Homework 3 Conservation law and Noether's Theorem

Let us consider a particle with mass m in $\mathbb{R}^3$ which is moving under the influence of a conservative field with potential $U = U(x_1, x_2, x_3)$.
Let $\gamma : I \to \mathbb{R}^3$ be the trajectory of the particle (here I is an interval) and $\tilde{\gamma} : I \to T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$, $\tilde{\gamma}(t) = (\gamma(t), \gamma'(t))$, $t \in I$, be its lift to $T\mathbb{R}^3$.
On $T\mathbb{R}^3$ we use the standard coordinates $(x_1, x_2, x_3, p_1, p_2, p_3)$.
We view U and $T = T(p_1, p_2, p_3) = \frac{1}{2} m (p_1^2 + p_2^2 + p_3^2)$ as functions on $T\mathbb{R}^3$
(note that
$$T(\tilde{\gamma}(t)) = \frac{1}{2} m \left(\gamma_1'(t)^2 + \gamma_2'(t)^2 + \gamma_3'(t)^2 \right)$$
is the kinetic energy of γ at time t and so T is usually called the kinetic energy). Then
$$L = T - U : T\mathbb{R}^3 \to \mathbb{R}$$
is called Lagrangian and $X \in \mathfrak{X}(\mathbb{R}^3)$ is called a symmetry of L if
$$\tilde{X} L = \left(\sum_{i=1}^{3} a_i(x) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{3} p_j \frac{\partial a_i(x)}{\partial x_j} \frac{\partial}{\partial p_i} \right) L = 0.$$

1) Show that Newton's equations of motion
$$m \gamma'' = F = -\nabla U$$
are equal to the Euler–Lagrange equations
$$\frac{\partial L}{\partial x_i}(\gamma(t), \gamma'(t)) = \frac{d}{dt} \left(\frac{\partial L}{\partial p_i} (\gamma(t), \gamma'(t)) \right), \quad i = 1, 2, 3.$$

2) Show that if $X = \sum_{i=1}^{3} a_i(x) \frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathbb{R}^3)$ is a symmetry of L, then $\sum_{i=1}^{3} a_i(x) \frac{\partial L}{\partial p_i}$ is a conserved quantity, that is, the function
$$I \ni t \longmapsto \sum_{i=1}^{3} a_i(\gamma(t)) \frac{\partial L}{\partial p_i}(\gamma(t), \gamma'(t))$$
is constant if γ is a solution of the Euler–Lagrange equation.

3) If $U = 0$, show that $\frac{\partial}{\partial x_i}$, $i = 1, 2, 3$, are symmetries of L and compute the corresponding conserved quantities.

4) If U depends only on the distance from $0 \in \mathbb{R}^3$, show that X_i, $i = 1, 2, 3$, (from Example on page 1) are symmetries of L and compute the corresponding conserved quantities.

(In this exercise we assume that $U \in C^1(\mathbb{R}^3)$ and that $\gamma \in C^2(I)$.)

Solution

ad 1)
$$\frac{\partial L}{\partial x_i} = -\frac{\partial U}{\partial x_i}, \quad \frac{\partial L}{\partial p_i} = m p_i \quad \text{and so}$$
$$-\frac{\partial U}{\partial x_i}(\gamma(t), \gamma'(t)) = \frac{\partial L}{\partial x_i}(\gamma(t), \gamma'(t)) = \frac{d}{dt} \frac{\partial L}{\partial p_i}(\gamma(t), \gamma'(t)) = \frac{d}{dt} m \gamma_i'(t) = m \gamma_i''(t), \quad i = 1, 2, 3.$$
This equations are equivalent to Newton's equation of motion
$$-\nabla U(\gamma(t)) = F(\gamma(t)) = m \gamma''.$$

ad 2) Consider
$$t \longmapsto \sum_{i=1}^{3} a_i(\gamma(t)) \frac{\partial L}{\partial p_i}(\gamma(t), \gamma'(t)) = \sum_{i=1}^{3} a_i(\gamma(t))\, m \gamma_i'(t).$$
If we differentiate w.r.t. to t, we get
$$\sum_{i,j=1}^{3} \frac{\partial a_i}{\partial x_j}(\gamma(t))\, \gamma_j'(t)\, m \gamma_i'(t) + \sum_{i=1}^{3} a_i(\gamma(t))\, m \gamma_i''(t) =$$
$$= \sum_{i,j=1}^{3} \frac{\partial a_i}{\partial x_j}(\gamma(t))\, \gamma_j'(t)\, m \gamma_i'(t) - \sum_{i=1}^{3} a_i(\gamma(t)) \frac{\partial U}{\partial x_i}(\gamma(t)) =$$
$$= (\tilde{X} L)(\gamma(t), \gamma'(t)) = 0.$$
This shows that the function $\sum_{i=1}^{3} a_i \frac{\partial L}{\partial p_i}$ is constant along any solution of the Euler–Lagrange equation.

ad 3) Note that for fixed $i=1,2,3$ we have $\widetilde{\frac{\partial}{\partial x_i}} = \frac{\partial}{\partial x_i} \in \mathfrak{X}(T\mathbb{R}^3)$ and that

$$\widetilde{\frac{\partial}{\partial x_i}} L = \frac{\partial}{\partial x_i} T = \frac{\partial}{\partial x_i} \sum_{j=1}^{3} \frac{1}{2} m p_j^2 = 0.$$

Hence $\sum_{j=1}^{3} a_j(x) \frac{\partial L}{\partial p_j} = \sum_{j=1}^{3} \delta_{ij} m p_j^2 = m p_i^2$ is constant along any solution of the Euler–Lagrange equation. It follows that the momentum

$$\vec{p} = m(p_1^2, p_2^2, p_3^2)$$

is a conserved quantity if $U = 0$. (This is so called free particle.)

ad 4) If U depends only on $r^2 = x_1^2 + x_2^2 + x_3^2$, there is a function of one variable u such that $U(x_1, x_2, x_3) = u(x_1^2 + x_2^2 + x_3^2)$. Then for

$$\mathbb{X}_1 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \quad \text{we have} \quad \widetilde{\mathbb{X}}_1 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + p_2 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial p_2}.$$

Hence

$$\widetilde{\mathbb{X}}_1 L = \widetilde{\mathbb{X}}_1 (T - U) = m p_2 p_1 - m p_1 p_2 - \left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}\right) U$$

$$= - x_2 u' 2x_1 + x_1 u' 2x_2 = 0 \quad . \text{ (Here } u' \text{ is the derivative of } u.\text{)}$$

The conserved quantity is

$$\sum_{i=1}^{3} a_i(x) \frac{\partial L}{\partial p_i} = x_2 m p_1 - x_1 m p_2 = m(x_2 p_1 - x_1 p_2)$$

which is the angular momentum w.r. to the axis z.

Similarly, for $\widetilde{\mathbb{X}}_2$ and $\widetilde{\mathbb{X}}_3$ we find that the conserved quantities are

$$m(x_1 p_3 - x_3 p_1) \quad \text{and} \quad m(x_2 p_3 - x_3 p_2).$$

Hence, the angular momentum

$$m \vec{x} \times \vec{p} = m(x_1, x_2, x_3) \times (p_1, p_2, p_3)$$

$$= m(x_2 p_3 - x_3 p_2, -x_1 p_3 + x_3 p_1, x_1 p_2 - x_2 p_1)$$

is constant along any solution of the Euler–Lagrange equations in this case (of a rotationaly symmetric potential).