# Homework 9

Calculate the Riemann curvature (or also the Gauss curvature) of the hyperbolic metric

$$g \equiv g_{\mathbb{H}^+} = \frac{1}{y^2}\left(dx \otimes dx + dy \otimes dy\right)$$

on $\mathbb{H}^+ = \{(x,y) \in \mathbb{R}^2 : y > 0\}$. That is she find the function

$$K : \mathbb{H}^+ \to \mathbb{R}$$

such that

$$R_{abcd}(z) = K(z)\left(g_{ad}(z)\, g_{bc}(z) - g_{ac}(z)\, g_{bd}(z)\right)$$

for any $a,b,c,d \in \{x,y\}$, $z \in \mathbb{H}^+$. (You may use calculations from week 10.)

Solution: From week 10 we know that

$$\Gamma^x_{xx} = \Gamma^y_{xy} = \Gamma^y_{yx} = 0, \quad \Gamma^y_{xx} = \frac{1}{y}, \quad \Gamma^x_{yx} = -\frac{1}{y} = \Gamma^x_{xy}, \quad \Gamma^y_{yy} = -\frac{1}{y}.$$

We also know that

$$R_{xyxy} = -R_{yxxy} = -R_{xyyx}$$

and all other components of the Riemannian curvature vanish. To compute these components, we find that

$$R_{xyx}{}^{y} = \frac{\partial}{\partial x}\Gamma^y_{yx} - \frac{\partial}{\partial y}\Gamma^y_{xx} + \sum_{u}\left(\Gamma^u_{yx}\Gamma^y_{xu} - \Gamma^u_{xx}\Gamma^y_{yu}\right)$$

$$= 0 - \frac{\partial}{\partial y}\left(\frac{1}{y}\right) + \Gamma^x_{yx}\Gamma^y_{xx} + \Gamma^y_{yx}\Gamma^y_{xy} - \Gamma^x_{xx}\Gamma^y_{yx} - \Gamma^y_{xx}\Gamma^y_{yy}$$

$$= y^{-2} - \frac{1}{y}\frac{1}{y} + 0.0 - 0.0 + \frac{1}{y}\frac{1}{y}$$

$$= y^{-2}.$$

Hence

$$R_{xyxy} = \sum_{u} R_{xyx}{}^{u}\, g_{uy} = R_{xyx}{}^{x}\, g_{xy} + R_{xyx}{}^{y}\, g_{yy}$$

$$= 0.0 + y^{-2}\, y^{-2} = y^{-4}.$$

This shows that since $g_{xy} = g_{yx} = 0$, $g_{xx} = g_{yy} = y^{-2}$ that

$$R_{xyxy} = y^{-4} = -\left(-y^{-2}y^{-2}\right) = -\left(g_{xy}\,g_{yx} - g_{xx}\,g_{yy}\right).$$

Thus the Gauss curvature of $g_{\mathbb{H}^+}$ is $K(z) = -1$, $z \in \mathbb{H}^+$.