Homework 8

Let V^n be a real vector space of dimension n with basis $\{e_1, \dots, e_n\}$. Let S_n be the subset of $T^{4,0}(V^n)$ of all tensors of type $(4,0)$ on V^n that have symmetries of Riemannian curvature tensors, that is, $R \in S_n$ iff

(RS1) $R(v_1, v_2, v_3, v_4) = -R(v_2, v_1, v_3, v_4)$,

(RS2) $R(v_1, v_2, v_3, v_4) + R(v_2, v_3, v_1, v_4) + R(v_3, v_1, v_2, v_4) = 0$,

(RS3) $R(v_1, v_2, v_3, v_4) = -R(v_1, v_2, v_4, v_3)$ and

(RS4) $R(v_1, v_2, v_3, v_4) = R(v_3, v_4, v_1, v_2)$.

for all $v_1, v_2, v_3, v_4 \in V^n$. Now it is straightforward to verify that S_n is a vector subspace of $T^{4,0}(V^n)$ and you may assume that this is true. It is now easy to show that $\dim S_1 = 0$ and in Theorem B from week 11 it is showed that $\dim S_2 = 1$ and that the tensor

$$(\varepsilon_1 \wedge \varepsilon_2) \odot (\varepsilon_1 \wedge \varepsilon_2) = \frac{1}{4}(\varepsilon_{1212} - \varepsilon_{2112} - \varepsilon_{1221} + \varepsilon_{2121})$$

is a generator of S^2. Here

$$\varepsilon_{ijkl} := \varepsilon_i \otimes \varepsilon_j \otimes \varepsilon_k \otimes \varepsilon_l$$

where $i, j, k, l \in \{1,2\}$ and $\{\varepsilon_1, \dots, \varepsilon_n\}$ is the dual basis to $\{e_1, \dots, e_n\}$.

Calculate the dimension of S_3 and find its basis.

Solution: Put

$$R_{ijkl} := \varepsilon_i \wedge \varepsilon_j \odot \varepsilon_k \wedge \varepsilon_l =$$
$$= \frac{1}{8}(\varepsilon_{ijkl} - \varepsilon_{ijlk} - \varepsilon_{jikl} + \varepsilon_{jilk} + \varepsilon_{klij} - \varepsilon_{klji} - \varepsilon_{lkij} + \varepsilon_{lkji}), \quad i,j,k,l = 1,2,3.$$

It is clear that R_{ijkl} satisfies (RS1), (RS3) and (RS4). Moreover, we have

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$$

for any i, j, k, l. We see that altogether we have $\binom{4}{2}$ linearly independent elements, for example $R_{1212}, R_{1313}, R_{2323}, R_{1213}, R_{1223}$ and R_{1323}. (To see that these elements are really linearly independent, note that $R_{ijkl}(e_a, e_b, e_c, e_d) \neq 0$ iff (i,j,k,l) is a permutation of (a,b,c,d).) Now these tensors constitute a basis of the vector space of tensors of type $(4,0)$ on V^3 that have the symmetries (RS1), (RS3) and (RS4). Hence, the dimension of this subspace is 6.

It remains to show that all tensors have also the symmetry (RS2), then $\dim S_3 = \binom{4}{2} = 6$. In order to prove this, consider

$$\text{(BI)} \quad R_{ijkl} + R_{jkil} + R_{kijl}.$$

But now since $i, j, k, l \in \{1,2,3\}$, there have to be at least two indices, say i and l, that agree. But then (BI) is

$$R_{ijki} + R_{jkii} + R_{kiji} = -R_{ijik} + 0 + R_{jiki} = -R_{ijik} - R_{ijki} = -R_{ijik} + R_{ijik} = 0.$$

Note that this applies for any such pair. The proof is complete. Hence, for example

$$\{R_{1212}, R_{1213}, R_{1223}, R_{1313}, R_{1323}, R_{2323}\}$$

is a basis of S_3 and $\dim S_3 = 6$.