Symbol of differential operators

<u>Definition</u> A smooth vector bundle of dimension n is a triple (E, M, π) (or $\pi : E \to M$) where

(VB1) E is a smooth manifold of dimension $m+n$,

(VB2) M is a — " — m,

(VB3) $\pi : E \to M$ is smooth map such that for every $x \in M$ the preimage $\pi^{-1}(x)$ is a real vector space of dimension n

which satisfies the following local triviality condition:

(LTC) $\forall x \in M$ there exists an open set $\mathcal{U}_x \subseteq M$ together with a diffeomorphism

$$\Phi_x : \mathcal{U}_x \times \mathbb{R}^n \longrightarrow \pi^{-1}(\mathcal{U}_x)$$

such that for every $x' \in \mathcal{U}_x$ we have that

$$\Phi_x(x', v) \in \pi^{-1}(x')$$

and that the map

$$\mathbb{R}^n \ni v \longmapsto \Phi_x(x', v) \ni \pi^{-1}(x')$$

is a linear isomorphism.

The space E is called the total space, M is called the base and π is called the canonical projection.

Remark: $E = \dot{\bigcup\limits_{x \in M}} \mathbb{R}^n$ and locally E looks as the product $\mathcal{U}_x \times \mathbb{R}^n$. However E need not be a global product $M \times \mathbb{R}^n$.

Example: If M is a smooth manifold, then TM, T^*M are smooth vector bundles of dimension m over M. Here TM as a vector bundle is (TM, M, π_{TM}). Other examples are $\Lambda^k T^*M$, $T^{k,l} M$ for $k = 0, 1, \ldots$, $l = 0, 1, \ldots$.

<u>Definition</u> Let (E, M, π) be a vector bundle of dimension n. Then a section of (E, M, π) is a smooth map

$$f : M \to E \quad \text{such that} \quad \pi \circ f : M \to M$$

is the identity map on M. We denote by $\Gamma(E)$ the set of all smooth sections of (E, M, π).

Example $\Gamma(TM) \cong \mathfrak{X}(M)$, $\Gamma(T^*M) \cong \Omega^1(M)$,

$\Gamma(\Lambda^k T^*M) \cong \Omega^k(M)$, $k = 0, 1, \ldots$.

Example $(E = M \times \mathbb{R}^n, M, \pi)$ is an example of an n-dimensional vector bundle over M which is called the trivial vector bundle.

$$\Gamma(E) \cong \mathcal{C}^\infty(M, \mathbb{R}^n) = \{ f : M \to \mathbb{R}^n \mid f \text{ smooth} \}$$

Lemma Let (E, M, π) be a smooth vector bundle over M. Then $\Gamma(E)$ is a vector space (of possible infinite dimension) with operations:

1) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
2) $(\lambda f)(x) = \lambda \cdot f(x)$

where $f, f_1, f_2 \in \Gamma(E)$, $\lambda \in \mathbb{R}$, $x \in M$.

Let us fix $x \in M$ and $\Phi_x : \mathcal{U}_x \times \mathbb{R}^n \to \pi^{-1}(\mathcal{U}_x)$ be the map as in (LTC). Assume that $f \in \Gamma(E)$. Then we may restrict f to the open set $\mathcal{U}_x$ and we get a composition

$$\mathcal{U}_x \hookrightarrow M \xrightarrow{f} E$$

which takes values in $\pi^{-1}(\mathcal{U}_x)$. And so using the inverse of Φ^{-1}_x we may consider

$$\mathcal{U}_x \hookrightarrow M \xrightarrow{f} E \xrightarrow{\Phi_x^{-1}} \mathcal{U}_x \times \mathbb{R}^n$$

$$\mathcal{U}_x \ni x' \longmapsto (x', (f_1(x'), \ldots, f_n(x')).$$

We see that in the trivialization Φ_x over $\mathcal{U}_x$ the section f can be viewed as a function

$$\mathcal{U}_x \longrightarrow \mathbb{R}^n, \quad x' \longmapsto (f_1(x'), \ldots, f_n(x')).$$

However note that this function depends on the choice of Φ_x.

Example If $\varphi : \mathcal{U} \to \mathbb{R}^m$ is a chart on a smooth manifold M with coordinate vector fields $\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \}$, then we know that $X \in \mathfrak{X}(M)$ is over $\mathcal{U}$ given by

$$X|_{\mathcal{U}} = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i} \quad \text{for some uniquely determined}$$

functions $a_i : \mathcal{U} \longrightarrow \mathbb{R}$, $i = 1, \ldots, m$. These functions $a_1, \ldots, a_m$ are precisely the functions prescribing the section X of the vector bundle TM over the set $\mathcal{U}$.

If $k \in \mathbb{N}_0$ and f, Φ_x are as above with f given by the collection

$$\mathcal{U}_x \longrightarrow \mathbb{R}^n, \quad x' \longmapsto (f_1(x'), \ldots, f_n(x')),$$

then it can be verified that the following definition does not depend on the choice of Φ_x:

<u>Definition</u> We write

$$j^k_x f = 0 \text{ if}$$

all partial derivatives of all $f_1, \ldots, f_n$ up to order k vanish at x.

Given $f_1 \in \Gamma(E)$, then we denote by $j^k_x f_1$ the equivalence class

$$\{ f_2 \in \Gamma(E) : \text{all partial derivatives of } f_1 - f_2 \in \Gamma(E) \text{ up to order } k \text{ vanish at } x \}.$$

<u>Definition</u> We call $j^k_x f$ the k-jet of f at point x.

Example: $M = \mathbb{R}^m$, $E = M \times \mathbb{R}^n$ so that $\Gamma(E) \cong \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$, then for $x = 0 \in \mathbb{R}^m$ we have that

$$j^0_0 f = 0 \quad \text{iff} \quad f = (f_1, \ldots, f_n), \; f_i(0) = 0, \; i = 1, \ldots, n$$

$$j^1_0 f = 0 \quad \text{iff} \quad j^0_0 f = 0, \; \frac{\partial f_j}{\partial x_i}(0) = 0, \; i = 1, \ldots, m, \; j = 1, \ldots, n$$

$$j^2_0 f = 0 \quad \text{iff} \quad j^1_0 f = 0, \; \frac{\partial^2 f_\ell}{\partial x_i \partial x_j} = 0, \; i, j = 1, \ldots, m, \; \ell = 1, \ldots, n$$

$$\vdots$$

Example: $M = \mathbb{R}^n$, $E = TM$, $X \in \mathfrak{X}(M)$ then

$$j^k_0 X = 0 \quad \text{iff} \quad X = \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i}, \quad j^k_0 a_i = 0, \; i = 1, \ldots, m.$$

Definition Let (E, M, π_E) and (F, M, π_F) be vector bundles over the base M. A linear map

$$D: \Gamma(E) \longrightarrow \Gamma(F)$$

is called a differential operator of order at most k if

$$Df(x) = 0 \quad \text{whenever} \quad j^k_x f = 0 \quad \text{for any } x \in M, f \in \Gamma(E).$$

D is called a differential operator of order k if it is a differential operator of order at most k but not of order at most $k-1$.

Example: 1) $\quad d: \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, $k = 0, \dots, m-1$; $m = \dim M$, is a diff. operator of order 1 since in a local chart

$$d\Big(\sum_I{}' f_I \, dx_I\Big) = \sum_I{}' \sum_{i=1}^m \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I \, .$$

2) $\quad \Delta: \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ is a differential operator of order 2 since in a local chart

$$\Delta f = -\frac{1}{\sqrt{\det g}} \sum_{i=1}^m \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^m \frac{\partial f}{\partial x_j} g^{ij} \sqrt{\det g}\Big), \quad f \in \mathcal{C}^\infty(M)$$

Assume that

$$D: \Gamma(E) \longrightarrow \Gamma(F)$$

is a diff. op. of order k and that $f \in \mathcal{C}^\infty(M)$. If $\sigma \in \Gamma(E)$, then we can consider

$$\Gamma(E) \longrightarrow \Gamma(F), \ \sigma \longmapsto [D, f]\sigma := D(f\sigma) - f(D\sigma).$$

It can be verified $[D, f]$ is a differential operator of order $k-1$. We can now iterate this construction k times and so if $f_1, \dots, f_k$ are smooth functions on M, then

$$\Gamma(E) \longrightarrow \Gamma(F); \ \sigma \longmapsto [\dots[[D, f_1], f_2], \dots, f_k]\,\sigma$$

is a differential of order 0. But this means that

$$[\dots, [[D, f_1], f_2], \dots, f_k]\,\sigma(x)$$

depends only on $j^0_x \sigma = \sigma(x)$. Moreover it can be verified that it depends only on $df_1(x), \dots, df_k(x) \in T^*_x M$ and it is symmetric in $df_1, \dots, df_k$. We see that for fixed $x \in M$ and D we get a map

$$(\text{SoD}) \quad S^k T_x^* M \otimes E_x \longrightarrow F_x$$

$$((df_1, \dots, df_k), \sigma(x)) \longmapsto \sigma_k(D)(df, \sigma) := [\dots [[D, f_1], \dots, f_k] \sigma(x).$$

$$df = (df_1, \dots, df_k).$$

<u>Definition</u> The map (SoD) is called the symbol of D.

Details can be found in

L. Nicolaescu: Geometry of Manifolds, Chapter 10, see RG I.

If we now fix $v \in T_x^* M$, $x \in M$ and choose $f \in \mathcal{C}^\infty(M)$ so that $v = df_0(x)$ then by the symbol of D determined by v and x is also understood the map

$$(\text{SoDI}) \qquad E_x \longrightarrow F_x, \quad \sigma_k(D)(v, \sigma)$$

with $df = (df_0, \dots, df_0)$.

<u>Definition</u> The map (SoDI) is called the symbol of D determined by $v \in T_x^* M$.

The map in (SoDI) is given by a homogeneous polynomial of degree k in components of v.