Homework 6

Calculate Christoffel symbols $\Gamma^r_{r\varphi}, \Gamma^\varphi_{r\varphi}, \Gamma^r_{\varphi\varphi}, \Gamma^\varphi_{\varphi\varphi}$ of the flat connection ∇ on $\mathbb{R}^2$ in polar coordinates. Here

$$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \varphi} = \Gamma^r_{r\varphi} \frac{\partial}{\partial r} + \Gamma^\varphi_{r\varphi} \frac{\partial}{\partial \varphi}$$

$$\nabla_{\frac{\partial}{\partial \varphi}} \frac{\partial}{\partial \varphi} = \Gamma^r_{\varphi\varphi} \frac{\partial}{\partial r} + \Gamma^\varphi_{\varphi\varphi} \frac{\partial}{\partial \varphi}.$$

Solution: We know from lecture 7 that

$$\frac{\partial}{\partial r} = \cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \varphi} = -r\sin\varphi \frac{\partial}{\partial x} + r\cos\varphi \frac{\partial}{\partial y}.$$

•) $$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \varphi} = \nabla_{\left(\cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y}\right)} \left(-r\sin\varphi \frac{\partial}{\partial x} + r\cos\varphi \frac{\partial}{\partial y}\right)$$

$$= \cos\varphi \left(\frac{\partial}{\partial x}(-r\sin\varphi)\right) \frac{\partial}{\partial x} + \sin\varphi \left(\frac{\partial}{\partial y}(-r\sin\varphi)\right) \frac{\partial}{\partial x}$$

$$+ \cos\varphi \left(\frac{\partial}{\partial x}(r\cos\varphi)\right) \frac{\partial}{\partial y} + \sin\varphi \left(\frac{\partial}{\partial y} r\cos\varphi\right) \frac{\partial}{\partial y} = (*)$$

$$r = \sqrt{x^2+y^2}, \quad \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\varphi, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\varphi$$

$$\varphi = \operatorname{arctg}\frac{y}{x}, \quad \frac{\partial \varphi}{\partial x} = \frac{-y/x^2}{1+y^2/x^2} = -\frac{r\sin\varphi}{r^2} = -\frac{\sin\varphi}{r}, \quad x>0$$

$$\frac{\partial \varphi}{\partial y} = \frac{\frac{1}{x}}{1+y^2/x^2} = \frac{r\cos\varphi}{r^2} = \frac{\cos\varphi}{r}$$

$$(*) = \left[\cos\varphi\left(-\frac{\partial r}{\partial x}\sin\varphi - r\cos\varphi\frac{\partial\varphi}{\partial x}\right) + \sin\varphi\left(-\frac{\partial r}{\partial y}\sin\varphi - r\cos\varphi\frac{\partial\varphi}{\partial y}\right)\right]\frac{\partial}{\partial x}$$

$$+ \left[\cos\varphi\left(\frac{\partial r}{\partial x}\cos\varphi - r\sin\varphi\frac{\partial\varphi}{\partial x}\right) + \sin\varphi\left(\frac{\partial r}{\partial y}\cos\varphi - r\sin\varphi\frac{\partial\varphi}{\partial y}\right)\right]\frac{\partial}{\partial y}$$

$$= \left[\cos\varphi(-\cos\varphi\sin\varphi + \cos\varphi\sin\varphi) + \sin\varphi(-\sin^2\varphi - \cos^2\varphi)\right]\frac{\partial}{\partial x}$$

$$+ \left[\cos\varphi(\cos^2\varphi + \sin^2\varphi) + \sin\varphi(\sin\varphi\cos\varphi - \sin\varphi\cos\varphi)\right]\frac{\partial}{\partial y}$$

$$= -\sin\varphi\frac{\partial}{\partial x} + \cos\varphi\frac{\partial}{\partial y} = \frac{1}{r}\frac{\partial}{\partial\varphi} \quad \Rightarrow \quad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \ \Gamma^r_{r\varphi} = 0$$

•) $$\nabla_{\frac{\partial}{\partial \varphi}} \frac{\partial}{\partial \varphi} = \nabla_{\left(-r\sin\varphi \frac{\partial}{\partial x} + r\cos\varphi \frac{\partial}{\partial y}\right)} \left(-r\sin\varphi \frac{\partial}{\partial x} + r\cos\varphi \frac{\partial}{\partial y}\right)$$

$$= -r\sin\varphi\left(\frac{\partial}{\partial x}(-r\sin\varphi)\right)\frac{\partial}{\partial x} + r\cos\varphi\left(\frac{\partial}{\partial y}(-r\sin\varphi)\right)\frac{\partial}{\partial x}$$

$$- r\sin\varphi\left(\frac{\partial}{\partial x}(r\cos\varphi)\right)\frac{\partial}{\partial y} + r\cos\varphi\left(\frac{\partial}{\partial y} r\cos\varphi\right)\frac{\partial}{\partial y} = (*)$$

$$= \left[-r\sin\varphi\left(-\frac{\partial r}{\partial x}\sin\varphi - r\cos\varphi\frac{\partial\varphi}{\partial x}\right) + r\cos\varphi\left(-\frac{\partial r}{\partial y}\sin\varphi - r\cos\varphi\frac{\partial\varphi}{\partial y}\right)\right]\frac{\partial}{\partial x}$$

$$+ \left[-r\sin\varphi\left(\frac{\partial r}{\partial x}\cos\varphi - r\sin\varphi\frac{\partial\varphi}{\partial x}\right) + r\cos\varphi\left(\frac{\partial r}{\partial y}\cos\varphi - r\sin\varphi\frac{\partial\varphi}{\partial y}\right)\right]\frac{\partial}{\partial y}$$

$$= \left[-r\sin\varphi(-\cos\varphi\sin\varphi + \cos\varphi\sin\varphi) + r\cos\varphi(-\sin^2\varphi - \cos^2\varphi)\right]\frac{\partial}{\partial x}$$

$$+ \left[-r\sin\varphi(\cos^2\varphi + \sin^2\varphi) + r\cos\varphi(\sin\varphi\cos\varphi - \sin\varphi\cos\varphi)\right]\frac{\partial}{\partial y}$$

$$= -r\cos\varphi\frac{\partial}{\partial x} - r\sin\varphi\frac{\partial}{\partial y} = -r\frac{\partial}{\partial r} \quad \Rightarrow \quad \Gamma^r_{\varphi\varphi} = -r, \ \Gamma^\varphi_{\varphi\varphi} = 0$$