Homework 2 Let $G := GL(n,\mathbb{R})$ be the group of all real invertible $n\times n$ matrices.

1) Show that G is an open subset of the vector space of $n\times n$ matrices $M(n,\mathbb{R})$ (and hence G is a smooth manifold in $\mathbb{R}^{n^2} \cong M(n,\mathbb{R})$.)

2) Show that $T_e G$ is (as a real vector space) isomorphic to $M(n,\mathbb{R})$. Here e is the identity $n\times n$ matrix.

3) If $g \in G$, let $R_g : G \to G$, $R_g(h) = h\cdot g$. Show that R_g is smooth. Let $T_e R_g : T_e G = M(n,\mathbb{R}) \to T_g G$ be its tangent map at the identity element (indeed $R_g(e) = e\cdot g = g$).

4) If $A \in M(n,\mathbb{R}) = T_e G$, let $X_A : G \to TG$, $X_A(g) = T_e R_g(A)$. Show that X_A is a smooth vector field.

5) For $B \in M(n,\mathbb{R})$ calculate $[X_A, X_B]$ and show that
$$[X_A, X_B] = -X_{[A,B]}$$
where $[A,B] = AB - BA$.

Solution : ad 1) Consider $\det : M(n,\mathbb{R}) \to \mathbb{R}$
$$\det (x_{ij})_{i=1,\dots,n}^{j=1,\dots,n} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)}.$$
Here S_n is the permutation group on $\{1,\dots,n\}$ and $\operatorname{sgn}(\sigma)$ is the sign of $\sigma \in S_n$. Now we view $M(n,\mathbb{R})$ as the vector space $\mathbb{R}^{n^2}$ with coordinate functions x_{ij}, $i,j = 1,\dots,n$. Now $\det$ is smooth (it is even polynomial), thus it is also continuous and so
$$\det^{-1}(\mathbb{R}\setminus\{0\}) = GL(n,\mathbb{R})$$
is open in $M(n,\mathbb{R})$ as $\mathbb{R}\setminus\{0\}$ is open in $\mathbb{R}$. (Here we are using well known fact that $A \in GL(n,\mathbb{R}) \Leftrightarrow \det A \neq 0$.)

ad 2) By Example 2) at the end of notes for week 1, we know that $GL(n,\mathbb{R})$ is a smooth manifold (inside $M(n,\mathbb{R})$) with atlas
$$\mathcal{A} = \{ \iota : GL(n,\mathbb{R}) \to M(n,\mathbb{R}) = \mathbb{R}^{n^2} \}$$
where ι is the canonical inclusion.

Now the tangent space $T_A M(n,\mathbb{R})$ at any $A \in M(n,\mathbb{R})$ is isomorphic to $M(n,\mathbb{R})$ via the map
$$M(n,\mathbb{R}) \ni B \mapsto [\gamma_B] \in T_A M(n,\mathbb{R})$$
where $\gamma_B(t) = A + tB$, $t\in\mathbb{R}$. From week 2 we know that the map
$$T_e\iota : T_e G \to T_e M(n,\mathbb{R}) \cong M(n,\mathbb{R})$$
is a linear isomorphism. This proves the second point.

ad 3) By definition, we have to verify that the map
$$(*) \quad M(n,\mathbb{R}) \supseteq \iota(G) \xrightarrow{\iota^{-1}} G \xrightarrow{R_g} G \xrightarrow{\iota} M(n,\mathbb{R}),$$
is smooth. (Note that $R_g : G\to G$, $R_g(h) = hg$ is well defined since product of two regular matrices is regular.) The map $(*)$ is
$$M(n,\mathbb{R}) \ni (x_{ij})_{i=1,\dots,n}^{j=1,\dots,n} \mapsto \Big(\sum_{k=1}^{n} x_{ik} g_{kj}\Big)_{i=1,\dots,n}^{j=1,\dots,n}$$
where $g = (g_{ij})_{i=1,\dots,n}^{j=1,\dots,n}$. Since the coefficients of $\Big(\sum_{k=1}^{n} x_{ik} g_{kj}\Big)_{i=1,\dots,n}^{j=1,\dots,n}$ depend smoothly on x_{ij}, the map R_g is smooth.

ad 4) Fix $A = (a_{ij})_{i=1,\dots,n}^{j=1,\dots,n} \in M(n,\mathbb{R})$. We have to verify that

a) $X_A(g) = T_e R_g(A) \in T_g G$ for every $g \in G$ and

b) if we write $\mathbb{X}_A(g) = \sum_{i,j=1}^{n} f_{ij}(g) \frac{\partial}{\partial x_{ij}}$, then the coefficient functions $f_{ij}(g)$ depend smoothly on matrix coefficients of g.

Write $g = (x_{kl})_{k=1,\dots,n}^{l=1,\dots,n}$.

ad a) By definition, $T_e R_g : T_e G \to T_{R_g(e)} G = T_{e\cdot g} G = T_g G$ and so indeed $\mathbb{X}_A(g) \in T_g G$.

ad b) $\mathbb{X}_A(g)$ is by definition the tangent vector to the curve $t \mapsto (At+e)g$ at $t=0$.

We have
$$\frac{d}{dt}(At+e)g\Big|_{t=0} = Ag = \Big(\sum_{k=1}^{n} a_{ik} x_{kj}\Big)_{i=1,\dots,n}^{j=1,\dots,n}$$

and so
$$\mathbb{X}_A(g) = \sum_{i,j=1}^{n} \Big(\overbrace{\sum_{k=1}^{n} a_{ik} x_{kj}}^{f_{ij}(g)}\Big) \frac{\partial}{\partial x_{ij}}.$$

ad 5) Let $B = (b_{ij})_{i=1,\dots,n}^{j=1,\dots,n}$ and write $\mathbb{X}_B(g) = \sum_{i,j=1}^{n} \Big(\overbrace{\sum_k b_{ik} x_{kj}}^{g_{ij}(g)}\Big) \frac{\partial}{\partial x_{ij}}$.

Then
$$[\mathbb{X}_A, \mathbb{X}_B](g) = \sum_{i,j=1}^{n} \Big(\sum_{k,l=1}^{n} \Big[f_{kl}(g) \frac{\partial g_{ij}}{\partial x_{kl}}() - g_{kl}() \frac{\partial f_{ij}}{\partial x_{kl}}(g) \Big] \Big) \frac{\partial}{\partial x_{ij}}$$
$$= \sum_{i,j=1}^{n} \Big(\sum_{k,l=1}^{n} \Big[f_{kl}(g) \sum_{m=1}^{n} b_{im} \delta_{km} \delta_{lj} - g_{kl}(g) \sum_{m=1}^{n} a_{im} \delta_{km} \delta_{lj} \Big] \Big) \frac{\partial}{\partial x_{ij}}$$
$$= \sum_{i,j=1}^{n} \Big(\sum_{k=1}^{n} \big[f_{kj}(g) b_{ik} - g_{kj}(g) a_{ik} \big] \Big) \frac{\partial}{\partial x_{ij}}$$
$$= \sum_{i,j=1}^{n} \Big(\sum_{k,l=1}^{n} \big(a_{kl} x_{lj} b_{ik} - b_{kl} x_{lj} a_{ik} \big) \Big) \frac{\partial}{\partial x_{ij}}$$
$$= -\sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \big(a_{ik} b_{kl} - b_{ik} a_{kl} \big) x_{lj} \Big) \frac{\partial}{\partial x_{ij}}$$
$$= -\sum_{i,j=1}^{n} \sum_{l=1}^{n} \Big(\sum_{k=1}^{n} \big(a_{ik} b_{kl} - b_{ik} a_{kl} \big) x_{lj} \Big) \frac{\partial}{\partial x_{ij}}$$
$$= -\mathbb{X}_{[A,B]}.$$

$\square$