<u>Example</u> The symbol of the Laplace – Beltrami operator

In coordinates the Laplace – Beltrami operator is given by

$$\Delta u = -\sum_{i,j=1}^{m} g^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \underbrace{l.o.t.}$$

terms that depends on partial derivatives of $u \in \mathcal{C}^\infty(M)$ of order at most 1

let $f \in \mathcal{C}^\infty(M)$, then the symbol $\sigma_2(\Delta)(v,-)$ for $v = df_x$ with $x \in M$ can be computed from

$$\sigma_2(\Delta)(v, u) = \frac{1}{2!}[[\Delta, f], f]\, u\ (x)$$

$$[[\Delta, f], f]\, u = [\Delta, f](f u) - f([\Delta, f] u)$$

$$= (\Delta f - f\Delta)(f u) - f((\Delta f - f\Delta) u)$$

$$= \Delta(f^2 u) - f\Delta(f u) - f\Delta(f u) + f^2 \Delta u$$

$$= -g^{ij}\frac{\partial^2}{\partial x_i \partial x_j}(f^2 u) + f g^{ij}\frac{\partial^2}{\partial x_i \partial x_j}(f u)$$

$$+ f g^{ij}\frac{\partial^2}{\partial x_i \partial x_j}(f u) - f^2 g^{ij}\frac{\partial^2}{\partial x_i \partial x_j} u = (*)$$

$$\frac{\partial^2}{\partial x_i \partial x_j}(f^2 u) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} f^2 u\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f^2}{\partial x_j} u + f^2 \frac{\partial u}{\partial x_j}\right)$$

$$= \frac{\partial^2 f^2}{\partial x_i \partial x_j} u + \frac{\partial f^2}{\partial x_j}\frac{\partial u}{\partial x_i} + \frac{\partial f^2}{\partial x_i}\frac{\partial u}{\partial x_j} + f^2 \frac{\partial^2 u}{\partial x_i \partial x_j} = (\square)$$

$$\frac{\partial^2 f^2}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\frac{\partial f^2}{\partial x_j} = \frac{\partial}{\partial x_i}\left(2f\frac{\partial f}{\partial x_j}\right) = 2\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} + 2f\frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$(\square) = 2\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} u + 2f\frac{\partial^2 f}{\partial x_i \partial x_j} u + \frac{\partial f^2}{\partial x_j}\frac{\partial u}{\partial x_i} + \frac{\partial f^2}{\partial x_i}\frac{\partial u}{\partial x_j} +$$

$$+ f^2\frac{\partial^2 u}{\partial x_i \partial x_j}$$

$$(*) = -g_{ij}\frac{\partial^2}{\partial x_i \partial x_j}(f^2 u) + 2f g^{ij}\frac{\partial^2}{\partial x_i \partial x_j}(f u) - f^2 g_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}$$

$$= -g^{ij}\Big(2\,\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\,u + 2f\,\cancel{\frac{\partial^2 f}{\partial x_i \partial x_j}}\,u + 2f\,\cancel{\frac{\partial f}{\partial x_j}\frac{\partial u}{\partial x_i}} + 2f\,\cancel{\frac{\partial f}{\partial x_i}\frac{\partial u}{\partial x_j}} +$$

$$+ f^2\,\cancel{\frac{\partial^2 u}{\partial x_i \partial x_j}}\Big)$$

$$+ 2f\,g^{ij}\,\frac{\partial^2}{\partial x_i \partial x_j}(fu) - f^2 g_{ij}\,\cancel{\frac{\partial^2 u}{\partial x_i \partial x_j}} = (\Delta)$$

$$\frac{\partial^2}{\partial x_i \partial x_j}(fu) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} f u\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}u + f\,\frac{\partial u}{\partial x_j}\right)$$

$$= \cancel{\frac{\partial^2 f}{\partial x_i \partial x_j}}\,u + \cancel{\frac{\partial f}{\partial x_j}\frac{\partial u}{\partial x_i}} + \cancel{\frac{\partial f}{\partial x_i}\frac{\partial u}{\partial x_j}} + f\,\cancel{\frac{\partial^2 u}{\partial x_i \partial x_j}}$$

$$(\Delta) = -2g^{ij}\,\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\,u = -2\,\|df\|^2 u = -2\,\|v\|^2 u$$

We see that the symbol of Δ is

$$\sigma_2(\Delta)(v, u(x)) = -\|v\|^2 u(x), \text{ that is,}$$

$$\sigma_2(\Delta)(v,-) : \mathbb{R}_x \longrightarrow \mathbb{R}_x$$

is multiplication by $-\|v\|^2$. Here $\mathbb{R}_x$ stands for the fiber of the trivial bundle $\mathbb{R}\times M$ over $x \in M$.

<u>Definition</u> The differential operator $D : \Gamma(E) \to \Gamma(F)$ as above is called elliptic if the symbol map $\sigma_k(D)(v,-) : E_x \to F_x$ is for every $x \in M$ and $0 \neq v \in T_x^* M$ linear isomorphism.

We see that Δ fulfills this definition.

Another elliptic operator on (M,g) is

$$\Delta = dd^* + d^*d = (d+d^*)^2 : \Omega^k(M) \to \Omega^k(M)$$

since $d^2 = 0$ and $(d^*)^2 = 0$. Operator $d + d^*$ is called the Hodge-Dirac operator and Δ is called the Hodge-Laplace operator.

Example Symbol of d

In coordinate chart $\varphi: U \to \mathbb{R}^m$ as usual and for $df = v \in T_x^* M$ we have that

$$\sigma_1(d)(v,-) \quad \text{is defined as}$$

$$[d, f]\, h = \quad d(fh) - f\,dh, \quad h \in \Omega^k(M).$$

Assume that

$$h = \sum_{I: |I| = k}' h_I \, dx_I, \text{ then}$$

$$d(fh) - f\,dh = d\Big(\sum' f\, h_I \, dx_I\Big) - f\, d\Big(\sum' h_I \, dx_I\Big)$$

$$= \sum' \big(d(f h_I \, dx_I) - f\, d(h_I \, dx_I)\big)$$

$$= \sum' \big(df \wedge h_I \, dx_I + f\, d(h_I \, dx_I) - f\, d(h_I \, dx_I)\big)$$

$$= \sum' df \wedge h_I \, dx_I = df \wedge h.$$

We see that

$$\sigma_1(d)(v,-) : \Lambda^k T_x^* M \longrightarrow \Lambda^{k+1} T_x^* M,$$

$$\sigma_1(d)(v, h(x)) = v \wedge h(x).$$

<u>Theorem</u> Let F_1, F_2, F_3 be (the total spaces) of vector bundles over the smooth manifold M and assume that

$$D_1 : \Gamma(F_1) \longrightarrow \Gamma(F_2) \text{ and } D_2 : \Gamma(F_2) \longrightarrow \Gamma(F_3)$$

are differential operators of degree k and l, respectively.

Then $$D_2 \circ D_1 : \Gamma(F_1) \longrightarrow \Gamma(F_3)$$

is a differential operator of degree at most $k+l$ and

$$\sigma_{k+l}(D_2 \circ D_1)(v,-) : (F_1)_x \longrightarrow (F_3)_x$$

for $v \in T_x^* M$ is given by

$$\sigma_{k+l}(D_2 \circ D_1)(v,-) = \sigma_l(D_2)(v,-) \circ \sigma_k(D_1)(v,-).$$

Proof can be found in lecture notes Geometry of Manifolds by L. Nicolaescu. $\square$

Example   Symbol of d^*

Assume that $\omega_1, \dots, \omega_m$ is an orthonormal coframing in the chart φ. Then by Stoke's theorem, the operator d^* is given by

$$d^*(h) = d^*\Big(\sum_{I:|I|=k}{}' \tilde{h}_I \, \omega_I\Big)$$

$$= d^*\Big(\sum_{I:|I|=k-1}{}' \sum_{i=1}^{m} \tilde{\tilde{h}}_{iI} \, \omega_{iI}\Big)$$

$$= -\Big(\sum_{I:|I|=k-1}{}' \sum_{i=1}^{m} \frac{\partial \tilde{\tilde{h}}_{iI}}{\partial \omega_i} \omega_I + \underbrace{l.o.t.}\Big)$$

there is no differentiation of the coefficient function $\tilde{h}_{iI}$

Here $\frac{\partial}{\partial \omega_1}, \dots, \frac{\partial}{\partial \omega_m}$ are the dual framing over φ w.r.t. to $\omega_1, \dots, \omega_m$. The symbol of d^* is given by

$$\sigma_1(d^*)(v, h(x)) = \big([d^*, f]\, h\big)(x) =$$

$$= (d^* fh)(x) - \big(f(d^* h)\big)(x)$$

$$= \Big(-\sum_{I:|I|=k-1} \sum_i \frac{\partial f \tilde{\tilde{h}}_{iI}}{\partial \omega_i} \omega_I\Big) + f\Big(\sum_{I:|I|=k-1} \sum_i \frac{\partial \tilde{\tilde{h}}_i}{\partial \omega_i} \omega_I\Big)$$

$$= -\sum_{I:I} \sum_{i=1}^{m} \frac{\partial f}{\partial \omega_i} \tilde{h}_{iI}\, \omega_I = -\, i_{v^\#} h$$

i is the insertion operator

$$i_{v^\#} : \Lambda^k T_x^* M \longrightarrow \Lambda^{k-1} T_x^* M.$$

Example   Symbol of $\Delta = dd^* + d^* d$ is then

$$\sigma_2(\Delta)(v, -) = \sigma_1(d)(v,-) \circ \sigma_1(d^*)(v,-)$$
$$+ \sigma(d^*)(v,-) \circ \sigma_1(d)(v,-)$$
$$= -(v \wedge -) \circ i_{v^\#} - i_{v^\#} \circ (v \wedge -)$$
$$= - \|v\|^2 (-)$$

$\sigma_2(\Delta)(v, h(x)) = -\|v\|^2 h(x)$ with $h(x) \in \Lambda^k T_x^* M$.

In the previous calculation we have used the following lemma

<u>Lemma</u> Let V be a finite dimensional real vector space with inner product g and $v \in V^*$. Then the sequence

$$(C) \qquad \cdots \to \Lambda^{k-1} V^* \xrightarrow{v\wedge -} \Lambda^k V^* \xrightarrow{v\wedge -} \Lambda^{k+1} V^* \xrightarrow{v\wedge -} \cdots$$

is exact and the adjoint operator to $v\wedge -$ is precisely $i_{v^\#}$.

Proof: It is clear that (C) is a complex. It is also easy to see that $i_{v^\#}$ is the adjoint operator to $v\wedge -$ (it is enough to verify under the assumption that $\|v\| = 1$ and that it is the vector vector in some orthonormal basis). It remains to show that (C) is exact.

We have decomposition

$$\Lambda^k V^* = \underbrace{\mathrm{Im}\,(v\wedge -) \oplus H^k(C)}_{\mathrm{Ker}\,(v-)} \oplus \mathrm{Im}(i_{v^\#}),$$

This decomposition is orthogonal w.r. to the inner product on $\Lambda^k V^*$ induced by g. Let us assume that $v = v_1$ has unit length and that $(v_1, \dots, v_m)$ is an orthonormal basis of V^*. Then

$$\Lambda^k V^* \ni w = \sum_{I : |I| = k}' a_I v_I, \quad a_I \in \mathbb{R}.$$

$$= \sum_{\substack{I : |I| = k-1 \\ 1 \notin I}}' a_{1I} v_{1I} + \sum_{\substack{I : |I| = k \\ 1 \notin I}}' b_I v_I.$$

Then $$i_{v^\#} \circ v \wedge w = \|v\|^2 \sum_{\substack{I : |I| = k \\ 1 \notin I}}' b_I v_I \in \mathrm{Im}\, i_{v^\#},$$

and $$v \wedge (i_{v^\#} w) = \|v\|^2 \sum_{\substack{I : |I| = k-1 \\ 1 \notin I}}' a_{1I} v_{1I} \in \mathrm{Im}\, v\wedge. \qquad \square$$

Theorem Let (M, g) be a closed Riemannian manifold and Δ be the Laplace – Beltrami operator

$$\Delta : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M).$$

1) Then there exists a complete orthonormal basis $\{f_n\}_{n \in \mathbb{N}}$ of $L^2(M)$ (= the Hilbert space of all measurable and square integrable function on M w.r. to ω_g) which consists of eigenvectors of Δ, that is $-\Delta f_n = \lambda_n f_n$.

2) The eigenvectors f_n are smooth $M \longrightarrow \mathbb{R}$ and $\lim_{n \to +\infty} \lambda_n = +\infty$ (we already know that $\lambda_n \geq 0$).

3) There $\exists\, C > 0$ and $\delta > 0$ such that $\lambda_n \geq C \cdot n^\delta$ if n is big enough ($\exists\, n_0 \in \mathbb{N}: n \geq n_0$).

Lemma 1.6.3 from Gilkey: Invariance Theory, the Heat Equation and the Atiyah-Singer index theorem.

The previous theorem holds for any self-adjoint and elliptic differential operator D on M.

(D is self-adjoint if the continuous extension of D to $L^2(M)$ is self-adjoint.)