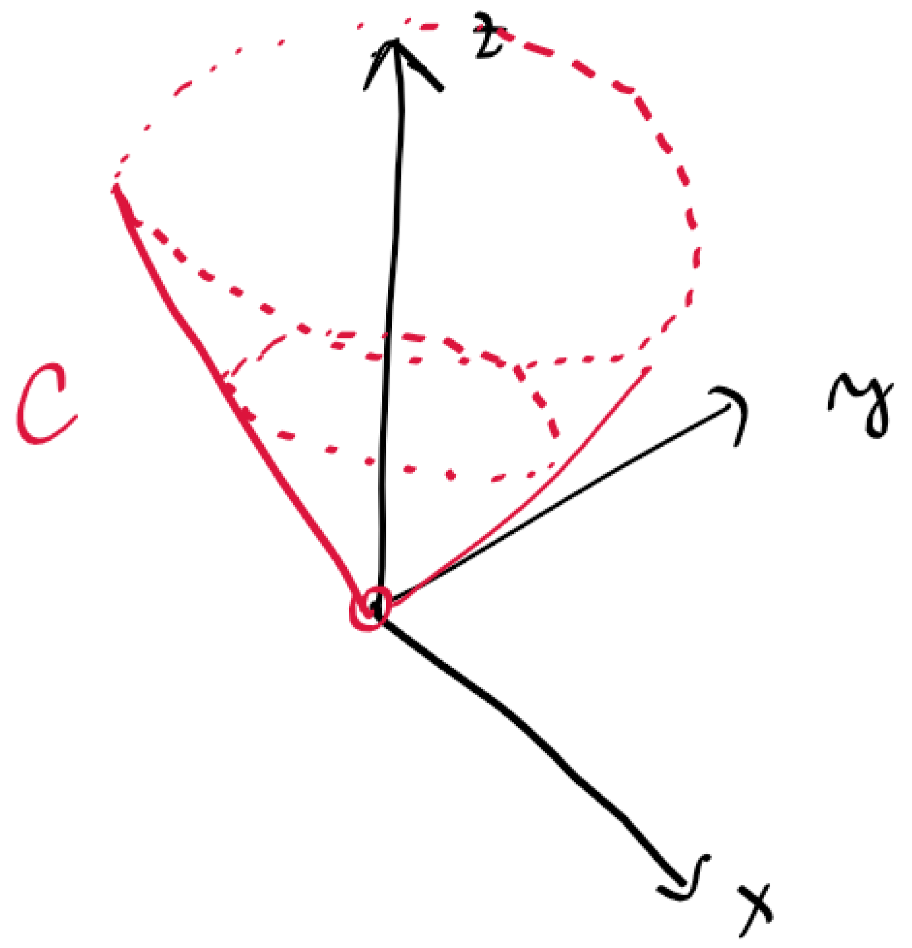
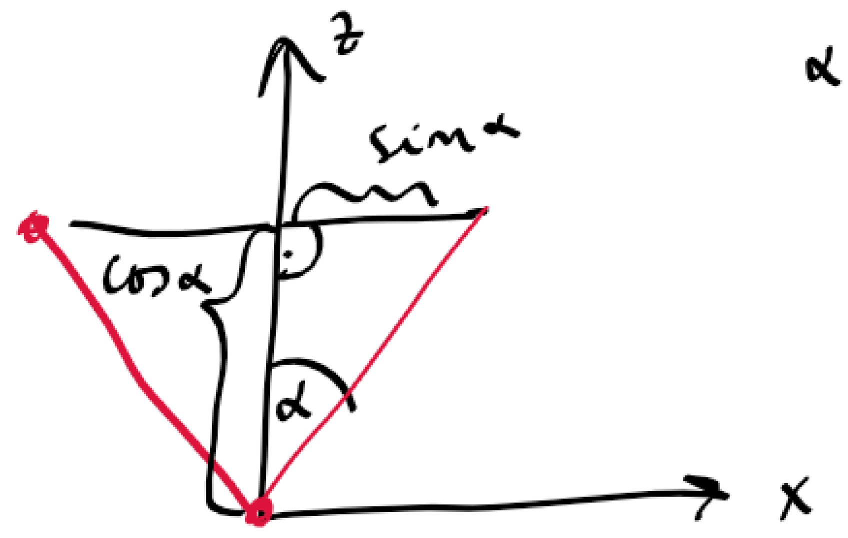
Homework 7

Geodesics on a cone

Let $C = \{(x,y,z) \in \mathbb{R}^3 : \frac{z^2}{\cos^2\alpha} = \frac{x^2+y^2}{\sin^2\alpha}, z > 0\}$ be a cone in $\mathbb{R}^3$ with $\alpha \in (0, \pi/2)$.



$y = 0$



Now C has a canonical structure of smooth manifold so that the embedding $\imath : C \hookrightarrow \mathbb{R}^3$ is an immersion. On C we consider the induced Riemannian metric $g_C = \imath^* g$ where $\imath : C \hookrightarrow \mathbb{R}^3$ is the embedding and g is the Euclidean metric on $\mathbb{R}^3$. Let

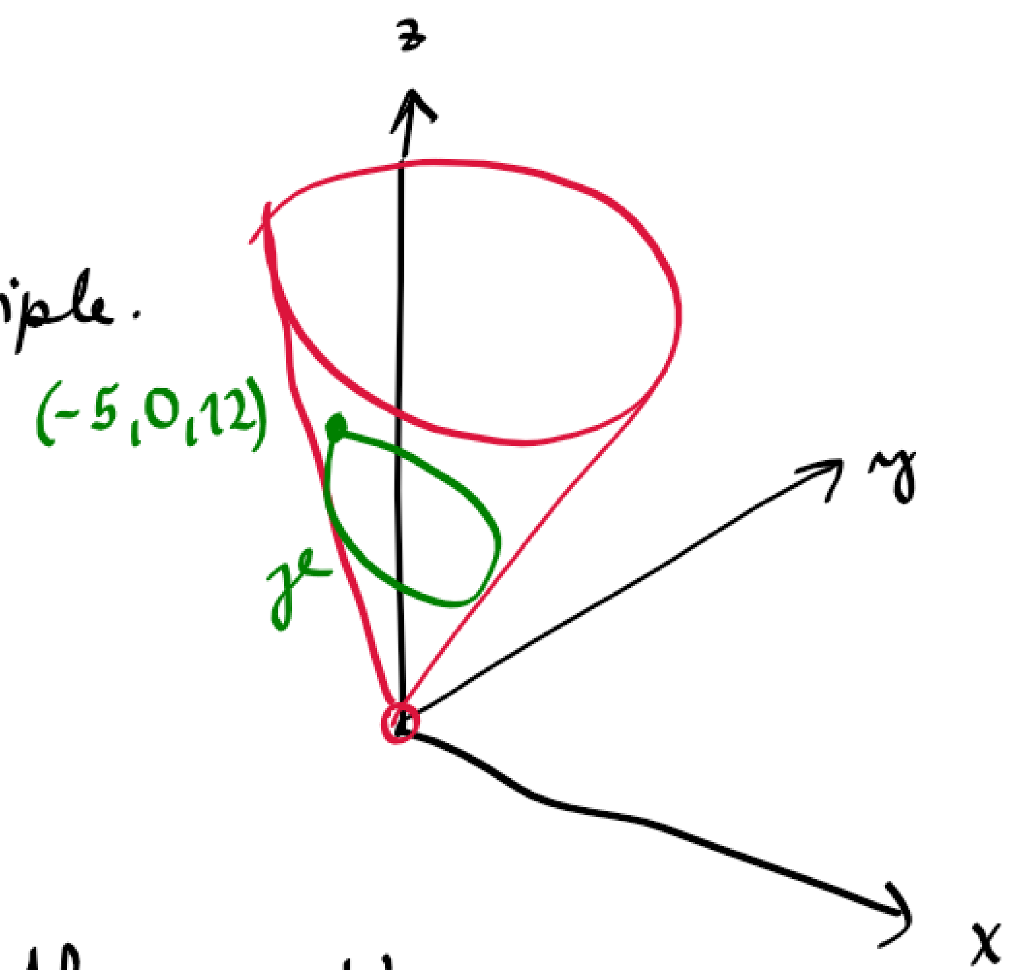
$$\Phi : (0, +\infty) \times (-\pi, \pi) \longrightarrow \mathbb{R}^3$$
$$\Phi(u,v) = (u \sin\alpha \cos v,\ u \sin\alpha \sin v,\ u \cos\alpha)$$

be a parametrization of C. Let $U \subseteq C$ be the image of Φ and $\varphi = \Phi^{-1} : U \longrightarrow (0,+\infty) \times (-\pi,\pi) \subseteq \mathbb{R}^2$ be the inverse of Φ. Now $\varphi : U \to \mathbb{R}^2$ is a chart on C which is compatible with the canonical smooth structure on C.

1. Compute g_C in the coordinate chart $\varphi : U \to \mathbb{R}^2$, that is, compute $\Phi^* g_C$.
2. Compute the associated Christoffel symbols write down the geodesic equation.
3. Verify that meridians (these are curves of the form $(0,+\infty) \ni t \longmapsto \Phi(t, v_0)$ with $v_0 \in (-\pi, \pi)$ fixed) are geodesics.

Bonus questions (for extra 100%)

Consider now $\alpha \in (0, \pi/2)$ so that $\sin\alpha = \frac{5}{13}$ and $\cos\alpha = \frac{12}{13}$ is a Pythagorian triple.



4. Show that there is a closed geodesic γ on C which starts and ends at the point $(-5, 0, 12)$.
5. Calculate the length of γ and the z-th coordinate of the lowest point on this curve.

Hint: Use in a smart way polar coordinates in $\mathbb{R}^2$ to find another parametrization of C and the fact that a (local) isometry of Riemannian manifolds maps geodesics to geodesics.

Solution: (ad 1) Let $g \equiv g_3$ be the Euclidean metric on $\mathbb{R}^3$. We have

$$
\begin{aligned}
\Phi^* g_c = \Phi^*\Big(\sum_{i=1}^{3} dx_i \otimes dx_i\Big) &= d(u\sin\alpha\cos v)\otimes d(u\sin\alpha\cos v) \\
&\quad + d(u\sin\alpha\sin v)\otimes d(u\sin\alpha\sin v) + d(u\cos\alpha)\otimes d(u\cos\alpha) \\
&= (\sin\alpha\cos v\,du - u\sin\alpha\sin v\,dv)\otimes(\sin\alpha\cos v\,du - u\sin\alpha\sin v\,dv) \\
&\quad + (\sin\alpha\sin v\,du + u\sin\alpha\cos v\,dv)\otimes(\sin\alpha\sin v\,du + u\sin\alpha\cos v\,dv) \\
&\quad + \cos\alpha\,du\otimes\cos\alpha\,du \\
&= (\sin^2\alpha\cos^2 v + \sin^2\alpha\sin^2 v + \cos^2\alpha)\,du\otimes du \\
&\quad + (u^2\sin^2\alpha\sin^2 v + u^2\sin^2\alpha\cos^2 v)\,dv\otimes dv \\
&= du\otimes du + u^2\sin^2\alpha\,dv\otimes dv
\end{aligned}
$$

$$
\begin{pmatrix} 1 & 0 \\ 0 & u^2\sin^2\alpha \end{pmatrix}
\qquad
\begin{aligned}
&g_{uu} = 1,\quad g_{uv} = g_{vu} = 0,\quad g_{vv} = u^2\sin^2\alpha \\
&g^{uu} = 1,\quad g^{uv} = g^{uv} = 0,\quad g^{vv} = u^{-2}\sin^{-2}\alpha
\end{aligned}
$$

(ad 2)

$$\Gamma^u_{uu} = \tfrac12 g^{uu}\Big(\tfrac{\partial}{\partial u}g_{uu} + \tfrac{\partial}{\partial u}g_{uu} - \tfrac{\partial}{\partial u}g_{uu}\Big) + \tfrac12 g^{uv}\Big(\tfrac{\partial}{\partial u}g_{uv} + \tfrac{\partial}{\partial u}g_{uv} - \tfrac{\partial}{\partial v}g_{uu}\Big) = 0$$

$$\Gamma^v_{uu} = \tfrac12 g^{vu}\Big(\tfrac{\partial}{\partial u}g_{uu} + \tfrac{\partial}{\partial u}g_{uu} - \tfrac{\partial}{\partial u}g_{uu}\Big) + \tfrac12 g^{vv}\Big(\tfrac{\partial}{\partial u}g_{uv} + \tfrac{\partial}{\partial u}g_{uv} - \tfrac{\partial}{\partial v}g_{uu}\Big) = 0$$

$$\Gamma^u_{vu} = \Gamma^u_{uv} = \tfrac12 g^{uu}\Big(\tfrac{\partial}{\partial u}g_{vu} + \tfrac{\partial}{\partial v}g_{uu} - \tfrac{\partial}{\partial u}g_{uv}\Big) + \tfrac12 g^{uv}\Big(\tfrac{\partial}{\partial u}g_{vv} + \tfrac{\partial}{\partial v}g_{uv} - \tfrac{\partial}{\partial v}g_{uv}\Big) = 0$$

$$
\begin{aligned}
\Gamma^v_{vu} = \Gamma^v_{uv} &= \tfrac12 g^{vu}\Big(\tfrac{\partial}{\partial u}g_{vu} + \tfrac{\partial}{\partial v}g_{uu} - \tfrac{\partial}{\partial u}g_{uv}\Big) + \tfrac12 g^{vv}\Big(\tfrac{\partial}{\partial u}g_{vv} + \tfrac{\partial}{\partial v}g_{uv} - \tfrac{\partial}{\partial v}g_{uv}\Big) \\
&= 0 + \tfrac12 u^{-2}\sin^{-2}\alpha\,(2u\sin^2\alpha) = \frac{1}{u}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^u_{vv} &= \tfrac12 g^{uu}\Big(\tfrac{\partial}{\partial v}g_{vu} + \tfrac{\partial}{\partial v}g_{vu} - \tfrac{\partial}{\partial u}g_{vv}\Big) + \tfrac12 g^{uv}\Big(\tfrac{\partial}{\partial v}g_{vv} + \tfrac{\partial}{\partial v}g_{vv} - \tfrac{\partial}{\partial v}g_{vv}\Big) \\
&= \tfrac12\cdot 1\cdot(-2u\sin^2\alpha) + 0 = -u\sin^2\alpha
\end{aligned}
$$

$$\Gamma^v_{vv} = \tfrac12 g^{vu}\Big(\tfrac{\partial}{\partial v}g_{vu} + \tfrac{\partial}{\partial v}g_{vu} - \tfrac{\partial}{\partial u}g_{vv}\Big) + \tfrac12 g^{vv}\Big(\tfrac{\partial}{\partial v}g_{vv} + \tfrac{\partial}{\partial v}g_{vv} - \tfrac{\partial}{\partial v}g_{vv}\Big) = 0$$

Geodesic equations:

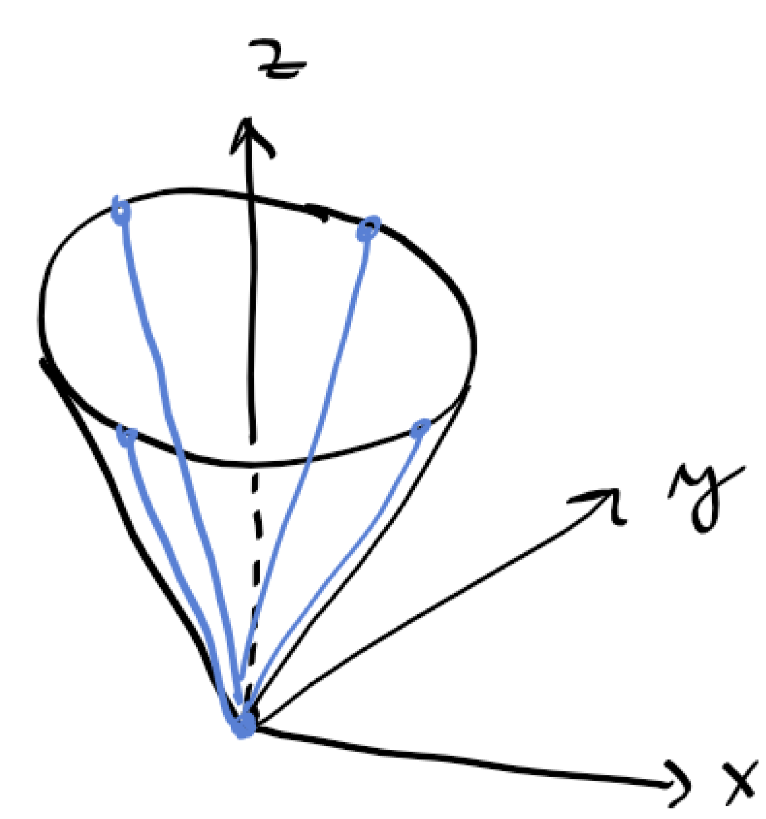
$$u'' + \Gamma^u_{uu}(u')^2 + 2\Gamma^u_{uv}\,u'v' + \Gamma^u_{vv}(v')^2 = 0$$

$$v'' + \Gamma^v_{uu}(u')^2 + 2\Gamma^v_{uv}\,u'v' + \Gamma^v_{vv}(v')^2 = 0$$

$$u'' - u\sin^2\alpha\,(v')^2 = 0$$

$$v'' + 2\frac{1}{u}u'v' = 0$$

Meridians on cone



ad (3) Meridians satisfy $u'' = 0$, $v' = 0$ and so they are geodesics

Bonus questions

ad (4) Consider the map

$$\Psi : (0,+\infty)\times(-\pi,\pi) \longrightarrow \mathbb{R}^2,$$

$$\Psi(u,v) = (u\cos(\sin\alpha\, v),\; u\sin(\sin\alpha\, v)).$$

The pullback of the Euclidean metric g_2 on $\mathbb{R}^2$ is

$$\Psi^* g_2 = \Psi^*(dx\otimes dx + dy\otimes dy) = d(u\cos(\sin\alpha\, v))\otimes d(u\cos(\sin\alpha\, v))$$
$$+\, d(u\sin(\sin\alpha v))\otimes d(u\sin(\sin\alpha v))$$

$$= (\cos(\sin\alpha v)\,du - \sin\alpha\cdot u\cdot \sin(\sin\alpha\, v)\,dv)\otimes(\cos(\sin\alpha v)\,du - \sin\alpha\, u\, \sin(\sin\alpha\, v)\,dv)$$

$$+ (\sin(\sin\alpha v)\,du + \sin\alpha\cdot u\cdot\cos(\sin\alpha\, v)\,dv)\otimes(\sin(\sin\alpha v)\,du + \sin\alpha\cdot u\cdot\cos(\sin\alpha\, v)\,dv)$$

$$= (\cos^2(\sin\alpha\, v) + \sin^2(\sin\alpha v))\,du\otimes du\, +$$

$$+ (\sin^2\alpha\cdot u^2\cdot(\sin^2(\sin\alpha v) + \cos^2(\sin\alpha v)))\,dv\otimes dv$$

$$= du\otimes du + \sin^2\alpha\cdot u^2\cdot dv\otimes dv = \Phi^* g_3$$

We see that Ψ is a local isometry of g_2 and $\Phi^* g_3$. Hence Ψ maps geodesics to of $\Phi^* g_3$ to geodesics of g_2 and Φ maps geodesics of $\Phi^* g_c$ to geodesics of g_C. But geodesics of g_2 are precisely line segments (with parametrization with constant speed).
Considr the following diagram

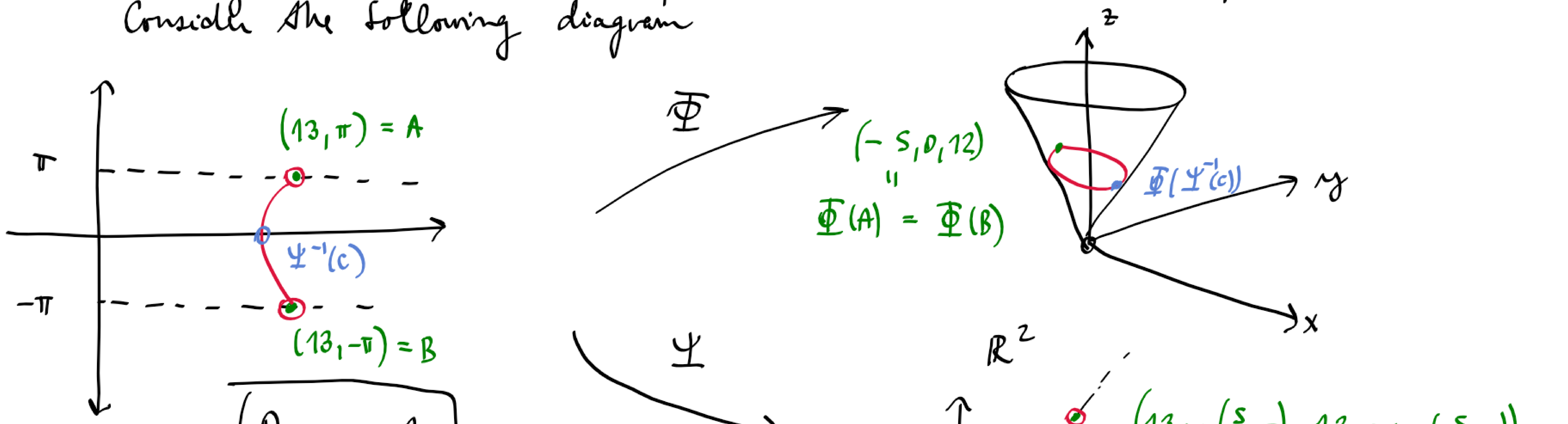
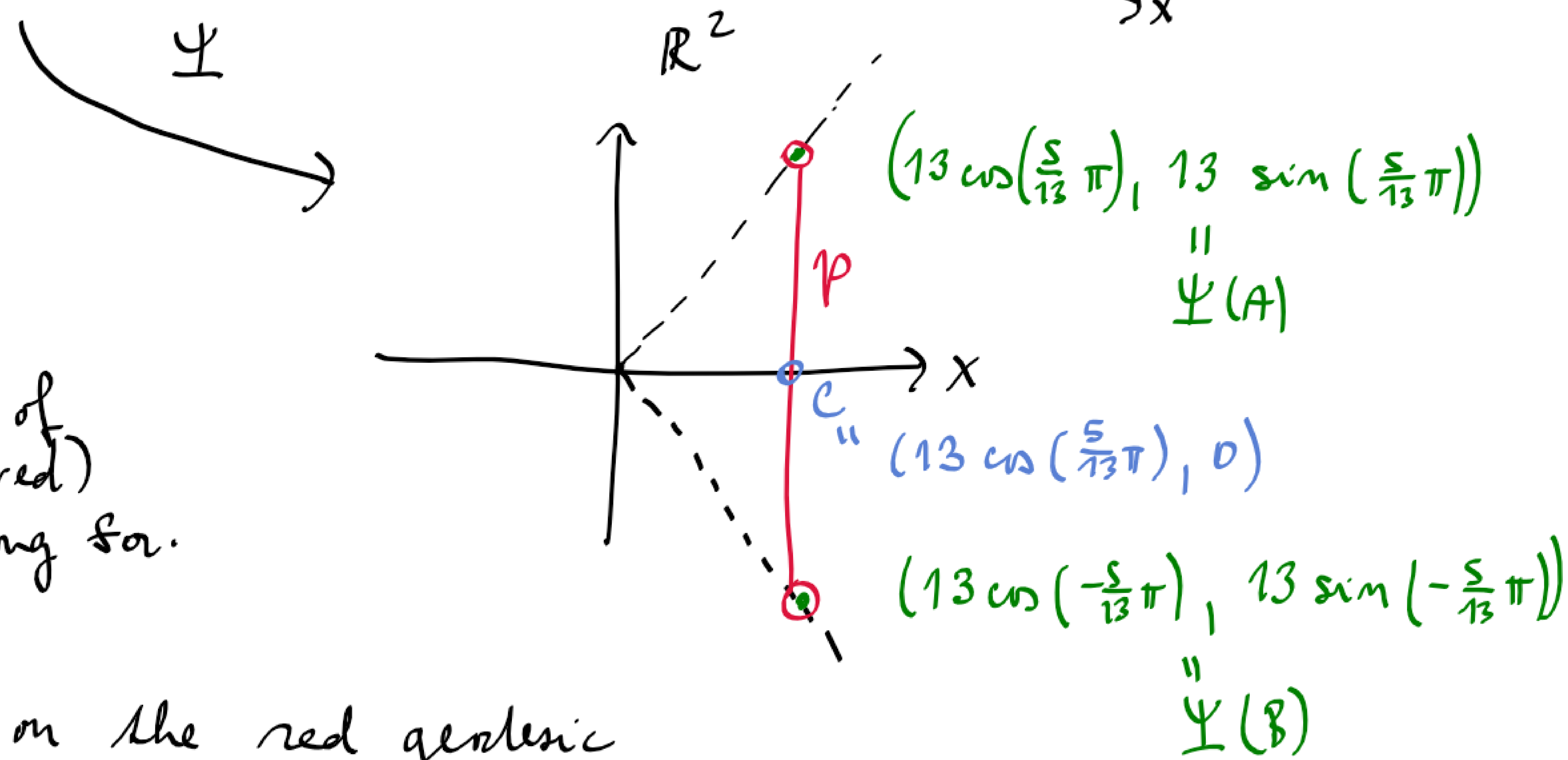


Diagram 1.



Then the $\Phi\circ\Psi^{-1}$ - image of the line segment p (in red) is the geodesic we are looking for.

ad ⑤ The lowest point on the red geodesic on C is the image of the point c (blue) (This should be clear from the diagram 1.) The inverse map to Ψ is

$$\Psi^{-1}(x,y) = \left(\sqrt{x^2+y^2},\ \tfrac{13}{5}\arctg\tfrac{y}{x}\right)$$

and so

$$\Psi^{-1}(13\cos(\tfrac{5}{13}\pi), 0) = (13\cos(\tfrac{5}{13}\pi), 0).$$

Finally
$$\Phi(13\cos(\tfrac{5}{13}\pi), 0) = \left(13\cos\tfrac{5}{13}\pi\cdot\tfrac{5}{13},\ 0,\ 13\cos(\tfrac{5}{13}\pi)\,\tfrac{12}{13}\right)$$
$$= (5\cos(\tfrac{5}{13}\pi), 0, 12\cos(\tfrac{5}{13}\pi)).$$

So $z = 12\cos(\frac{5}{13}\pi)$. The length of γ is clearly the length of the line segment p which is $26\cdot\sin(\frac{5}{13}\pi)$.