Geodesics as distance minimizing curves

About HW: there will be 9 HW assignments, to obtain 1 as a grade from this course one can obtain at least 80% out of 9 points (each HW is for 1 point).
It is required to obtain at least 20% out of 9 points to obtain "zápočet". In HW 7 there is an extra bonus point.

Geodesics as distance minimizing curves

Let (M,g) be a Riemannian manifold and $\gamma: [a,b] \to M$ be a curve on M. Then the length of γ is defined as

$$L(\gamma) := \int_a^b \|\gamma'(t)\|_{\gamma(t)}\, dt \quad \text{where} \quad \|v\|_x := \sqrt{g_x(v,v)}\,, \; v \in T_x M,\; x \in M.$$

In order that the length of γ is defined, let us assume that γ is piecewise $\mathcal{C}^1$, that is, there is a subdivision $a = t_0 < t_1 < \dots < t_n = b$ of $[a,b]$ such that $\gamma|_{[t_i,t_{i+1}]}$ is $\mathcal{C}^1$ for every $i = 0,\dots,n-1$.

We know from (calculus) that the length of γ does not depend on parametrization, that is, if $\phi: [a',b'] \to [a,b]$ is a diffeomorphism with $\phi' > 0$, then $L(\gamma) = L(\gamma \circ \phi)$.

If $A, B \in M$, then the distance between A, B, denoted by $d(A,B)$, is defined as

$$d(A,B) = \inf \{ L(\gamma) \mid \gamma: [a,b] \to M \text{ is piecewise } \mathcal{C}^1,\ \gamma(a) = A,\ \gamma(b) = B \}.$$

It can be verified that M with d is a metric space. If A, B belong to different connected components of M, then $d(A,B) = +\infty$.

<u>Definition</u> Let (M,g) be a Riemannian manifold and $A, B \in M$. Then a curve $\gamma_0: [a,b] \to M$ is a distance minimizing curve between A, B if

(DM1) $\gamma_0(a) = A$, $\gamma_0(b) = B$ and

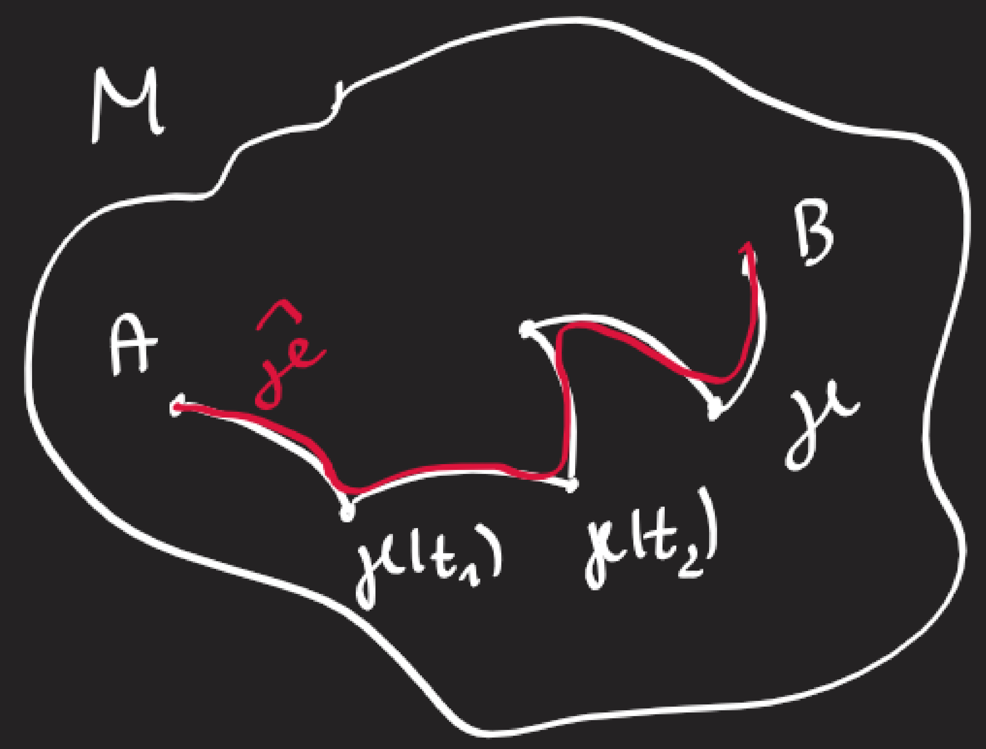
(DM2) $L(\gamma_0) = d(A,B)$.

Remark: It is possible to show that

$$d(A,B) = \inf\{L(\gamma) \mid \gamma : [a,b] \to M \text{ is } \mathcal{C}^1,\ \gamma(a) = A,\ \gamma(b) = B\}.$$

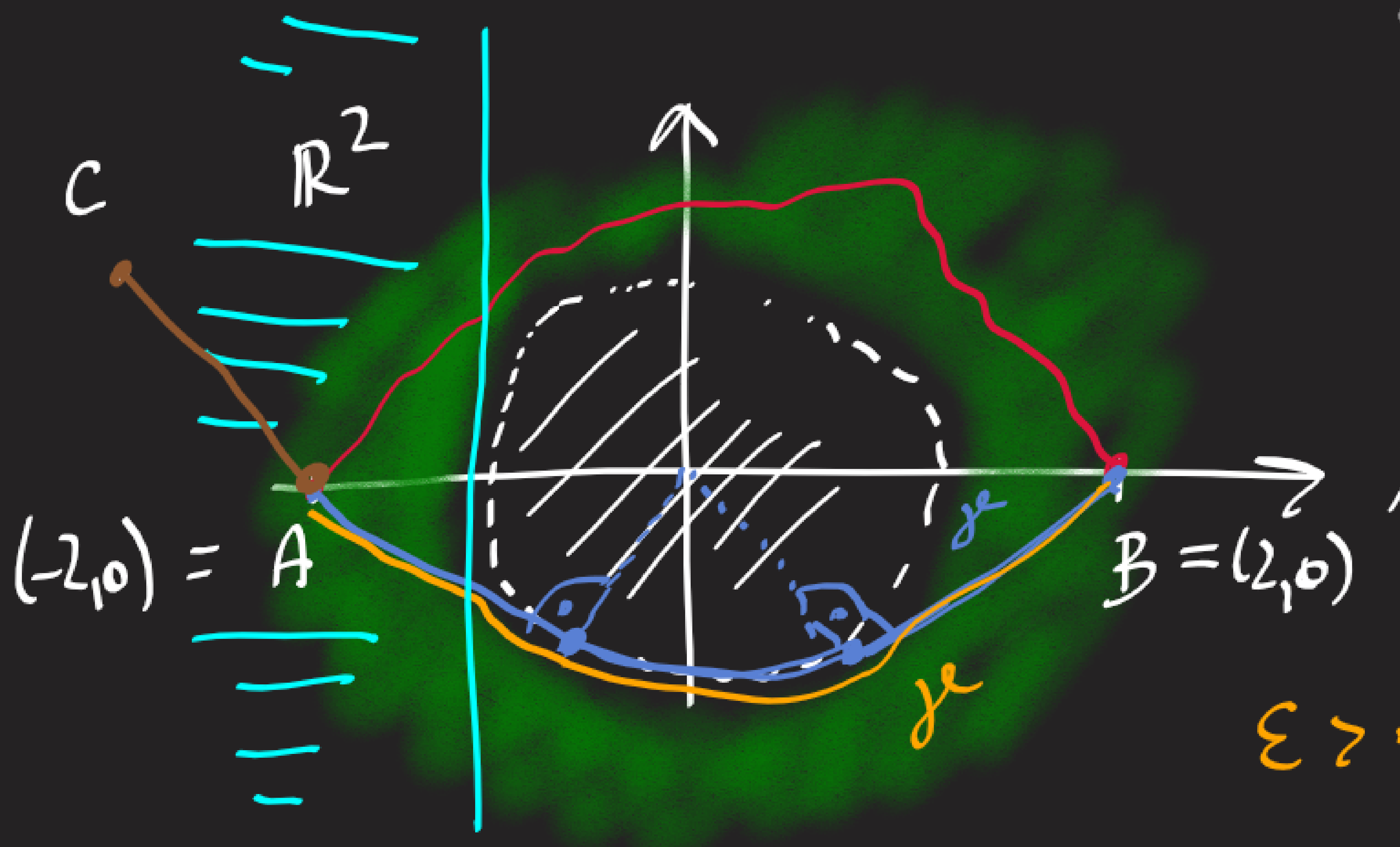
$\gamma' \neq 0$ on $[a,b]$

We will work with this definition of $d(A,B)$.



$\varepsilon > 0$ $\quad L(\gamma) + \varepsilon > L(\hat{\gamma}) > L(\gamma)$

Example •) In general, a distance minimizing curve need not exist. Consider $M = \mathbb{R}^2 \setminus \{(x,y) : x^2+y^2 \leq 1\}$, let $\imath : M \hookrightarrow \mathbb{R}^2$ be the canonical inclusion and $g_M = \imath^* g$ where g is the Euclidean metric on $\mathbb{R}^2$. It is easy to see that there is no distance minimizing curve in M between A, B, since $d(A,B)$ is the length of the blue curve.



$L(\gamma) < L(\gamma) < L(\gamma) + \varepsilon$

$\varepsilon > 0$

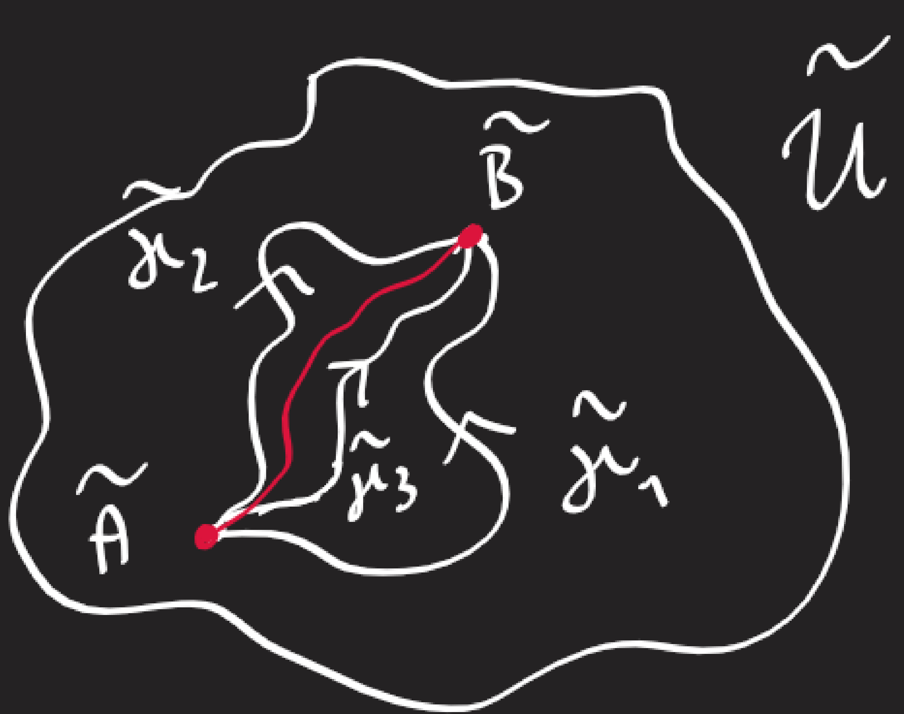
•) However it can be shown that if (M,g) is any Riemannian manifold and $A \in M$, then there is an open neighbourhood $\mathcal{U}$ of A such that any point $C \in \mathcal{U}$ can be connected to A by a unique distance minimizing curve.

Short overview of Calculus of variations

Let $\tilde{\mathcal{U}}$ is an open subset of $\mathbb{R}^m$ and $\tilde{A}, \tilde{B} \in \tilde{\mathcal{U}}$. Consider

$$X = X_{\tilde{A},\tilde{B}} = \{\tilde{\gamma} : [a,b] \to \tilde{\mathcal{U}} \mid \tilde{\gamma}(a) = \tilde{A},\ \tilde{\gamma}(b) = \tilde{B},\ \tilde{\gamma} \text{ is } \mathcal{C}^1\}.$$

Here we work with fixed $a, b \in \mathbb{R}$, $a < b$.



This space can be endowed with metric

$$d(\tilde{\gamma}_1, \tilde{\gamma}_2) = \sup_{[a,b]} \left(\|\tilde{\gamma}_1(t) - \tilde{\gamma}_2(t)\| \right) + \sup_{[a,b]} \left(\|\tilde{\gamma}_1'(t) - \tilde{\gamma}_2'(t)\| \right)$$

$\|\ldots\|$ is the Euclidean norm on $\mathbb{R}^m$

(X,d) is a metric space. The metric topology is generated by the collection of all open balls

$$\mathcal{U}_\varepsilon(\hat\gamma_0) = \{\hat\gamma \in X : d(\hat\gamma_0, \hat\gamma) < \varepsilon\}.$$

Let us assume that $\mathcal{L} : T\tilde{\mathcal{U}} \to \mathbb{R}$ is a smooth function. We will view $T\tilde{\mathcal{U}} = \tilde{\mathcal{U}} \times \mathbb{R}^m$ with coordinates $x_1, \dots, x_m, p_1, \dots, p_m$. Now the function

$$S : X \to \mathbb{R}, \quad S(\hat\gamma) = \int_a^b \mathcal{L}(\tilde\gamma(t), \tilde\gamma'(t))\, dt$$

is continuous (w.r.t. the metric topology).
S is usually called a functional on X.

<u>Example</u> Let (M,g) be a Riemannian manifold. Let $\varphi : \mathcal{U} \to \mathbb{R}^m$ be a chart on M and assume that $A, B \in \mathcal{U}$. Now if $\gamma : [a,b] \to M$ such that $\gamma(t) \in \mathcal{U}$ for every $t \in [a,b]$, put $\varphi \circ \gamma = \tilde\gamma$ and $\varphi(A) = \tilde A$, $\varphi(B) = \hat B$. Let $\varphi(\mathcal{U}) = \tilde{\mathcal{U}}$ and $\tilde g := (\varphi^{-1})^* g$. Then we have that

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt = \int_a^b \sqrt{\tilde g_{\tilde\gamma(t)}(\hat\gamma'(t), \tilde\gamma'(t))}\, dt = L(\hat\gamma).$$

Our goal is to find local or global minimizer ($\equiv$ minimum) of L on X. Notice that

$$\mathcal{L}(\tilde\gamma(t), \tilde\gamma'(t)) = \sqrt{\tilde g_{\tilde\gamma(t)}(\hat\gamma'(t), \hat\gamma'(t))}.$$

<u>Theorem</u> (Euler–Lagrange equations)
Let $X, \tilde A, \hat B, \tilde{\mathcal{U}}, S, \mathcal{L}$ be as above. Now if a curve $\hat\gamma_0$ is a local minimizer of S on X, then $\hat\gamma_0$ satisfies Euler–Lagrange equations on $[a,b]$:

$$\text{(EL)} \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial p_k}(\tilde\gamma(t), \hat\gamma'(t))\right) = \frac{\partial \mathcal{L}}{\partial x_k}(\tilde\gamma(t), \tilde\gamma'(t)), \quad k = 1, \dots, m.$$

(A curve $\hat\gamma_0$ is a local minimizer of S if there exists $\varepsilon > 0$ such $S(\tilde\gamma_0) \le S(\hat\gamma)$ for every $\hat\gamma \in \mathcal{U}_\varepsilon(\hat\gamma_0)$.)

Proof can be found in any textbook on Calculus of variations. $\square$

Euler–Lagrange equations for the functional L

$$\mathcal{L}(x,v) = \sqrt{\tilde{g}_x(v,v)} \quad \text{where } x \in \tilde{M},\ v \in T_x \tilde{M}.$$

In the coordinates $x_1,\dots,x_m, p_1,\dots,p_m$ with

$$\tilde{g}(x) = \sum_{i,j=1}^{m} \tilde{g}_{ij}(x)\, dx_i \otimes dx_j, \quad \text{then}$$

$$\mathcal{L}(x_1,\dots,x_m,p_1,\dots,p_m) = \sqrt{\sum_{i,j=1}^{m} \tilde{g}_{ij}(x)\, p_i p_j}\ .$$

($\mathcal{L}$ is usually called the Lagrangian.) The Euler–Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial p_k}(\tilde{\gamma}(t), \tilde{\gamma}'(t))\right) = \frac{\partial \mathcal{L}}{\partial x_k}(\tilde{\gamma}(t), \tilde{\gamma}'(t)), \quad t \in [a,b],\ k = 1,\dots,m.$$

$$\frac{\partial \mathcal{L}}{\partial p_k}(\underbrace{x_1,\dots,x_m}_{x},p_1,\dots,p_m) = 2\sum_{j=1}^{m} \tilde{g}_{kj}\, p_j \Big/ \sqrt{\sum_{i,j=1}^{m} \tilde{g}_{ij}(x)\, p_i p_j}\ ,$$

$$\frac{\partial \mathcal{L}}{\partial x_k}(x_1,\dots,x_m,p_1,\dots,p_m) = \sum_{i,j=1}^{m} \frac{\partial \tilde{g}_{ij}}{\partial x_k}(x)\, p_i p_j \Big/ \sqrt{\sum_{i,j=1}^{m} \tilde{g}_{ij}(x)\, p_i p_j}\ .$$

These differential equations are difficult. The problematic part are the denominators and we would like to get rid of them, if possible. Let us at this moment recall that L is invariant under reparametrization of curves. If γ is a curve on M which minimizes the distance between A, B, then we can without loss of generality assume that $\gamma'(t) \neq 0$ for every $t \in [a,b]$. For such a curve we can find a reparametrization by a new parameter s so that $\frac{d\gamma}{ds}$ has constant velocity, this means that $\| \frac{d\gamma}{ds}(s) \|_{\gamma(s)}$ is constant on $[a,b]$. (More precisely, given $\gamma : [a,b] \to M$ with $\gamma'(t) \neq 0$ on $[a,b]$, then I can find $\phi : [a,b] \to [a,b]$ a diffeomorphism with $\phi' > 0$ such $\gamma(s) = (\gamma \circ \phi)(s)$, then $\| \frac{d\gamma}{ds}(s) \|_{\gamma(s)} = \text{const.}$)

We see that if γ is a distance minimizing curve between A, B, then we can without loss of generality assume that γ is parametrized with constant velocity.

Lemma A curve $\tilde{\gamma}_0$ is a local (or global) minimizer of L on X and $\tilde{g}_{\tilde{\gamma}_0(t)}(\tilde{\gamma}_0'(t), \tilde{\gamma}_0'(t))$ is constant on $[a,b]$
iff
$\tilde{\gamma}_0$ is a local (or global) minimizer of
$$E(\tilde{\gamma}_0) = \int_a^b \tilde{g}_{\tilde{\gamma}_0(t)}(\tilde{\gamma}_0'(t), \tilde{\gamma}_0'(t))\, dt \quad \text{on } X.$$

Remark: The functional E is called the energy functional as it is closely related to the kinetic energy of a moving particle.

Proof: We will show only "$\Rightarrow$". By Cauchy-Schwartz we have that (write $\sqrt{\tilde{g}_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t), \tilde{\gamma}'(t))} =: \|\tilde{\gamma}'(t)\|_{\tilde{\gamma}(t)}$)

$$L(\tilde{\gamma})^2 = \Big(\int_a^b \|\tilde{\gamma}'(t)\|_{\tilde{\gamma}(t)} \cdot \underset{\substack{\uparrow \\ \text{constant function on } [a,b]}}{1}\, dt \Big)^2 \leq$$

$$\leq \int_a^b \|\tilde{\gamma}'(t)\|^2_{\tilde{\gamma}(t)}\, dt \cdot \int_a^b 1^2\, dt$$
$$= \int_a^b \|\tilde{\gamma}'(t)\|^2_{\tilde{\gamma}(t)}\, dt\, (b-a) = (b-a)\, E(\tilde{\gamma}).$$

Moreover, $=$ holds iff $\|\tilde{\gamma}'(t)\|_{\tilde{\gamma}(t)}$ is proportional to 1, that is, $\|\tilde{\gamma}'(t)\|_{\tilde{\gamma}(t)}$ is constant on $[a,b]$.

Let us assume that $\tilde{\gamma}_0$ is a local minimizer of L and that $\tilde{\gamma}_0$ is parametrized with constant velocity. Then we have that
$$(b-a)E(\tilde{\gamma}_0) = L(\tilde{\gamma}_0)^2 \leq L(\tilde{\gamma})^2 \leq (b-a)\, E(\tilde{\gamma}).$$
where $\varepsilon > 0$ is such that $L(\gamma) \geq L(\tilde{\gamma}_0)$ for every $\tilde{\gamma} \in U_\varepsilon(\tilde{\gamma}_0)$

Let us now consider the Euler-Lagrange equations for the functional E.
$$E(\tilde{\gamma}) = \int_a^b \tilde{g}_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t), \tilde{\gamma}'(t))\, dt.$$
$$\mathcal{L}(\tilde{\gamma}(t), \tilde{\gamma}'(t)) = \tilde{g}_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t), \tilde{\gamma}'(t))$$
$$\mathcal{L}(x_1, \dots, x_m, p_1, \dots, p_m) = \sum_{i,j=1}^m \tilde{g}(x)_{ij}\, p_i\, p_j.$$

The Euler – Lagrange equations are

$$\frac{\partial L}{\partial p_k}(x_1,\dots,x_m,p_1,\dots,p_m) = 2\sum_{j=1}^{m} \tilde g(x)_{kj}\, p_j$$

$$\frac{\partial L}{\partial x_k}(x_1,\dots,x_m,p_1,\dots,p_m) = \sum_{i,j=1}^{m} \frac{\partial \tilde g_{ij}}{\partial x_k}(x)\, p_i p_j$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial p_k}(\tilde\gamma(t),\tilde\gamma'(t))\right) = \frac{\partial L}{\partial x_k}(\tilde\gamma(t),\tilde\gamma'(t)) \quad , k=1,\dots,m \quad \text{are}$$

$$\frac{d}{dt}\left(2\sum_{j=1}^{m} \tilde g(\hat\gamma(t))_{kj}\, \tilde\gamma_j'(t)\right) = \sum_{i,j=1}^{m} \frac{\partial \tilde g_{ij}}{\partial x_k}(\tilde\gamma(t))\, \tilde\gamma_i'(t)\, \tilde\gamma_j'(t)$$

$$2\sum_{i,j=1}^{m} \frac{\partial \tilde g_{kj}}{\partial x_i}(\hat\gamma(t))\, \tilde\gamma_i'(t)\, \tilde\gamma_j'(t) + 2\sum_{j=1}^{m} \tilde g(\hat\gamma(t))_{kj}\, \tilde\gamma_j''(t)$$

$$= \sum_{i,j=1}^{m} \frac{\partial \tilde g_{ij}}{\partial x_k}(\tilde\gamma(t))\, \tilde\gamma_i'(t)\, \tilde\gamma_j'(t) \qquad / :2$$

$$\sum_{j=1}^{m} \tilde g_{kj}(\hat\gamma(t))\, \tilde\gamma_j''(t) + \frac{1}{2}\sum_{i,j=1}^{m}\left(\frac{\partial \tilde g_{kj}}{\partial x_i}(\hat\gamma(t))\, \tilde\gamma_i'(t)\, \tilde\gamma_j'(t) + \frac{\partial \tilde g_{ki}}{\partial x_j}(\hat\gamma(t))\, \tilde\gamma_j'(t)\, \tilde\gamma_i'(t)\right)$$

$$-\frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial \tilde g_{ij}}{\partial x_k}(\hat\gamma(t))\, \tilde\gamma_i'(t)\, \tilde\gamma_j'(t) = 0 \qquad / \cdot\sum_{k=1}^{m} \tilde g^{lk}(\tilde\gamma(t))$$

$$\boxed{\delta_{lj} = \sum_{k=1}^{m} \tilde g^{lk}(\tilde\gamma(t))\, \tilde g_{kj}(\hat\gamma(t))}$$

$$\tilde\gamma_l''(t) + \frac{1}{2}\sum_{i,j,k=1}^{m} g^{lk}(\hat\gamma(t))\left(\frac{\partial \tilde g_{kj}}{\partial x_i}(\hat\gamma(t))\, \tilde\gamma_i'(t)\, \tilde\gamma_j'(t)\right.$$

$$\left. + \frac{\partial \tilde g_{ki}}{\partial x_j}(\hat\gamma(t))\, \tilde\gamma_j'(t)\, \hat\gamma_i'(t) - \frac{\partial \hat g_{ij}}{\partial x_k}(\tilde\gamma(t))\, \tilde\gamma_i'(t)\, \hat\gamma_j'(t)\right) = 0$$

$$l = 1,\dots,m$$

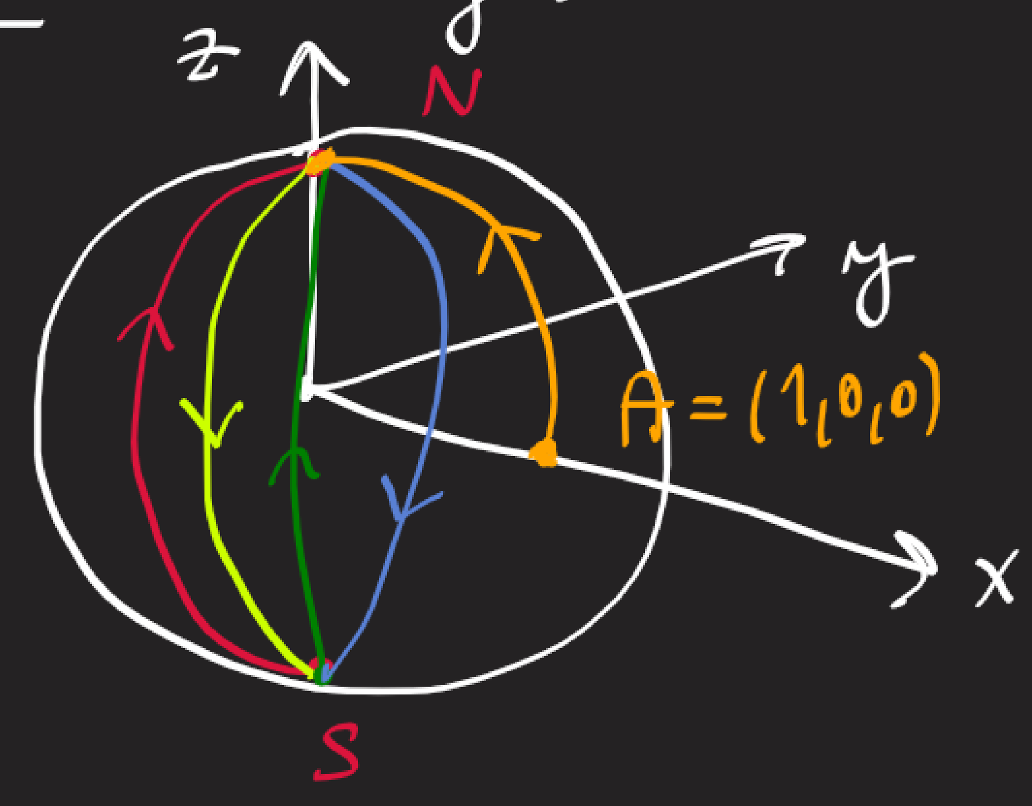
$$\tilde\gamma_l''(t) + \sum_{i,j=1}^{m} \Gamma_{ij}^{l}(\hat\gamma(t))\, \hat\gamma_i'(t)\, \hat\gamma_j'(t) = 0 \quad , \; l=1,\dots,m$$

Geodesic equations for the Levi-Civita connection.

Corollary The geodesic equation for the Levi-Civita connection are just the Euler–Lagrange equations for the functional E. (And so local minimizers of E and thus by the previous Lemma of L are geodesics).

Example Geodesics on sphere

$\mathbb{R}^3 \supseteq S^2$



$d(N,S) = \pi$

there are infinitely many distance minimizing curves between N, S

$d(A,N) = \frac{\pi}{2}$

Upper half-plane model of hyperbolic geometry

Let $\mathbb{H}^+ = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$. On $\mathbb{H}^+$ we consider

$$g = g_{\mathbb{H}^+} = \frac{dx \otimes dx + dy \otimes dy}{y^2}.$$ So we have that

$$g_{xx} = y^{-2} = g_{yy}, \quad g_{xy} = g_{yx} = 0, \quad \text{or in matrix notation}$$

$$\frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad \text{We need} \quad g^{xx} = y^2 = g^{yy}, \quad g^{xy} = g^{yx}.$$

Christoffel symbols are

$$\Gamma^x_{xx} = \Gamma^y_{xy} = \Gamma^y_{yx} = 0 \qquad \Gamma^y_{xx} = \frac{1}{y}$$

$$\Gamma^y_{yy} = \frac{1}{2} g^{yx}(\ldots) + \frac{1}{2} g^{yy}\left(\frac{\partial g_{yy}}{\partial y} + \frac{\partial g_{yy}}{\partial y} - \frac{\partial g_{yy}}{\partial y}\right)$$

$$= \frac{1}{2} y^2 \left(\frac{\partial g_{yy}}{\partial y}\right) = \frac{1}{2} y^2 (-2y^{-3}) = -y^{-1} = -\frac{1}{y}$$

$$\Gamma^x_{yx} = \Gamma^x_{xy} = \frac{1}{2} g^{xx}\left(\frac{\partial g_{yx}}{\partial x} + \frac{\partial g_{xx}}{\partial y} - \frac{\partial g_{xy}}{\partial x}\right) + \frac{1}{2} \underbrace{g^{xy}}_{0}(\ldots)$$

$$= \frac{1}{2} y^2(-2y^{-3}) = -\frac{1}{y}.$$

Geodesics equations for unknown functions $x(t), y(t)$

$$x'' - 2\frac{1}{y} x'y' = 0, \quad y'' + \frac{1}{y}(x'^2 - y'^2) = 0$$

$$x'' - 2\frac{1}{y}x'y' = 0, \quad y'' + \frac{1}{y}(x'^2 - y'^2) = 0.$$

Let us first assume that $x' = 0$. The second equation reduces to

$y'' - y'^2/y = 0$. Now we have

$$\left(\frac{y'}{y}\right)' = \frac{y''}{y} - \frac{y'^2}{y^2} = \frac{1}{y}\left(y'' - \frac{y'}{y}\right) = 0.$$

Since $y > 0$, so $\frac{y'}{y} = c \in \mathbb{R} \Rightarrow$

$$\ln y = \int \frac{dy}{y} \qquad \int c\,dt = ct + d, \; d \in \mathbb{R}$$

$$y(t) = e^{d} e^{ct} \quad \text{is a solution with } c, d \in \mathbb{R}.$$

Let us now assume that $x' \neq 0$. Then x has an inverse function and so t and hence also y are functions of x. Let us compute

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{y''x' - x''y'}{(x')^2} \cdot \frac{1}{x'} =$$

$$= \frac{1}{(x')^3}(y''x' - x''y') = \frac{1}{(x')^3}\left(\frac{1}{y}(y'^2 - x'^2)x' - 2\frac{x'y'}{y}y'\right) =$$

$$= \frac{1}{x'^3}\frac{1}{y}(-x'^3 - y'^2x') = -\frac{1}{y}\left(1 + \frac{y'^2}{x'^2}\right) = -\frac{1}{y}\left(1 + \left(\frac{y'}{x'}\right)^2\right)$$

$$= -\frac{1}{y}\left(1 + \left(\frac{dy}{dx}\right)^2\right). \text{ We get}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{y}\left(1 + \left(\frac{dy}{dx}\right)^2\right), \quad \text{ODE for unknow function } y \equiv y(x).$$

$$-y\frac{d^2y}{dx^2} = 1 + \left(\frac{dy}{dx}\right)^2$$

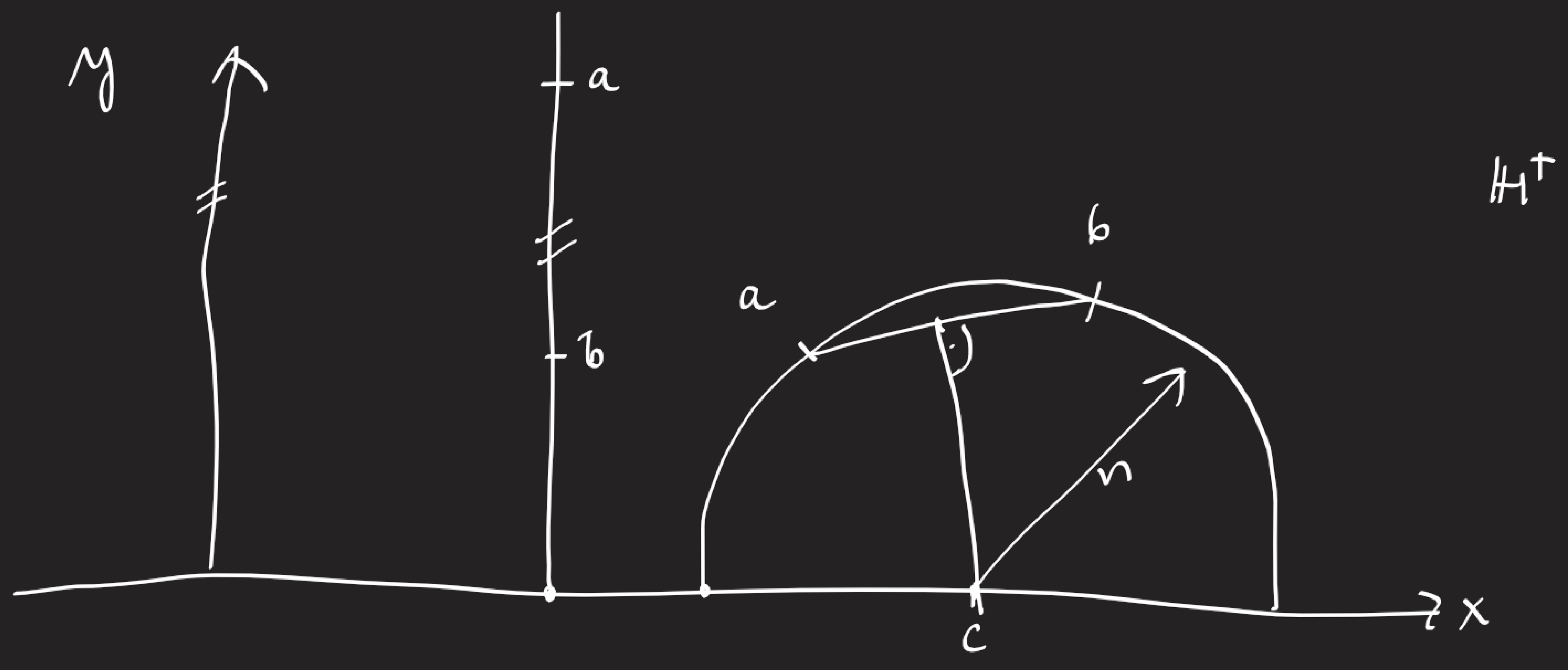
$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -1$$

$$\frac{d}{dx}\left(y\frac{dy}{dx}\right) = -1 \Rightarrow y\frac{dy}{dx} = -x + c \Rightarrow$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + 2xc + d, \quad c, d \in \mathbb{R}$$

$$y^2 + (x - c)^2 = r^2, \quad r > 0$$

We see that the geodesics on $\mathbb{H}^+$ are



Note that given two point $a, b \in \mathbb{H}^+$, then there is a unique geodesic in $\mathbb{H}^+$ that passes through a, b.

<u>Remark</u> Upper-half plane model is an example of non-Euclidean geometry. This is a geometry where first four Euclid axioms hold but the fifth Euclid axiom is violated:

"Given a line p and a point $x \notin p$, then there is a unique line p' such $p \parallel p'$ and $x \in p'$."

In the upper-half plane model lines are geodesics and we say that two lines are parallel if they do not intersect.