Homework 1. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, $M := \{(x,y) : f(x,y) = 0\}$ and assume that $\nabla f \neq 0$ on M. Show that M has a canonical structure of smooth manifold.
(Use Implicit Function Theorem.)

Solution: Since f is continuous, $f^{-1}\{0\} = M$ is closed. On M we consider the subspace topology given by $M \subseteq \mathbb{R}^2$. It is well known that then M is Hausdorff and second countable as these properties are hereditary for subspace topology.
It remains to construct atlas on M which turns M into a topological and smooth manifold.
Let

$$M_1 = \{ z \in M : \frac{\partial f}{\partial x_1}(z) \neq 0\} \quad \text{and}$$
$$M_2 = \{ z \in M : \frac{\partial f}{\partial x_2}(z) \neq 0\}.$$

Now M_1, M_2 are open in M and $M_1 \cup M_2 = M$. Now we have to consider two cases

1) If $z = (x,y) \in M_1$, then by Implicit Function Theorem (IFT) there is
- an open interval U_x^1 containing x and
- an open interval U_y^1 containing y and
- a unique smooth function $g_z^1 : U_x^1 \to U_y^1$ such that

$$V_z^1 := M \cap (U_x^1 \times U_y^1) = \{ (u, g_z^1(u)) : u \in U_x^1 \}.$$

2) If $z = (x,y) \in M_2$, then by IFT there is
- an open interval U_x^2 containing x and
- an open interval U_y^2 containing y and
- a unique smooth function $g_z^2 : U_y^2 \to U_x^2$ such that

$$V_z^2 := M \cap (U_x^2 \times U_y^2) = \{ (g_z^2(v), v) : v \in U_y^2 \}.$$

Now if $z \in M_i$, let $\varphi_z^i : V_z^i \to \mathbb{R}$, $\varphi_z^i(u,v) = \begin{cases} u, & i = 1, \\ v, & i = 2. \end{cases}$

Note that φ_z^i is injective and

$$\varphi_z^i(V_z^i) = \begin{cases} U_x^1, & i = 1, \\ U_y^2, & i = 2. \end{cases}$$

We claim that $\mathcal{A} = \{ \varphi_z^i : V_z^i \to \mathbb{R} \mid z \in M_i, i = 1,2\}$ is a smooth atlas on M.

- By definition of subspace topology, V_z^i is open in M (if defined) and $\bigcup_{i=1,2} \bigcup_{z \in M_i} V_z^i = M$.
- Now $\varphi_z^1 : U_z^1 \to U_x^1$, $(u,v) \mapsto u$ is bijective. Its inverse is $U_x^1 \ni u \mapsto (u, g_z^1(u))$. Now this map is smooth and regular, i.e. its derivative is everywhere nonvanishing. Now it is well known that any such map is a homeomorphism onto its image. This shows that φ_z^1 is a homeomorphism onto its image. Similarly for φ_z^2.

•) It remains to verify that all transition maps are smooth.

If $\emptyset \neq V^1_{z_1} \cap V^1_{z_2}$, then by uniqueness in IFT

$$\varphi^1_z \circ (\varphi^1_z)^{-1} : \mathcal{U}^1_{x_1} \cap \mathcal{U}^1_{x_2} \longrightarrow \mathcal{U}^1_{x_1} \cap \mathcal{U}^1_{x_2}$$

is the identity map.

If $\emptyset \neq V^2_{z_1} \cap V^2_{z_2}$, then again

$$\varphi^2_{z_2} \circ (\varphi^2_{z_1})^{-1} : \mathcal{U}^2_{y_1} \cap \mathcal{U}^2_{y_2} \longrightarrow \mathcal{U}^2_{y_1} \cap \mathcal{U}^2_{y_2}$$

is the identity map

If $\emptyset \neq V^1_{z_1} \cap V^2_{z_2}$, then

$$\varphi^2_{z_2} \circ (\varphi^1_{z_1})^{-1} : \mathcal{U}^1_{x_1} \cap \mathcal{U}^1_{x_2} \longrightarrow \mathcal{U}^2_{y_1} \cap \mathcal{U}^2_{y_2}$$

$$u \longmapsto g^1_{z_1}(u) .$$

Finally if $\emptyset \neq V^2_{z_2} \cap V^1_{z_1}$, then

$$\varphi^1_{z_1} \circ (\varphi^2_{z_2})^{-1} : \mathcal{U}^2_{y_1} \cap \mathcal{U}^2_{y_2} \longrightarrow \mathcal{U}^1_{x_1} \cap \mathcal{U}^1_{x_2}$$

$$v \longmapsto g^2_{z_2}(v).$$

All these maps are smooth which completes the proof.