Homework 5

Flow of incompressible fluid

By a flow of incompressible fluid (or also incompressible flow) is understood a 1-parametric family Φ_t of local diffeomorphisms of $\mathbb{R}^n$ (as in week 4) such that for every $t \in \mathbb{R}$ and every $K \subseteq \mathbb{R}^n$ (such that $\Phi_t(x)$ is defined for every $x \in K$) the Lebesgue measure of K is the same as the Lebesgue measure of $\Phi_t(K)$.

If $\mathbb{X} \in \mathfrak{X}(\mathbb{R}^n)$, then $Fl_t^{\mathbb{X}}$ is an incompressible flow iff the Lebesgue measure $dx^n = dx_1 \wedge \dots \wedge dx_n$ (which can be viewed as a skew-symmetric tensor field of type $(n,0)$ on $\mathbb{R}^n$) is invariant under the flow of $\mathbb{X}$, that is

$$\mathcal{L}_{\mathbb{X}}\, dx^n = 0.$$

Let us consider for simplicity $n=2$ so that $dx^2 = dx_1 \wedge dx_2$.

(1) Show that

$$dx_1 \wedge dx_1 = dx_2 \wedge dx_2 = 0 \quad \text{and} \quad dx_1 \wedge dx_2 = -dx_2 \wedge dx_1.$$

(2) Show that

$$\mathcal{L}_{\mathbb{X}}(dx_1 \wedge dx_2) = (\mathcal{L}_{\mathbb{X}} dx_1) \wedge dx_2 + dx_1 \wedge (\mathcal{L}_{\mathbb{X}} dx_2).$$

(Hint express $dx_1 \wedge dx_2$ using $dx_1 \otimes dx_2$ and $dx_2 \otimes dx_1$ and use (the last) Theorem in week 5.

(3) Assume that $\mathbb{X} = a_1(x_1,x_2)\frac{\partial}{\partial x_1} + a_2(x_1,x_2)\frac{\partial}{\partial x_2}$. Show that

$$\mathcal{L}_{\mathbb{X}}\, dx^2 = \mathcal{L}_{\mathbb{X}}(dx_1 \wedge dx_2) = 0$$

iff the divergence $\operatorname{div}(\mathbb{X})$ of $\mathbb{X}$ vanishes, that is

$$\operatorname{div}(\mathbb{X}) = \frac{\partial a_1}{\partial x_1}(x_1,x_2) + \frac{\partial a_2}{\partial x_2}(x_1,x_2) = 0.$$

(4) Let $f(x+iy) = f_1(x+iy) + i f_2(x+iy)$ be a holomorphic function on $\mathbb{C}$ (here f_1 is the real part and f_2 is the imaginary part of f). Show that

$$\mathbb{X} = a_1(x_1,x_2)\frac{\partial}{\partial x_1} + a_2(x_1,x_2)\frac{\partial}{\partial x_2}$$

has zero divergence when

$$a_1(x_1,x_2) = f_1(x_1+ix_2),\ -a_2(x_1,x_2) = f_2(x_1+ix_2) \text{ for } i=1,2.$$

(Hint: consider the Cauchy–Riemann equations.)

Solution: 1. By definition, for $x \in \mathbb{R}^2$ and $v_1, v_2 \in T_x\mathbb{R}^2 \cong \mathbb{R}^2$ we have

$$(dx_i \wedge dx_j)(v_1, v_2) = \frac{1}{2}\left(dx_i(v_1) \cdot dx_j(v_2) - dx_i(v_2) \cdot dx_j(v_1)\right) =$$

$$= \frac{1}{2}\left(dx_i \otimes dx_j - dx_j \otimes dx_i\right)(v_1, v_2)$$

This shows that $dx_i \wedge dx_j = \frac{1}{2}\left(dx_i \otimes dx_j - dx_j \otimes dx_i\right)$ and so

$$dx_i \wedge dx_i = \frac{1}{2}\left(dx_i \otimes dx_i - dx_i \otimes dx_i\right) = 0, \quad i = 1, 2, \text{ and}$$

$$dx_i \wedge dx_j = \frac{1}{2}\left(dx_i \otimes dx_j - dx_j \otimes dx_i\right) = -dx_j \wedge dx_i.$$

2. $$\mathcal{L}_X\, dx_1 \wedge dx_2 = \mathcal{L}_X\left(\frac{1}{2}\left(dx_1 \otimes dx_2 - dx_2 \otimes dx_1\right)\right) =$$

$$= \frac{1}{2}\left(\mathcal{L}_X dx_1\right) \otimes dx_2 + \frac{1}{2} dx_1 \otimes \mathcal{L}_X dx_2$$

$$- \frac{1}{2}\left(\mathcal{L}_X dx_2\right) \otimes dx_1 - \frac{1}{2} dx_2 \otimes \left(\mathcal{L}_X dx_1\right)$$

$$= \frac{1}{2}\left(\mathcal{L}_X dx_1\right) \wedge dx_2 + \frac{1}{2} dx_1 \wedge \mathcal{L}_X(dx_2).$$

Here $X \in \mathfrak{X}(\mathbb{R}^2)$.

3. Now $$\left(\mathcal{L}_X\, dx_i\right)(x) = \mathcal{L}_{a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2}}\, dx_i =$$

$$= \frac{\partial a_i}{\partial x_1}(x)\, dx_1 + \frac{\partial a_i}{\partial x_2}(x)\, dx_2.$$

So $$\mathcal{L}_X (dx_1 \wedge dx_2)(x) = \left[\left(\mathcal{L}_X dx_1\right) \wedge dx_2 + dx_1 \wedge \left(\mathcal{L}_X dx_2\right)\right](x)$$

$$= \left(\frac{\partial a_1}{\partial x_1}(x)\, dx_1 + \frac{\partial a_1}{\partial x_2}(x)\, dx_2\right) \wedge dx_2$$

$$+ dx_1 \wedge \left(\frac{\partial a_2}{\partial x_1}(x)\, dx_1 + \frac{\partial a_2}{\partial x_2}(x)\, dx_2\right)$$

$$= \left(\frac{\partial a_1}{\partial x_1}(x) + \frac{\partial a_2}{\partial x_2}(x)\right) dx_1 \wedge dx_2.$$

Since $dx_1 \wedge dx_2 \neq 0$ at every point $x \in \mathbb{R}^2$, $\mathcal{L}_X(dx_1 \wedge dx_2) = 0$

iff $$\operatorname{div} X = \left(\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2}\right)(x) = 0.$$

4. If $f(x+iy) = f_1(x+iy) + i f_2(x+iy)$ is holomorphic on $\mathbb{C}$, then Cauchy–Riemann equation imply

$$\frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} = 0.$$

Hence, if we put $a_1(x_1, x_2) := f_1(x_1 + i x_2)$ and $a_2(x_1, x_2) := -f_2(x_1 + i x_2)$, then

$$\operatorname{div} X = \frac{\partial a_1}{\partial x_1}(x) + \frac{\partial a_2}{\partial x_2}(x) = \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} = 0 \quad \text{where}$$

$$X = a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2}.$$