Homework 4

Let V be a real vector space with basis $M = \{e_1, \dots, e_n\}$ and dual basis $M^* = \{\varepsilon_1, \dots, \varepsilon_n\}$.

1) Show that the map

$$T^{1,1}(V) \longrightarrow \mathrm{End}(V)$$

$$T = \sum_{i,j=1}^{n} T^i_j\, \varepsilon_j \otimes e_i \longmapsto \varphi \in \mathrm{End}(V),\ [\varphi]_M = (\varphi_{ij})_{i=1,\dots,n}^{j=1,\dots,n}$$

$$\varphi_{ij} = T^i_j$$

does not depend on the choice of basis of V.

2) Show that the map

$$T^{2,0}(V) \longrightarrow \{ f : V \times V \to \mathbb{R} \mid f \text{ is bilinear} \} = \mathcal{B}(V)$$

$$T = \sum_{i,j=1}^{n} T_{ij}\, \varepsilon_i \otimes \varepsilon_j \longmapsto f \in \mathcal{B}(V),\ [f]_M = (T_{ij})_{i=1,\dots,n}^{j=1,\dots,n}$$

does not depend on the choice of basis of V.

Solution: ad 1) First proof. In lecture 4 we defined the map

$$T^{1,1}(V) \longrightarrow \mathrm{End}(V) = \{ \varphi : V \to V \mid \varphi \text{ is linear} \}$$

$$T \longmapsto (V \ni v \longmapsto \varphi_T(v) \in V)$$

where $w := \varphi_T(v) \in V$ is the uniquely determined by the requirement

(*) $V^* \ni \alpha \longmapsto T(v, \alpha) = \alpha(w) \in \mathbb{R}$

The existence and uniquenes of w follows from the fact that (*) is linear and that $(V^*)^*$ is canonicaly isomorphic through the map

$$V \ni v \longmapsto (V^* \ni \alpha \longmapsto \alpha(v)).$$

Hence, if $M = \{e_1, \dots, e_n\}$ is a basis of V and $M^* = \{\varepsilon^1, \dots, \varepsilon^n\}$ is the dual basis so that

$$\varepsilon^i_i(e_j) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

and $T = \sum_{i=1}^{n} T_i{}^j\, \varepsilon^i \otimes e_j \in T^{1,1}(V)$ and $\alpha = \sum_{i=1}^{n} \alpha_i \varepsilon^i$, then

$$T(e_j, \alpha) = \sum_{i=1}^{n} T_j{}^i \alpha_i = \alpha\Big(\sum_{i=1}^{n} T_j{}^i e_i\Big).$$

This shows that $\varphi_T(e_j) = \sum_{i=1}^{n} T^i_j e_i$ and so by definition, the matrix $[\varphi_T]_M$ of φ_T w.r. to M is

$$[\varphi_T]_M = (\varphi_{ij})_{i=1,\dots,n}^{j=1,\dots,n} \quad \text{where } \varphi_{ij} = T^i_j.$$

(Here i is indexing rows and j is indexing columns of $[\varphi_T]_M$.)

This proves the claim.

Second proof. Let $M' = \{e'_1, \dots, e'_n\}$ be a second basis of M and $(M')^* = \{\varepsilon'^1, \dots, \varepsilon'^n\}$ be the dual basis. If

$$[\mathrm{Id}]^M_{M'} = (a_{ij})_{i=1,\dots,n}^{j=1,\dots,n} \quad \text{so that} \quad \varepsilon_j = \sum_{i=1}^{n} a_{ij} \varepsilon'_i, \text{ then}$$

$$[Id]^{M^*}_{(M')^*} = \left([Id]^{M'}_{M}\right)^T = \left((b_{ij})^{j=1,\dots,m}_{i=1,\dots,m}\right)^T$$

so that $\varepsilon^j = \sum_{i=1}^m b_{ji}\varepsilon'^i$. Write $\Lambda^i_j = a_{ij}$ and $(\Lambda^{-1})^i_j = b_{ij}$ for $i,j = 1,\dots,m$. Then we may write

$$e_j = \sum_{i=1}^m \Lambda^i_j e'_i \quad \text{and} \quad \varepsilon^j = \sum_{i=1}^m (\Lambda^{-1})^j_i \varepsilon'^i .$$

Hence

$$T = \sum_{i,j=1}^m T_i^{\ j}\varepsilon^i \otimes e_j = \sum_{i,j,l,k=1}^m T_i^{\ j}(\Lambda^{-1})^i_l \Lambda^k_j \,\varepsilon'^l \otimes e_k$$

$$= \sum_{k,l=1}^m T'^{k}_{\ l}\, \varepsilon'^l \otimes e'_k$$

where

$$(\square) \quad T'^k_{\ l} = \sum_{i,j=1}^m T_i^{\ j}(\Lambda^{-1})^i_l \Lambda^k_j = \sum_{i,j=1}^m a_{kj} T_i^{\ j} b_{ij}.$$

Hence, $[\varphi_T]_{M'} = (\varphi'_{ij})^{j=1,\dots,m}_{i=1,\dots,m}$ with $\varphi'_{ij} = T'^i_{\ j}$.

On the other hand, $[\varphi_T]_M = (\varphi_{ij})^{j=1,\dots,m}_{i=1,\dots,m}$ with $\varphi_{ij} = T_j^{\ i}$ and so by $(\square)$

$$\varphi'_{ij} = \sum_{k,l=1}^m a_{ik}\varphi_{kl} b_{lj}, \text{ or using matrix multiplication}$$

$$[\varphi_T]_{M'} = [Id]^M_{M'}[\varphi_T]_M [Id]^{M'}_M$$

which is the well known transformation rule for the matrix of an endomorphism. Hence, the map

$$T \mapsto \varphi_T, \ [\varphi_T]_M = (\varphi_{ij})^{j=1,\dots,m}_{i=1,\dots,m}$$

does not depend on the choice of basis.

ad2) Let $T = \sum_{i,j=1}^m T_{ij}\varepsilon^i\otimes\varepsilon^j$. Then using notation used above,

$$T = \sum_{i,j=1}^m T_{ij}\varepsilon^i\otimes\varepsilon^j = \sum_{i,j=1}^m T_{ij}\sum_{k,l=1}^m (\Lambda^{-1})^i_k(\Lambda^{-1})^j_l \,\varepsilon'^k\otimes\varepsilon'^l$$

$$= \sum_{k,l=1}^m T'_{kl}\,\varepsilon'^k\otimes\varepsilon'^l$$

with $T'_{kl} = \sum_{i,j=1}^m T_{ij}(\Lambda^{-1})^i_k(\Lambda^{-1})^j_l$.

Hence, $T \mapsto [f]_{M'} = (T'_{ij})^{j=1,\dots,m}_{i=1,\dots,m}$.

But

$$T'_{ij} = \sum_{k,l=1}^m T_{ij} b_{ik} b_{jl} = \sum_{k,l=1}^m b_{ik} T_{ij} b_{jl},$$

or using matrix multiplication

$$[f]_{M'} = \left([Id]^{M'}_M\right)^T [f]_M [Id]^{M'}_M$$

which is again the well known transformation law for matrix of a bilinear form.