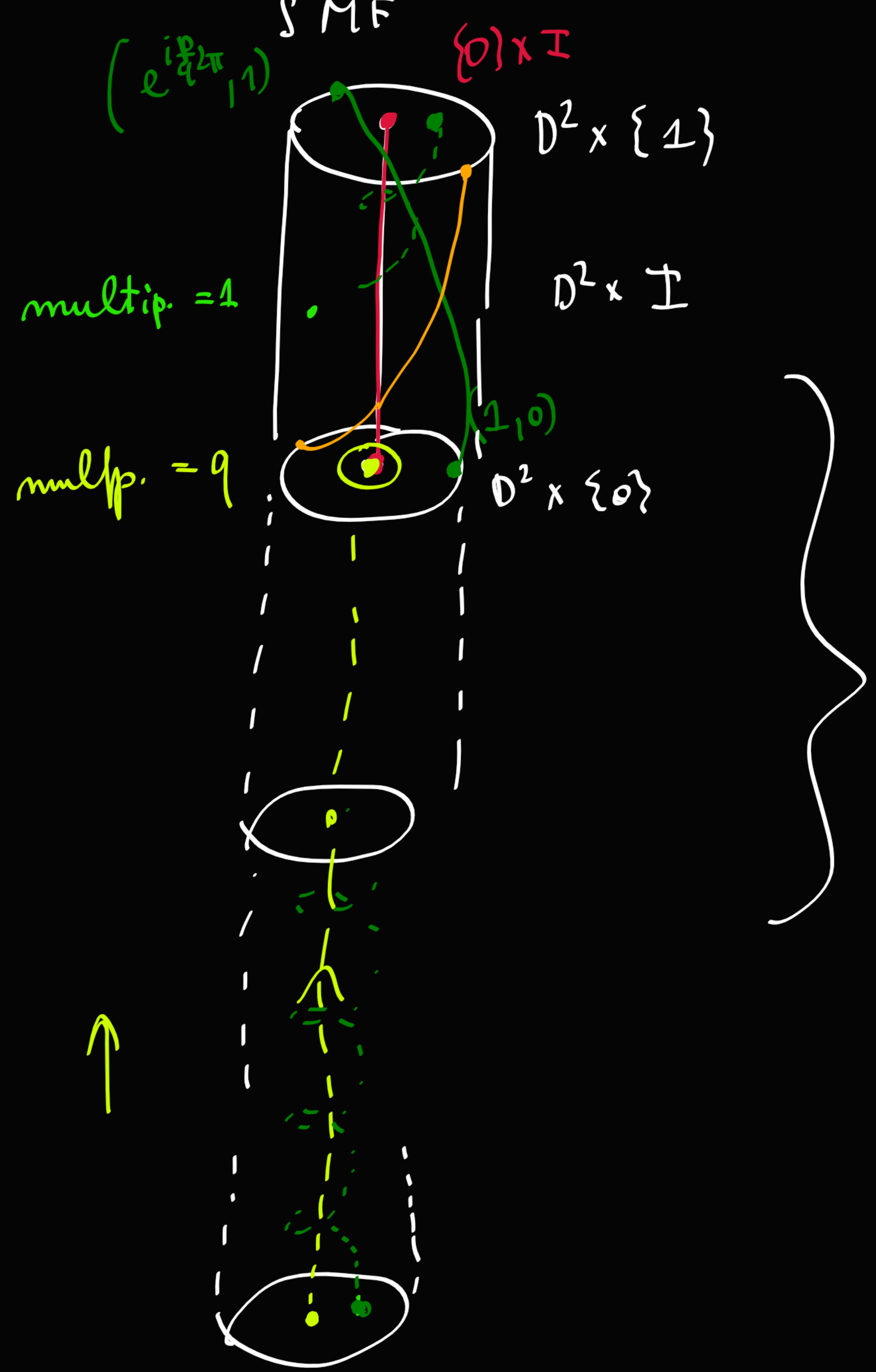
Seifert manifolds

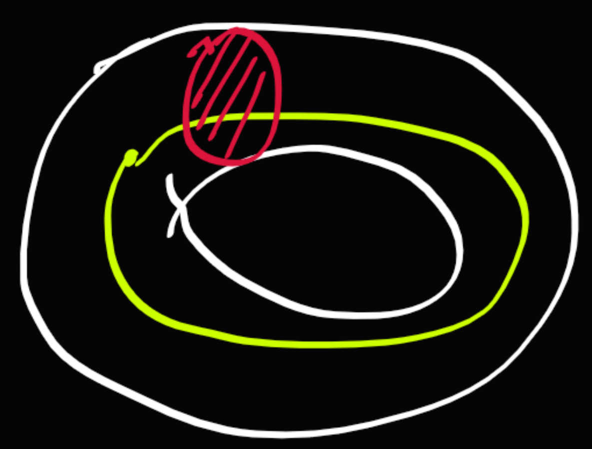
Seifert model fibering of $S^1 \times D^2$ is a twisted circle bundle over D^2 constructed as follows:

$$S^1 = [0,1]/\sim \quad , \quad 0 \sim 1$$

$$\underbrace{D^2 \times [0,1] / \sim}_{\text{SMF}} \qquad (x,0) \sim (e^{i\frac{p}{q}2\pi} x, 1) \quad , \quad p, q \in \mathbb{N} \text{ coprime}$$

$D^2 \subseteq \mathbb{C}$





$S^2 \times D^1$ $q = 1$, $p \in \mathbb{N}$

$q \neq 1$

Seifert 3-manifold is a manifold with a Seifert fibering

Seifert fibering of a 3-manifold M is a decomposition of M into disjoint union of circles, called fibers, such that each fiber has an open neigh. diffeomorphic, preserving fibers, to a neigh. of a fiber in some model Seifert fibering of $S^1 \times D^2$.

Each fiber C has a well defined multiplicity, the number of times a small disc around a point on C meets each nearby fiber.

multiplicity $= 1 \Rightarrow$ regular fibers

multiplicity $> 1 \Rightarrow$ singular fibers

<u>Theorem</u> For a compact irreducible 3-manifold M there $\exists$ a collection $T \subset M$ of disjoint incompressible tori such that each component of $M \setminus T$ is either toroidal or a Seifert manifold and a minimal collection of such tori is unique up to isotopy.