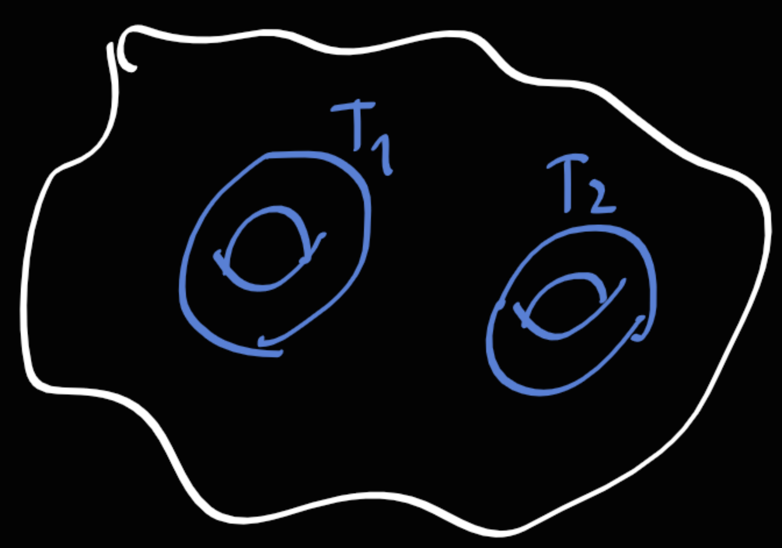
Thurston's geometrization conjecture

(TGC) Every prime, closed and oriented 3-dim. manifold can be cut along an embedded disjoint union of 2-tori such that any connected component of their complement admits a locally homogeneous Riemannian metric of finite volume.

M



$T = \dot{\bigcup}_{i \in A} T_i$

$M \setminus T$

William Paul Thurston (1946 - 2012)

received Fields medal for TGC in 1982, Thurston did not prove TGC in full generality but only for Haken manifolds

TGC $\Rightarrow$ Poincaré conjecture

"every closed and simply connected 3-man. is homeomorphic to S^3"

TGC was proved by G. Perelman (2002-03) who completed the proof proposed by R. Hamilton, "Ricci flow"

Perelman declined Fields medal in 2006

Remarks on TGC:

- This can be regarded as a generalization of Uniformazation Theorem: "Every closed 2-dimensional surface admits a Riemannian metric with constant scalar curvature".
- This cutting of the 3-manifold along the family of disjoint 2-tori is not quite the decomposition from the last week.

Locally homogeneous Riemannian metric

Assume that G is a Lie group of dimension m. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}$ is tangent space at the identity element $e \in G$. Let h be any positive definite bilinear form on G.

Then there is a unique (left or right)-invariant Riemannian metric g_G on G such that

$$h = g_G|_e : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}.$$

It can be shown that this a 1-1 correspondence between

$$\{ \text{positive definite bil. products on } \mathfrak{g} \}$$
$$\overset{1-1}{\longleftrightarrow} \{ \text{left-invariant Riemannian metrics on } G \}$$

Now assume that K is a compact subgroup of G such the quotient space

$$G/K = \{ gK \mid g \in G \}$$

with its canonical topology is Hausdorff. Then it can be proved that also G/K is a smooth manifold such that

$$G \longrightarrow G/K$$

is smooth. Moreover this smooth atlas on G/K is unique.

From the course on Representations of Lie groups and algebras you should know that $\mathfrak{g}$ is a representation of G by "adjoint action"

$$G \times \mathfrak{g} \to \mathfrak{g}, \quad (g, v) \longmapsto g^{-1} v g \quad (\text{"for a matrix group } G\text{"}).$$

Hence $\mathfrak{g}$ is also a representation of K just by restricting the adjoint action. Now as we assume that K is compact, then the set of K-invariant positive definite bilinear forms on $\mathfrak{g}$ is non-zero. (This follows from so called "averaging trick" or also "the Weyl trick".) If now h is a K-invariant positive definite bilinear form on $\mathfrak{g}$ $\Rightarrow$ g_G R. metric on G $\Rightarrow$ $\exists!$ a unique Riemannian metric g_M on $M := G/K$ such that:

$$\pi : G \longrightarrow G/K, \quad (\pi^* g_M)_g \big|_{\operatorname{Ker}(T_g\pi)^\perp} = (g_G) \big|_{\operatorname{Ker}(T_g\pi)^\perp}.$$

$g \in G$, $\quad T_g\pi : T_g G \longrightarrow T_{\pi(g)} G/K$ is a surjective linear map

$$T_g G = \operatorname{Ker}(T_g\pi) \oplus \operatorname{Ker}(T_g\pi)^\perp$$

$$T_g\pi\big|_{\operatorname{Ker}(T_g\pi)^\perp} : \operatorname{Ker}(T_g\pi)^\perp \xrightarrow{1-1} T_{\pi(g)} G/K$$ is a linear isomorphism

This construction is 1-1 correspondence

$\{$ K-invariant pos. def. sym. bil. forms on $\mathfrak{p}\} \leftrightarrow$

$\{$ G-invariant Riemannian metrics on $G/K = M\}$

a Riemannian metric g_M on $G/K = M$ is G-invariant if for every $g \in G$ the map

$(L_g) : G/K \longrightarrow G/K$, $g' \cdot K \longmapsto g \cdot g' \cdot K$

is a local isometry of g_M. (This map (L_g) is a diffeom.)

The space G/K is called a homogeneous space.

<u>Examples</u> 1) $\mathbb{R}^m = Euc(m)/O(m)$

$O(m) \ltimes \mathbb{R}^m = Euc(m)$.. the group of Euclidean motions

Any $O(m)$-invariant positive definite sym. bil. form on $\mathbb{R}^m$ gives rise to a $Euc(m)$-invariant Riemannian metric on $\mathbb{R}^m$

The standard inner product on $\mathbb{R}^m \longmapsto$ the standard (Euclidean) metric on $\mathbb{R}^m$

2) $S^m = O(m+1)/O(m) = SO(m+1)/SO(m) = Spin(m+1)/Spin(m)$

(under SO: orientability on S^m; under $Spin$: spin structure)

the standard round metric on S^m is $O(m+1)$-invariant metric

3) $\mathbb{H}^n_+ = SO_0(n,1)/SO(n)$

($\mathbb{H}^n_+$: n-dim hyperbolic space; $SO_0(n,1)$: the identity component of the identity element in $SO(n,1)$)

hyperbolic metric is an $SO_0(n,1)$-invariant metric on $\mathbb{H}^n_+$

These are all examples of Riemannian manifolds with constant sectional curvature

$\mathbb{R}^n$	0
S^m	1
$\mathbb{H}^n_+$	-1

Let us assume for a moment that (M, g) is a Riemannian manifold of dim. m. Then it is well known the set of all local isometries

$$M \longrightarrow M$$

is a Lie group G_M of dim $\leq m + \binom{m}{2}$. (The upper bound follows from the fact that any local isometry is completely determined by its tangent map at any given point $x \in M$ provided that M is connected.) In case that the group G_M of local isometries of (M, g) acts transitively on M, that is for any $x, y \in M$ $\exists g \in G$: $g \cdot x = y$, then (M, g) is isomorphic as a Riemannian geometry to

G_M / K where K is the stabilizer of fixed point $x \in M$.

Locally homogeneous Riemannian manifolds of finite volume (LHRM)

Def A model geometry is a simply connected smooth manifold M with a transitive action of a Lie group G on M with a compact stabilizer K (of some fixed point $x \in M$).

As we have seen above, then $M \cong G/K$ and M posses a G-invariant Riemannian metric g_M. Without loss of generality we may assume that G is a maximal group which acts on M by local isometries of g_M.

Def. LHRM is a quotient of a model geometry by some discrete subgroup Γ of G that acts on M freely and properly discontinuously.
(These two conditions guarantee that the space $\Gamma \backslash M$ is again a smooth manifold).

Model geometry $\quad M = G/K = \{gK : g \in G\}$

LHRM $\quad \Gamma \backslash M = \Gamma \backslash (G/K) = \{\Gamma g K : g \in G\}$

Example : $S^m = O(m+1)/O(m)$

$\{\pm Id_{\mathbb{R}^{m+1}}\} = \Gamma \subset O(m+1)$

$\Gamma \backslash S^m = \mathbb{RP}^m$ real projective space

8 model geometries in TGC

1. Spherical geometry

$S^3 = O(4)/O(3)$

LHRM -- Poincare sphere, lens spaces

Sectional curvature $+1$

2. Euclidean geometry

$\mathbb{R}^3 = \mathbb{R}^3 \rtimes O(3) / O(3)$

LHRM... 3-torus $(\mathbb{Z}^3 \backslash \mathbb{R}^3)$, mapping torus of a finite order aut. of T^2

in total 10 closed 3-manifolds with finite volume

Sectional curvature 0

3. Hyperbolic geometry

$\mathbb{H}^3_+ = SO(3,1)/O(3)$

LHRM are not completely classified in this case

Sectional curvature -1

4. $S^2 \times \mathbb{R}$

LHRM -- $S^2 \times S^1$, mapping torus of antipodal map $S^2 \to S^2$,
$S^2 \times I / (x,0) \sim (-x,1)$, $S^1 \times \mathbb{RP}^2$, ...

5. $\mathbb{H}^2_+ \times \mathbb{R}$

many examples of LHRM

6. Universal cover $\widetilde{SL}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$

LHRM are completely classified

7. Nil geometry: geometry on the Heisenberg group

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\} \subseteq GL(3, \mathbb{R})$$

LHRM are completely classified in this case

8. Sol geometry on $\mathbb{R}^3$

$$(x, y, z) \circ (x', y', z') = (x + e^{-z} x', \; y + e^{z} y', \; z + z')$$

$$e^{2x} \, dx \otimes dx + e^{-2z} \, dy \otimes dy + dz \otimes dz$$