Geodesics as distance minimizing curves

Let (M, g) be a Riemannian manifold and $\gamma : [a,b] \to M$ be a piecewise $\mathcal{C}^1$ curve, that is γ is continuous and there exists a division $a = t_0 < t_1 < \dots < t_m = b$ such that
$\gamma|_{[t_i, t_{i+1}]}$ is $\mathcal{C}^1$ for every $i = 0, \dots, m-1$.

The length of γ is defined as

$$L(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)}\, dt \quad \text{where} \quad \|\gamma'(t)\|_{\gamma(t)} = \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}.$$

The length of γ does not depend on parametrization, that is if $\phi : [a', b'] \to [a,b]$ is $\mathcal{C}^1$ with $\phi(a') = a$, $\phi(b') = b$ and $\phi' > 0$, then $L(\gamma) = L(\gamma \circ \phi)$.

Given two points $A, B \in M$, it is a natural question to find a curve $\gamma_0 : [a,b] \to M$ such that

(DM1) $\gamma_0(a) = A$, $\gamma_0(b) = B$

(DM2) $L(\gamma_0) = d(A,B)$

where

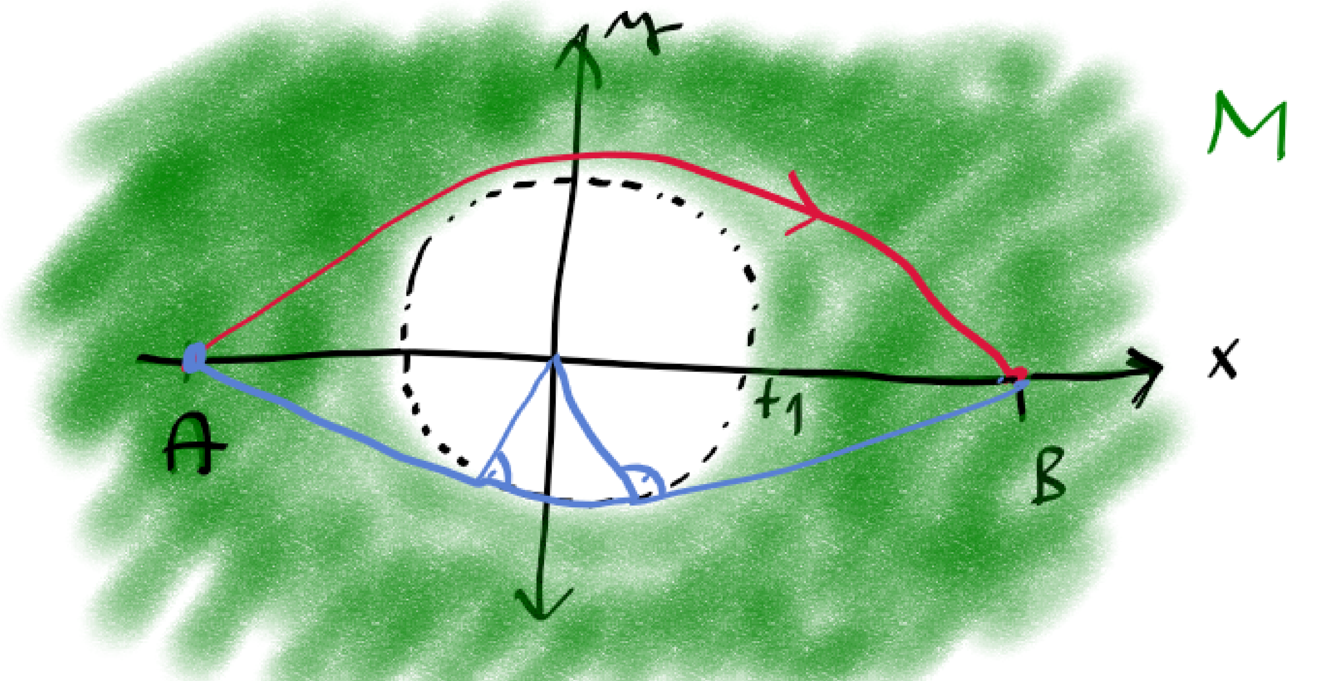
$$d(A,B) := \inf \{ L(\gamma) \mid \gamma : [a,b] \to M \text{ is piecewise } \mathcal{C}^1,\ a,b \in \mathbb{R},\ \gamma(a) = A,\ \gamma(b) = B \}$$

is called the distance of A and B. Now if $\varepsilon > 0$, then it can be shown that there is a $\mathcal{C}^1$ curve $\gamma : [a,b] \to M$ such that
$d(A,B) < L(\gamma) < d(A,B) + \varepsilon$. Hence,

$$d(A,B) := \inf \{ L(\gamma) \mid \gamma : [a,b] \to M \text{ is } \mathcal{C}^1,\ a,b \in \mathbb{R},\ \gamma(a) = A,\ \gamma(b) = B \}$$

Definition If (DM1) and (DM2) hold, we say that γ_0 is a distance minimazing curve (between A and B).

Example However, note that a distance minimazing curve need not exist. Consider for example $M = \mathbb{R}^2 \setminus \{ x \in \mathbb{R}^2 : \|x\| \le 1 \}$ and $A = (-2, 0)$ and $B = (2, 0)$. Now it is easy to see that there is no distance minimazing curve in M connecting A, B even though the distance of A, B is equal to the length of the blue curve.



We will now show that distance minimizing curves are geodesics for the Levi-Civita connection of g.

Crash course in Calculus of Variations

Let $\tilde{U}$ be open subset of $\mathbb{R}^m$; $\tilde{A}, \tilde{B} \in U$ be fixed and

$$X \equiv X_{\tilde{A}\tilde{B}} := \{ \tilde{\gamma} : [a,b] \to \tilde{U} \mid \gamma(a) = \tilde{A},\ \gamma(b) = \tilde{B},\ \tilde{\gamma} \text{ is } \mathcal{C}^1 \}.$$

Now X is a metric space with metric

$$d(\tilde{\gamma}_0, \tilde{\gamma}_1) = \|\tilde{\gamma}_0 - \tilde{\gamma}_1\|,\ \tilde{\gamma}_0, \tilde{\gamma}_1 \in X,$$

where

$$\|\tilde{\gamma}\| = \sup_{t \in [a,b]} (\|\tilde{\gamma}(t)\| + \|\tilde{\gamma}'(t)\|).$$

Now $U_\varepsilon(\tilde{\gamma}_0) := \{ \tilde{\gamma} \in X : d(\tilde{\gamma}_0, \tilde{\gamma}) < \varepsilon \}$ is the open ball of radius $\varepsilon > 0$ centered at $\tilde{\gamma}_0 \in X$. The collection of all such ball is a basis for the metric topology on X.

Now let $\mathcal{L} : T\tilde{\mathcal{U}} \longrightarrow \mathbb{R}$ be a smooth function, so that

$$\mathcal{S} : X \longrightarrow \mathbb{R}, \quad \mathcal{S}(\tilde{\gamma}) = \int_a^b \mathcal{L}(\tilde{\gamma}(t), \tilde{\gamma}'(t))\, dt$$

is a continuous functional on X. Here we view $T\tilde{\mathcal{U}} = \tilde{\mathcal{U}} \times \mathbb{R}^m$ and denote by $x_1, \dots, x_m, p_1, \dots, p_m$ the standard coordinates.

Example Let $\varphi : \mathcal{U} \to \mathbb{R}^m$ be a chart on the smooth manifold M with Riemannian metric g. Put $\tilde{\mathcal{U}} = \varphi(\mathcal{U})$. Assume that

$$\tilde{g}_x = ((\varphi^{-1})^* g)_x = \sum_{i,j=1}^m g_{ij}(x)\, dx_i \otimes dx_j$$

so that $g_{ij}(x)$ are smooth functions on $\mathcal{U}$. If $A, B \in \mathcal{U}$ and $\gamma : [a,b] \to M$ is a curve as above with $\gamma(a) = A$, $\gamma(b) = B$ whose image is contained in $\mathcal{U}$, then we put $\tilde{\gamma} = \varphi \circ \gamma$ and $\tilde{A} = \varphi(A)$, $\tilde{B} = \varphi(B)$. Then

$$L(\tilde{\gamma}) = \int_a^b \sqrt{\tilde{g}_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t), \tilde{\gamma}'(t))}\, dt = \int_a^b \sqrt{((\varphi^{-1})^* g)_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t), \tilde{\gamma}'(t))}\, dt$$

$$= \int_a^b \sqrt{g_{\varphi^{-1}(\tilde{\gamma}(t))}\left((T_{\tilde{\gamma}(t)}\varphi^{-1})(\tilde{\gamma}'(t)), (T_{\tilde{\gamma}(t)}\varphi^{-1})(\tilde{\gamma}'(t))\right)}\, dt$$

$$= \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt = L(\gamma),$$

since

$$(T_{\tilde{\gamma}(t)}\varphi^{-1})(\tilde{\gamma}'(t)) = (T_{\tilde{\gamma}(t)}\varphi^{-1})((\varphi \circ \gamma)'(t))$$

$$= (T_{\tilde{\gamma}(t)}\varphi^{-1}) \circ (T_{\gamma(t)}\varphi)(\gamma'(t)) =$$

$$= (T_{\gamma(t)}\varphi^{-1} \circ \varphi)(\gamma'(t)) = \gamma'(t).$$

Now we can use $\tilde{g}$ to define $\mathcal{L} : T\tilde{\mathcal{U}} \longrightarrow \mathbb{R}$

$$\mathcal{L}(x, v) = \|v\|_x := \sqrt{\tilde{g}_x(v,v)}, \quad x \in \tilde{\mathcal{U}}, \ v \in T_x\tilde{\mathcal{U}}.$$

The associated functional $\mathcal{S}$ on X is the length of $\tilde{\gamma}$, indeed

$$\mathcal{S}(\tilde{\gamma}) = \int_a^b \mathcal{L}(\tilde{\gamma}(t), \tilde{\gamma}'(t))\, dt = \int_a^b \|\tilde{\gamma}'(t)\|_{\tilde{\gamma}(t)}\, dt = L(\tilde{\gamma}).$$

We are interested in finding local minima of $\mathcal{S}$, a curve $\tilde{\gamma}_0 \in X$ is a local minimum or minimizer of $\mathcal{S}$ if there exists $\varepsilon > 0$ so that for every $\tilde{\gamma} \in U_\varepsilon(\tilde{\gamma}_0)$ we have

$$\mathcal{S}(\tilde{\gamma}) \geq \mathcal{S}(\tilde{\gamma}_0)$$

Theorem (Euler–Lagrange equations)

Let $\tilde{\gamma} : [a,b] \to \tilde{\mathcal{U}}$ be a local minimizer of $\mathcal{S}$ on X with $\mathcal{L}$ as above. Then $\tilde{\gamma}(t) = (\tilde{\gamma}_1(t), \dots, \tilde{\gamma}_m(t))$ solves the system of ODE's

(E-L) $\quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial p_k}(\tilde{\gamma}(t), \tilde{\gamma}'(t))\right) = \frac{\partial \mathcal{L}}{\partial x_k}(\tilde{\gamma}(t), \tilde{\gamma}'(t))$, $k = 1, \dots m$.

(These are so-called Euler-Lagrange equations.)

Proof can be found in any textbook on Calculus of variations. □

Example (continued) If $\mathcal{L}(x,v) = \sqrt{\tilde{g}_x(v,v)}$ with $x \in \tilde{U}$, $v \in T_x\tilde{U}$, then we may write

$$\mathcal{L}(x_1, \dots, x_m, p_1, \dots, p_m) = \sqrt{\sum_{i,j=1}^{m} g_{ij}(x)\, p_i p_j}$$

and so the Euler-Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial p_k}(\tilde{\gamma}(t), \tilde{\gamma}'(t))\right) = \frac{\partial \mathcal{L}}{\partial x_k}(\tilde{\gamma}(t), \tilde{\gamma}'(t)) \quad , i = 1, \dots, m$$

where

$$(EL)' \begin{cases} \frac{\partial \mathcal{L}}{\partial p_k}(x,p) = \left(2 \sum_{j=1}^{m} g_{kj}(x)\, p_j\right) \Big/ \sqrt{\sum_{i,j=1}^{m} g_{ij}(x)\, p_i p_j}\,, \\ \frac{\partial \mathcal{L}}{\partial x_k}(x,p) = \left(\sum_{i,j=1}^{m} \frac{\partial g_{ij}}{\partial x_k}(x)\, p_i p_j\right) \Big/ \sqrt{\sum_{i,j=1}^{m} g_{ij}(x)\, p_i p_j}\,. \end{cases}$$

However these equations are difficult to solve. Rather than trying to solve them let us go back to the beginning.

Now if γ is a distance minimizing curve between $\tilde{A}, \tilde{B} \in U$, then without loss of generality we may assume that $\tilde{\gamma}' \neq 0$ on $[a,b]$. Then we may reparametrize $\tilde{\gamma}$ by arc length, that is, we may find

$$\phi : [a', b'] \longrightarrow [a,b]$$

such that $\|(\tilde{\gamma}\circ\phi)'(t)\|_{\gamma\circ\phi(t)} = 1$ for every $t \in [a', b']$. Hence we may without loss of generality assume that $\|\tilde{\gamma}'(t)\|_{\gamma(t)}$ is constant on $[a,b]$.

Lemma A curve $\hat{\gamma}_0 : [a,b] \to \tilde{U}$ is

(MoL) a local (global) minimum of $\mathcal{S}$ on X and $\|\hat{\gamma}_0'(t)\|$ is constant on $[a,b]$

iff

(MoE) it is a local (global) minimum of the functional $\mathcal{E}(\tilde{\gamma}) = \int_a^b \|\tilde{\gamma}_0'(t)\|^2 dt$

Remark $\mathcal{E}$ is called the energy functional as it is closely related to the kinetic energy of a moving particle with mass m.

Proof of Lemma: By Cauchy-Schwartz inequality,

$$L(\hat{\gamma})^2 = \left(\int_a^b \|\hat{\gamma}'(t)\|_{\hat{\gamma}(t)} \cdot 1\, dt\right)^2 \leq \int_a^b \|\hat{\gamma}'(t)\|^2_{\hat{\gamma}(t)}\, dt \int_a^b 1 \cdot dt = (b-a)\, E(\hat{\gamma}).$$

Moreover, $=$ holds iff $\|\hat{\gamma}'(t)\|^2$ is proportional to the constant function 1 on $[a,b]$.

Hence, if $\tilde{\gamma}_0$ is a minimum of L then we may without loss of generality assume that $\|\tilde{\gamma}_0'(t)\|^2$ is constant. But then for any $\tilde{\gamma} \in X$ we have

$$(b-a)\, E(\tilde{\gamma}_0) = L(\tilde{\gamma}_0)^2 \leq L(\tilde{\gamma})^2 \leq (b-a)\, E(\tilde{\gamma}).$$

This shows equivalence of (MoL) and (MoE). □

Hence, we can now consider the Euler–Lagrange equations for L. These are

$$2\,\frac{d}{dt}\Big(\sum_{j=1}^{m} g_{kj}(\tilde{\gamma}(t))\,\tilde{\gamma}_j'(t)\Big) = \sum_{i,j=1}^{m} \frac{\partial g_{ij}}{\partial x_k}(\tilde{\gamma}(t))\,\tilde{\gamma}_i'(t)\,\tilde{\gamma}_j'(t), \quad k=1,\dots,m.$$

Now if we differentiate with respect to t on the left hand side we get

$$2\sum_{j=1}^{m} g_{kj}(\tilde{\gamma}(t))\,\tilde{\gamma}_j''(t) + 2\sum_{i,j=1}^{m} \frac{\partial g_{kj}}{\partial x_i}(\tilde{\gamma}(t))\,\tilde{\gamma}_i'(t)\,\tilde{\gamma}_j'(t) =$$

$$= \sum_{i,j=1}^{m} \frac{\partial g_{ij}}{\partial x_k}(\tilde{\gamma}(t))\,\tilde{\gamma}_i'(t)\,\tilde{\gamma}_j'(t), \quad k=1,\dots,m.$$

$$2\sum_{j=1}^{m} g_{kj}(\tilde{\gamma}(t))\,\tilde{\gamma}_j''(t) = \sum_{i,j=1}^{m}\Big[\frac{\partial g_{ij}}{\partial x_k}(\tilde{\gamma}(t))\,\tilde{\gamma}_i'(t)\,\tilde{\gamma}_j'(t) - 2\frac{\partial g_{kj}}{\partial x_i}(\tilde{\gamma}(t))\,\tilde{\gamma}_i'(t)\,\tilde{\gamma}_j'(t)\Big]$$

$$= \sum_{i,j=1}^{m}\Big[\Big(\frac{\partial g_{ij}}{\partial x_k} - 2\,\frac{\partial g_{kj}}{\partial x_i}\Big)(\tilde{\gamma}(t))\Big]\,\tilde{\gamma}_i'(t)\,\tilde{\gamma}_j'(t)$$

Now let $(g^{kl}(x))_{k=1,\dots,m}^{l=1,\dots,m}$ be the inverse matrix to $(g_{ij}(x))_{i=1,\dots,m}^{j=1,\dots,m}$ so that $\sum_{k=1}^{m} g^{ki}(x)\,g_{ij}(x) = \delta_{kj}$. Then we have

$$2\,\tilde{\gamma}_l''(t) = 2\sum_{k,j=1}^{m} g^{lk}(\tilde{\gamma}(t))\,g_{kj}(\tilde{\gamma}(t))\,\tilde{\gamma}_j''(t)$$

$$= \sum_{i,j,k=1}^{m} g^{lk}(\tilde{\gamma}(t))\Big(\frac{\partial g_{ij}}{\partial x_k} - 2\,\frac{\partial g_{kj}}{\partial x_i}\Big)(\tilde{\gamma}(t))\,\tilde{\gamma}_i'(t)\,\tilde{\gamma}_j'(t) = (*)$$

$$\left|\ \sum_{i,j=1}^{m}\frac{\partial g_{kj}}{\partial x_i}(\tilde{\gamma}(t))\,\tilde{\gamma}_i'\,\tilde{\gamma}_j'(t) = \sum_{j,i=1}^{m}\frac{\partial g_{ki}}{\partial x_j}(\tilde{\gamma}(t))\,\tilde{\gamma}_j'(t)\,\tilde{\gamma}_i'(t) = \sum_{i,j=1}^{m}\frac{\partial g_{ki}}{\partial x_j}(\tilde{\gamma}(t))\,\tilde{\gamma}_i'(t)\,\tilde{\gamma}_j'(t)\ \right)$$

$$(*) = \sum_{i,j=1}^{m}\sum_{k=1}^{m}\Big[g^{lk}\Big(\frac{\partial g_{ij}}{\partial x_k} - \frac{\partial g_{kj}}{\partial x_i} - \frac{\partial g_{ki}}{\partial x_j}\Big)\Big](\tilde{\gamma}(t))\,\tilde{\gamma}_i'(t)\,\tilde{\gamma}_j'(t)$$

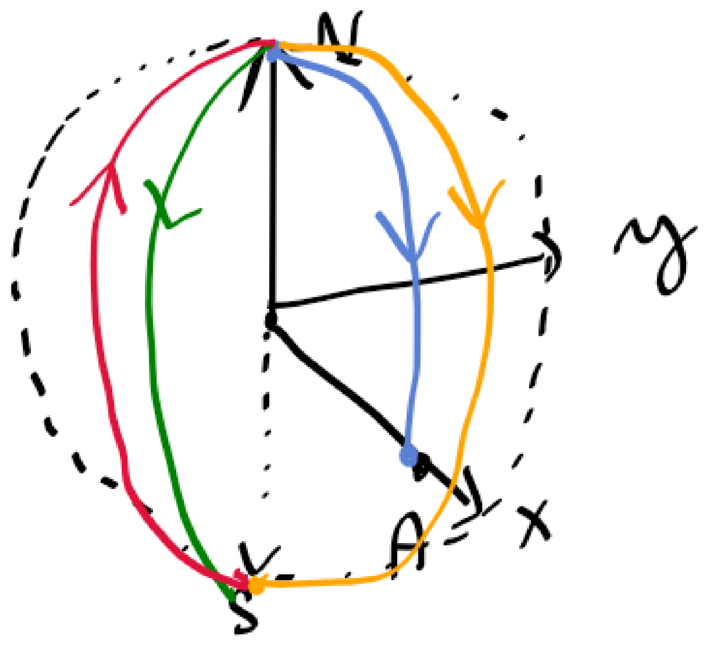
$$= -\sum_{i,j=1}^{m} 2\,\Gamma_{ij}^{l}(\gamma(t))\,\tilde{\gamma}_i'(t)\,\tilde{\gamma}_j'(t) \qquad (\text{by LC}).$$

Which is the geodesic equation (GE) for the Levi-Civita connection of g

$$\tilde{\gamma}_k''(t) + \sum_{i,j=1}^{m}\Gamma_{ij}^{k}(\gamma(t))\,\tilde{\gamma}_i'(t)\,\tilde{\gamma}_j'(t) = 0, \quad k=1,\dots,m.$$

<u>Example</u> (Geodesics on sphere S^2 - continued)

We have seen in previous week that great circles are geodesics for the Levi-Civita connection of g_{S^2} (parametrized with constant velocity). Thus parts of great circles are distance minimizing curves on S^2. So for example the distance between $N=(0,0,1)$ and $S=(0,0,-1)$ is π and any meridian is a shortest curve between N and S. Thus there are infinitely many shortest curves connecting N and S. On the other hand, There is a unique distance minimizing curve between N and $A = (1,0,0)$ (blue curve) and the distance between N and A is the length $\frac{\pi}{2}$ of this curve.



Upper half-plane model of hyperbolic geometry

Let $\mathbb{H}^+ := \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$. On $\mathbb{H}^+$ consider Riemannian metric $g_{\mathbb{H}^+} = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$ or in matrix notation

$$\frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_{xx} = \frac{1}{y^2} = g_{yy}, \quad g_{xy} = g_{yx} = 0.$$

The inverse matrix is $y^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and so $g^{xx} = g^{yy} = y^2$, $g^{xy} = g^{yx} = 0$.

The Christoffel symbols are

$$\Gamma^x_{xx} = 0, \quad \Gamma^y_{xx} = \frac{1}{2} g^{yy}\left(2\frac{\partial}{\partial x} g_{xy} - \frac{\partial}{\partial y} g_{yy}\right) + \frac{1}{2} g^{yx}(\dots) = \frac{1}{2} y^2 \left(\frac{2}{y^3}\right) = \frac{1}{y}$$

$$\Gamma^x_{yx} = \Gamma^x_{xy} = \frac{1}{2} g^{xx}\left(\frac{\partial}{\partial x} g_{yx} + \frac{\partial}{\partial y} g_{xx} - \frac{\partial}{\partial x} g_{xy}\right) + \frac{1}{2} g^{xy}(\dots) = \frac{1}{2} y^2 (-2) y^{-3} = -\frac{1}{y},$$

$$\Gamma^y_{xy} = \frac{1}{2} g^{yy}\left(\frac{\partial}{\partial x} g_{yy} + \frac{\partial}{\partial y} g_{xy} - \frac{\partial}{\partial y} g_{xy}\right) + \frac{1}{2} g^{yx}(\dots) = 0 = \Gamma^y_{yx},$$

$$\Gamma^y_{yy} = \frac{1}{2} g^{yy}\left(2\frac{\partial}{\partial y} g_{yy} - \frac{\partial}{\partial y} g_{yy}\right) + \frac{1}{2} g^{yx}(\dots) = \frac{1}{2} y^2\left(-\frac{2}{y^3}\right) = -\frac{1}{y}.$$

The geodesic equations are

$$x'' - 2\frac{1}{y} y' x' = 0, \quad y'' + \frac{1}{y}(x')^2 - \frac{1}{y}(y')^2 = 0$$

for unknown functions $x = x(t)$, $y = y(t)$. This is a pair of non-linear ODE's of second order.

It is clear that if $x' = 0$, then we get

$$0 = \frac{1}{y}\left(y'' - \frac{1}{y}(y')^2\right) = \frac{y''}{y} - \frac{y'^2}{y^2} = \left(\frac{y'}{y}\right)' \Rightarrow \frac{y'}{y} = c \in \mathbb{R} \Rightarrow$$

$y(t) = c_1 e^{c_2 t}$, $t \in \mathbb{R}$ is a solution.

Next let us assume that $x' \neq 0$. Then t is a function of x and thus we can also think of y as a function of x. Hence

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y'}{x'}\right) \underset{\text{chain rule}}{=} \frac{y''x' - x''y'}{(x')^2} \cdot \frac{1}{x'}$$

$$= \frac{1}{(x')^3}\left(x'\left(\frac{(y')^2}{y} - \frac{(x')^2}{y}\right) - y'\frac{2y'x'}{y}\right) = -\frac{1}{(x')^2}\frac{1}{y}\left(y'^2 + x'^2\right)$$

$$= -\frac{1}{y}\left(1 + \left(\frac{y'}{x'}\right)^2\right) = -\frac{1}{y}\left(1 + \left(\frac{dy}{dx}\right)^2\right).$$

We have found that

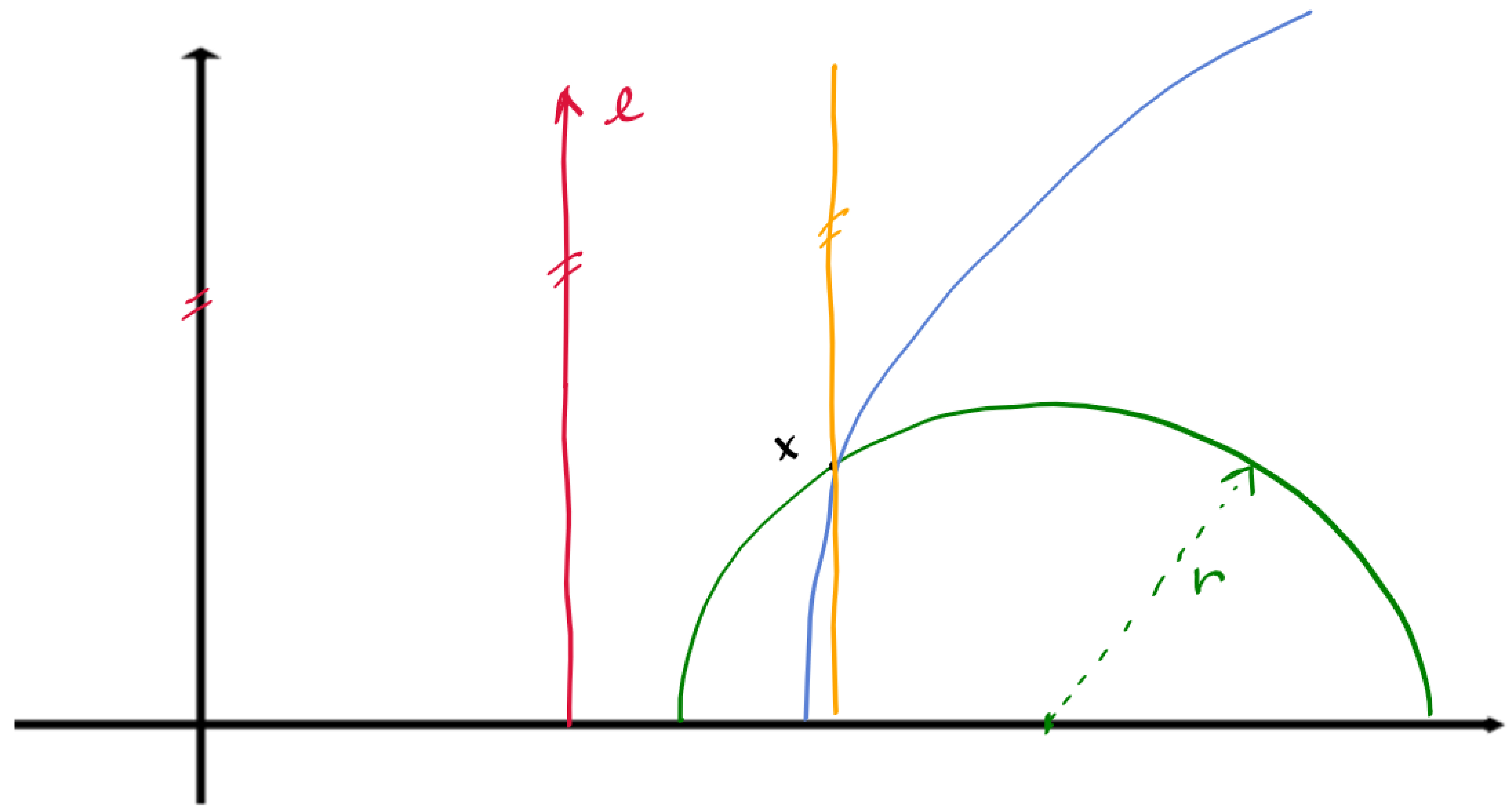
$$-1 = y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{d}{dx}\left(y\frac{dy}{dx}\right) \Rightarrow y\frac{dy}{dx} = -x + c, \; c \in \mathbb{R}$$

$$\Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + cx + d, \; d \in \mathbb{R} \Rightarrow y^2 + x^2 - 2cx = d \in \mathbb{R} \Rightarrow$$

$$(x-c)^2 + y^2 = d + \frac{c^2}{4} = r^2 \geq 0.$$

We see that geodesics are either lines parallel to y-axis and upper half circles whose center lies at x-axis (with parametrization by constant velocity).

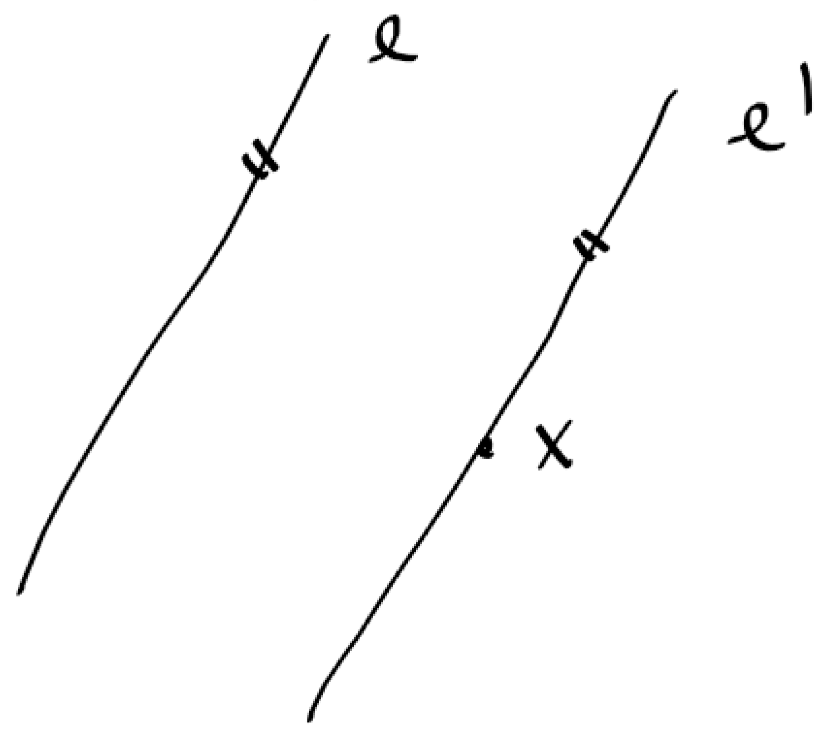
Pic. 1



Remark

This is a model of non-Euclidean geometry. In this geometry the first four Euclid axioms hold but the fifth postulate is violated:

"Given a line l and a point x with $x \notin l$, there is a unique parallel line l' to l such that $x \in l'$."



In the upper half-plane model, lines are geodesics and two geodesics are parallel if they do not intersect. If $x \in \mathbb{H}^+$ and l is a geodesic in $\mathbb{H}$, there are infinitely many parallel lines to l which contain x. See Pic. 1.