Curvature tensor

Theorem A Let M be a smooth manifold with affine connection ∇.
Then the map

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$$R(X,Y)Z \equiv R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is a tensor field of type $(3,1)$ on M.

Proof: We have to show that R is linear over $C^\infty(M)$ in every argument. By definition, it is obvious that R is linear over constants in every argument, it follows at once from linearity of ∇ and $[\,,\,]$ over constants. Thus it remains to verify:

$$R(fX,Y,Z) = R(X,fY,Z) = R(X,Y,fZ) = f\,R(X,Y,Z)$$

for every $f \in C^\infty(M)$, $X,Y,Z \in \mathfrak{X}(M)$. We have

$$\begin{aligned}
R(fX,Y,Z) &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z \\
&= f\nabla_X \nabla_Y Z - \nabla_Y (f\nabla_X Z) - \nabla_{(f[X,Y]-(Yf)X)} Z \\
&= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - \cancel{(Yf)\nabla_X Z} - f\nabla_{[X,Y]} Z \\
&\quad + \cancel{(Yf)\nabla_X Z} = f\,R(X,Y,Z),
\end{aligned}$$

$$\begin{aligned}
R(X,fY,Z) &= \nabla_X \nabla_{fY} Z - \nabla_{fY}\nabla_X Z - \nabla_{[X,fY]} Z \\
&= f\nabla_X\nabla_Y Z + \cancel{(Xf)\nabla_Y Z} - f\nabla_Y\nabla_X Z - \cancel{(Xf)\nabla_Y Z} \\
&\quad - f\nabla_{[X,Y]} Z = f\,R(X,Y,Z) \qquad \text{and}
\end{aligned}$$

$$\begin{aligned}
R(X,Y,fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ) = \\
&= \nabla_X\big((Yf)Z + f\nabla_Y Z\big) - \nabla_Y\big((Xf)Z + f\nabla_X Z\big) \\
&\quad - ([X,Y]f)Z - f\nabla_{[X,Y]} Z = \\
&= \cancel{(X(Yf))Z} + \cancel{(Yf)\nabla_X Z} + \cancel{(Xf)\nabla_Y Z} + f\nabla_X\nabla_Y Z \\
&\quad - \cancel{(Y(Xf))Z} - \cancel{(Xf)\nabla_Y Z} - \cancel{(Yf)\nabla_X Z} - f\nabla_Y\nabla_X Z \\
&\quad - \cancel{([X,Y]f)Z} - f\nabla_{[X,Y]} Z = f\,R(X,Y,Z). \qquad \square
\end{aligned}$$

Definition The tensor field R of the affine connection ∇ defined in Theorem A is called the curvature tensor (or just the curvature for short).

Theorem (Symmetries of the curvature)
Let ∇ be a torsion-free affine connection on a manifold M with curvature tensor R. Then

(CS1) $R(X,Y,Z) = -R(Y,X,Z)$

(CS2) $R(X,Y,Z) + R(Y,Z,X) + R(Z,X,Y) = 0$ (the first Bianchi identity)

for every $X,Y,Z \in \mathfrak{X}(M)$.

Proof: ad (i) Follows at once from definition and $[X,Y] = -[Y,X]$.

ad (ii) Is a consequence of Jacobi identity of vector fields

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

We have

$$R(X,Y,Z) + R(Y,Z,X) + R(Z,X,Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$+ \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y$$
$$= \nabla_X([Y,Z]) + \nabla_Y([Z,X]) + \nabla_Z([X,Y])$$
$$- \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y - \nabla_{[X,Y]} Z$$
$$= [[Y,Z],X] + [[Z,X],Y] + [[X,Y],Z] = 0. \qquad \square$$

Local description of curvature tensor

Let $\varphi : U \to \mathbb{R}^m$ be a chart on M with coordinate function $x_1, \dots, x_m$ and coordinate vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$. Let $\Gamma_{ij}^k(x)$ be the Christoffel symbols of an affine connection ∇ in the chart φ so that

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^m \Gamma_{ij}^k(x) \frac{\partial}{\partial x_k}.$$

The curvature tensor R is of a form

$$R(x) = \sum_{i,j,k,l=1}^m R_{ijk}^l(x)\, dx_i \otimes dx_j \otimes dx_k \otimes \frac{\partial}{\partial x_l}$$

where $R_{ijk}^l(x)$ are smooth functions of $x_1, \dots, x_m$. In order to compute these coefficient functions, consider

$$\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k} - \nabla_{\underbrace{[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}]}_{=0}} \frac{\partial}{\partial x_k}$$

$$= \nabla_{\frac{\partial}{\partial x_i}} \left(\sum_{l=1}^m \Gamma_{jk}^l(x) \frac{\partial}{\partial x_l} \right) - \nabla_{\frac{\partial}{\partial x_j}} \left(\sum_{l=1}^m \Gamma_{ik}^l(x) \frac{\partial}{\partial x_l} \right)$$

$$= \sum_{l=1}^m \left(\left(\frac{\partial}{\partial x_i} \Gamma_{jk}^l \right)(x) - \left(\frac{\partial}{\partial x_j} \Gamma_{ik}^l \right)(x) \right) \frac{\partial}{\partial x_l}$$

$$+ \sum_{l=1}^m \left(\Gamma_{jk}^l(x) \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_l} - \Gamma_{ik}^l(x) \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_l} \right)$$

$$= \sum_{l=1}^m \left(\left(\frac{\partial}{\partial x_i} \Gamma_{jk}^l \right)(x) - \left(\frac{\partial}{\partial x_j} \Gamma_{ik}^l \right)(x) \right) \frac{\partial}{\partial x_l}$$

$$+ \sum_{l,u=1}^m \left(\Gamma_{jk}^l(x)\, \Gamma_{il}^u(x) - \Gamma_{ik}^l(x)\, \Gamma_{jl}^u(x) \right) \frac{\partial}{\partial x_u}$$

$$= \sum_{l=1}^m \left[\left(\frac{\partial}{\partial x_i} \Gamma_{jk}^l \right) - \left(\frac{\partial}{\partial x_j} \Gamma_{ik}^l \right) + \sum_{u=1}^m \left(\Gamma_{jk}^u \Gamma_{iu}^l - \Gamma_{ik}^u \Gamma_{ju}^l \right) \right](x) \frac{\partial}{\partial x_l}$$

This shows that

$$\text{(RTC)} \qquad R_{ijk}{}^{l}(x) = \left(\frac{\partial}{\partial x_i} \Gamma^{l}_{jk}\right)(x) - \left(\frac{\partial}{\partial x_j} \Gamma^{l}_{ik}\right)(x) + \sum_{u=1}^{m} \left(\Gamma^{u}_{jk}(x)\, \Gamma^{l}_{iu}(x) - \Gamma^{u}_{ik}(x)\, \Gamma^{l}_{ju}(x) \right)$$

Example Let ∇ be the flat connection on $\mathbb{R}^m$. Since $\Gamma^{k}_{ij}(x) = 0$, the curvature tensor of ∇ vanishes identically on $\mathbb{R}^m$!

Riemannian curvature tensor

Definition Let (M,g) be a Riemannian manifold, ∇ be the Levi-Civita connection of g and R be its curvature tensor. We define a tensor field $R \in \mathcal{T}^{4,0}M$ by

$$\text{(RC)} \qquad R(\mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{W}) = g(R(\mathbb{X},\mathbb{Y},\mathbb{Z}),\mathbb{W}), \quad \mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{W} \in \mathfrak{X}(M),$$

and call it the Riemannian curvature tensor (or just the Riemannian curvature for short).

Remark The definition of the Riemannian curvature tensor is valid as (RC) is clearly a tensor field of type $(4,0)$ on M. The curvature tensor $R \in \mathcal{T}^{3,1}(M)$ of the Levi-Civita connection can be recovered from the Riemannian curvature (which is of type $(4,0)$) and so we denote these two tensor fields by the same letter. The type of the (curvature) tensor should be always clear from context and so there is no risk of confusion.

Theorem (Symmetries of Riemannian curvature)
Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇. Then the Riemannian curvature $R \in \mathcal{T}^{4,0}(M)$ has the following symmetries:

(RS1) $R(\mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{W}) = -R(\mathbb{Y},\mathbb{X},\mathbb{Z},\mathbb{W})$,
(RS2) $R(\mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{W}) + R(\mathbb{Y},\mathbb{Z},\mathbb{X},\mathbb{W}) + R(\mathbb{Z},\mathbb{X},\mathbb{Y},\mathbb{W}) = 0$,
(RS3) $R(\mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{W}) = -R(\mathbb{X},\mathbb{Y},\mathbb{W},\mathbb{Z})$ and
(RS4) $R(\mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{W}) = R(\mathbb{Z},\mathbb{W},\mathbb{X},\mathbb{Y})$

for every $\mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{W} \in \mathfrak{X}(M)$.

Proof: (RS1) follows at once from (CS1) and (RS2) is the first Bianchi identity (CS2).

ad (RS3):

$$R(\mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{W}) = g(\nabla_{\mathbb{X}}\nabla_{\mathbb{Y}}\mathbb{Z} - \nabla_{\mathbb{Y}}\nabla_{\mathbb{X}}\mathbb{Z} - \nabla_{[\mathbb{X},\mathbb{Y}]}\mathbb{Z},\mathbb{W})$$

$$= g(\nabla_{\mathbb{X}}\nabla_{\mathbb{Y}}\mathbb{Z},\mathbb{W}) - g(\nabla_{\mathbb{Y}}\nabla_{\mathbb{X}}\mathbb{Z},\mathbb{W}) - g(\nabla_{[\mathbb{X},\mathbb{Y}]}\mathbb{Z},\mathbb{W})$$

$$= \mathbb{X}g(\nabla_{\mathbb{Y}}\mathbb{Z},\mathbb{W}) - g(\nabla_{\mathbb{Y}}\mathbb{Z},\nabla_{\mathbb{X}}\mathbb{W}) - \mathbb{Y}g(\nabla_{\mathbb{X}}\mathbb{Z},\mathbb{W}) + g(\nabla_{\mathbb{X}}\mathbb{Z},\nabla_{\mathbb{Y}}\mathbb{W})$$
$$- [\mathbb{X},\mathbb{Y}]\,g(\mathbb{Z},\mathbb{W}) + g(\mathbb{Z},\nabla_{[\mathbb{X},\mathbb{Y}]}\mathbb{W})$$

$$= \mathbb{X}(\mathbb{Y}g(\cancel{\mathbb{Z},\mathbb{W}}) - g(\mathbb{Z},\cancel{\nabla_{\mathbb{Y}}}\mathbb{W})) - \mathbb{Y}g(\mathbb{Z},\cancel{\nabla_{\mathbb{X}}}\mathbb{W}) + g(\mathbb{Z},\nabla_{\mathbb{Y}}\nabla_{\mathbb{X}}\mathbb{W})$$
$$- \mathbb{Y}(\mathbb{X}g(\cancel{\mathbb{Z},\mathbb{W}}) - g(\mathbb{Z},\cancel{\nabla_{\mathbb{X}}\mathbb{W}})) + \mathbb{X}\cancel{g}(\mathbb{Z},\nabla_{\mathbb{Y}}\mathbb{W}) - g(\mathbb{Z},\nabla_{\mathbb{X}}\nabla_{\mathbb{Y}}\mathbb{W})$$
$$- [\mathbb{X},\mathbb{Y}]\,g(\cancel{\mathbb{Z}},\mathbb{W}) + g(\mathbb{Z},\nabla_{[\mathbb{X},\mathbb{Y}]}\mathbb{W}) =$$

$$= g(Z, D_Y D_X V - D_X D_Y V + D_{[X,Y]} V) = g(Z, -R(X,Y,V)) =$$
$$= -g(R(X,Y,V), Z) = -R(X,Y,V,Z).$$

ad (RS4). By (RS 2) we have:

$$R(X,Y,Z,V) + R(Y,Z,X,V) + R(Z,X,Y,V) = 0,$$
$$R(Y,Z,V,X) + R(Z,V,Y,X) + R(V,Y,Z,X) = 0,$$
$$R(Z,V,X,Y) + R(V,X,Z,Y) + R(X,Z,V,Y) = 0,$$
$$R(V,X,Y,Z) + R(X,Y,V,Z) + R(Y,V,X,Z) = 0.$$

We add these identities together:

$$\begin{aligned} 0 = {} & R(X,Y,Z,V) + R(Y,Z,X,V) + R(Z,X,Y,V) + \\ & R(Y,Z,V,X) + R(Z,V,Y,X) + R(V,Y,Z,X) + \\ & R(Z,V,X,Y) + R(V,X,Z,Y) + R(X,Z,V,Y) + \\ & R(V,X,Y,Z) + R(X,Y,V,Z) + R(Y,V,X,Z) \\ = {} & 2\,(R(Z,X,Y,V) + R(V,Y,Z,X)) \Rightarrow \end{aligned}$$
$$R(Z,X,Y,V) = R(Y,V,Z,X).$$

□

Example Riemannian curvature on S^2

Let us consider the spherical coordinates on S^2 given by (as in week 8)

$$\Phi : (-\pi,\pi) \times (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \longrightarrow S^2$$
$$\Phi(\varphi,\theta) = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta).$$

Then we know that

$$\Phi^* g_{S^2} = \cos^2\theta \, d\varphi \otimes d\varphi + d\theta \otimes d\theta, \quad g_{\varphi\varphi} = \cos^2\theta, \; g_{\theta\theta} = 1, \; g_{\varphi\theta} = g_{\theta\varphi}$$

and

$$\Gamma^{\theta}_{\varphi\varphi} = \cos\theta\sin\theta, \quad \Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = -\tan\theta, \quad \Gamma^{\varphi}_{\varphi\varphi} = \Gamma^{\theta}_{\theta\varphi} = \Gamma^{\theta}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\theta} = \Gamma^{\theta}_{\theta\theta} = 0.$$

By symmetries of Riemannian curvature tensor we know that

$$R_{\varphi\theta\varphi\theta} = -R_{\theta\varphi\varphi\theta} = R_{\theta\varphi\theta\varphi} = -R_{\varphi\theta\theta\varphi}$$

is in general non-zero while all other components of R vanish. By (RTC), we know that

$$R_{\varphi\theta\varphi}{}^{\theta} = \frac{\partial}{\partial\varphi}\Gamma^{\theta}_{\theta\varphi} - \frac{\partial}{\partial\theta}\Gamma^{\theta}_{\varphi\varphi} + \Gamma^{\varphi}_{\theta\varphi}\Gamma^{\theta}_{\varphi\varphi} + \Gamma^{\theta}_{\theta\varphi}\Gamma^{\theta}_{\varphi\theta} - \Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\theta}_{\theta\varphi} - \Gamma^{\theta}_{\varphi\varphi}\Gamma^{\theta}_{\varphi\theta}$$
$$= 0 - (-\sin^2\theta + \cos^2\theta) - \tan\theta\cos\theta\sin\theta + 0 \cdot 0 - 0 \cdot 0 - \cos\theta\sin\theta \cdot 0$$
$$= \sin^2\theta - \cos^2\theta - \sin^2\theta = -\cos^2\theta.$$

Thus

$$R_{\varphi\theta\varphi\theta} = g_{\theta\theta} R_{\varphi\theta\varphi}{}^{\theta} + g_{\varphi\theta} R_{\varphi\theta\varphi}{}^{\varphi} = -\cos^2\theta + 0 = -\cos^2\theta = -g_{\varphi\varphi}g_{\theta\theta}$$

Note that we have

$$R_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}$$

for any $i,j,k,l \in \{\varphi,\theta\}$.

Theorem B Let (M,g) be a Riemannian manifold of dimension 2 with Riemannian curvature tensor $R \in \mathcal{T}^{4,0}(M)$. Then there is a unique $K \in C^\infty(M)$ such that any chart

$$R_{ijkl}(x) = K(x)\big(g_{il}(x)\, g_{jk}(x) - g_{ik}(x)\, g_{jl}(x)\big)$$

where $i,j,k,l \in \{1,2\}$ or equivalently :

$$R(\mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{V}) = K\big(g(\mathbb{X},\mathbb{V})\, g(\mathbb{Y},\mathbb{Z}) - g(\mathbb{X},\mathbb{Z})\, g(\mathbb{Y},\mathbb{V})\big)$$

for every $\mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{V} \in \mathfrak{X}(M)$.

Proof: Fix $x \in M$. Since T_xM is a real vector space of dimension two, the set S_x of all tensors

$$T : T_xM \times T_xM \times T_xM \times T_xM \longrightarrow \mathbb{R}$$

of type $(4,0)$ that have the symmetries (RS1)-(RS4) of the Riemannian curvature tensor is a vector subspace of $T^{4,0}_xM$ of dimension 1. The tensor

$$G_x(X,Y,Z,V) = g_x(X,V)\, g_x(Y,Z) - g_x(X,Z)\, g_x(Y,V), \quad X,Y,Z,V \in T_xM,$$

has these symmetries. Since clearly $G_x \neq 0$, it follows that G_x is a generator of S_x. Hence, there is $K(x) \in \mathbb{R}$ such that $R_x = K(x)\, G_x$.

Now let $G \in \mathcal{T}^{4,0}(M)$ be given by

$$G(\mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{V}) = g(\mathbb{X},\mathbb{V})\, g(\mathbb{Y},\mathbb{Z}) - g(\mathbb{X},\mathbb{Z})\, g(\mathbb{Y},\mathbb{V}).$$

As we have shown above G_x is a generator of S_x and so there is a function $K : M \longrightarrow \mathbb{R}$ such that

$$R_x = K(x)\, G_x.$$

Since R and G are smooth tensor fields, it is easy to show that $K \in \mathcal{C}^\infty(M)$. $\square$

Theorem Let (M, g_M) be a surface in $\mathbb{R}^3$. Then the function K from Theorem B is the Gaussian curvature (from the course Differential geometry of curves and surfaces). In particular, the Gaussian curvature does not depend on the embedding $M \hookrightarrow \mathbb{R}^3$ but only on g_M.

Proof is any textbook on Riemannian geometry. $\square$

Theorem C Let (M, g_M) and (N, g_N) be two Riemannian manifolds of the same dimension m with Riemannian curvature tensors R_M and R_N, respectively. Assume that

$$\Phi : M \longrightarrow N$$

is a (local) isometry, that is, $\Phi^* g_N = g_M$. Then

$$\Phi^* R_N = R_M.$$

Proof: Let $x \in M$ and $u,v,z,w \in T_{x_0}M$. We have to show that

$$(R_M)_x(u,v,z,w) = (\Phi^* R_N)_{x_0}(u,v,z,v).$$

As $\Phi^* g_N = g_M$, we have that for $u \neq 0$

$$0 < (g_M)_x(u,u) = (\Phi^* g_N)_x(u,u) = (g_N)_{\Phi(x)}\big(T_x\Phi(u), T_x\Phi(u)\big).$$

This shows that $T_x\Phi$ is injective and since $\dim M = \dim N = m$, we have that $T_x\Phi : T_xM \longrightarrow T_{\Phi(x)}N$ is a linear isomorphism. Hence, Φ is a local diffeomorphism, that is, each point

$x \in M$ has an open neighbourhood $\mathcal{U}$ such that $\Phi|_{\mathcal{U}}$ is a diffeomorphism $\Phi|_{\mathcal{U}}: \mathcal{U} \to \Phi(\mathcal{U})$.

Let us fix $x_0 \in M$ and $\psi: \mathcal{V} \to \mathbb{R}^m$ be a chart on N around $\Phi(x_0)$ with coordinate functions $y_1, \dots, y_m$ and coordinate vector fields $\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m}\}$. Assume that

$$(g_N)(y) = \sum_{i,j=1}^{m} g^N_{ij}(y)\, dy_i \otimes dy_j .$$

Choose a small open neighbourhood $\mathcal{U}$ of x_0 such that $\Phi(\mathcal{U}) \subseteq \mathcal{V}$ and $\Phi|_{\mathcal{U}}$ is a diffeomorphism. Then the compisition

$$\varphi := \psi \circ \Phi : \mathcal{U} \to \mathbb{R}^m$$

is a chart on M which is compatible with the given smooth atlas. Let $x_1, \dots, x_m$ be the associated coordinate functions and $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$ be the coordinate vector fields. Note that

$$\Phi^* dy_i = d((\psi \circ \Phi)_i) = d(\varphi_i) = dx_i \quad , i = 1, \dots, m$$

where $(\dots)_i$ is the i-th component of the corresponding map $\mathcal{U} \to \mathbb{R}^m$. Assume that

$$(g_M)(x) = \sum_{i,j=1}^{m} g^M_{ij}(x)\, dx_i \otimes dx_j .$$

Since

$$(g_M)(x) = (\Phi^* g_N)(x) = \sum_{i,j=1}^{m} g^N_{ij}(\Phi(x))\, (\Phi^* dy_i) \otimes (\Phi^* dy_j)$$

$$= \sum_{i,j=1}^{m} g^N_{ij}(\Phi(x))\, dx_i \otimes dx_j ,$$

we conclude that

$$g^M_{ij}(x) = g^N_{ij}(\Phi(x))$$

for every $i,j = 1, \dots, m$ and $x \in \mathcal{U}$. But then it follows that also

$$(\Gamma_M)^k_{ij}(x) = (\Gamma_N)^k_{ij}(\Phi(x))$$

for every $i,j,k = 1, \dots, m$, $x \in \mathcal{U}$ and where $(\Gamma_M)^k_{ij}$ and $(\Gamma_N)^k_{ij}$ are the Christoffel symbols of g_M and g_N in the charts φ and ψ, respectively. But then we have that

$$(*) \qquad (R_M)^l_{ijk}(x) = (R_N)^l_{ijk}(\Phi(x))$$

for every $i,j,k,l = 1, \dots, m$ and $x \in \mathcal{U}$. This shows that

$$(\Phi^* R_N)_{x_0}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = (R_N)_{\Phi(x_0)}\left(T_{x_0}\Phi\left(\frac{\partial}{\partial x_i}\right), T_{x_0}\Phi\left(\frac{\partial}{\partial x_j}\right), T_{x_0}\Phi\left(\frac{\partial}{\partial x_k}\right)\right) =$$

$$= (R_N)_{\Phi(x_0)}\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}\right) = (R_M)_{x_0}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right).$$

In the last identity we have used $(*)$. Thus $(\Phi^* R_N)_{x_0}$ and $(R_M)_{x_0}$ are two tensors of type $(4,0)$ on $T_{x_0}M$ that agree on the basis $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$, it follows that they agree on $T_{x_0}M$. □

Corollary There is no chart $\varphi: \mathcal{U} \to \mathbb{R}^2$ on S^2 such that $(\varphi^{-1})^* g_{S^2} = g|_{\varphi(\mathcal{U})}$ where g_{S^2} is the round metric on S^2 and

g is the Euclidean metric on $\mathbb{R}^2$.

Proof: We have show in Example Riemannian curvature on S^2 that the Gaussian curvature of g_{S^2} is 1 and so the Riemannian curvature R_{S^2} of g_{S^2} is nowhere vanishing. If $\varphi: \mathcal{U} \to \mathbb{R}^2$ is a chart on S^2, then also
$$(\varphi^{-1})^* R_{S^2} = R_{(\varphi^{-1})^* g_{S^2}} \text{ is nowhere vanishing on } \mathcal{U}.$$

On the other hand, the Riemannian curvature R of g on $\mathbb{R}^2$ is zero everywhere. Hence, there is no map such that $(\varphi^{-1})^* g_{S^2} = g|_{\mathcal{U}}$. $\square$

Example

Consider the cone C in $\mathbb{R}^3$ prescribed by the conditions
$$x^2 + y^2 = z^2, \quad z > 0.$$

Now the map
$$\Phi: (0, +\infty) \times (-\pi, \pi) \to \mathbb{R}^3, \qquad \Phi(r, \varphi) = (r\cos\varphi, r\sin\varphi, r)$$
is a parametrization of $C \setminus \{(-x, 0, x) : x > 0\}$. Let g_C be the standard Riemannian metric on C given by the inclusion $C \subseteq \mathbb{R}^3$. Then
$$\begin{aligned}
\Phi^* g_C &= (\Phi^*) \left(\sum_{i=1}^{3} dx_i \otimes dx_i \right) \\
&= d(r\cos\varphi) \otimes d(r\cos\varphi) + d(r\sin\varphi) \otimes d(r\sin\varphi) \\
&\quad + dr \otimes dr \\
&= (\cos\varphi\, dr - r\sin\varphi\, d\varphi) \otimes (\cos\varphi\, dr - r\sin\varphi\, d\varphi) \\
&\quad + (\sin\varphi\, dr + r\cos\varphi\, d\varphi) \otimes (\sin\varphi\, dr + r\cos\varphi\, d\varphi) \\
&\quad + dr \otimes dr \\
&= (\cos^2\varphi + \sin^2\varphi)\, dr \otimes dr + r^2(\sin^2\varphi + \cos^2\varphi)\, d\varphi \otimes d\varphi \\
&= dr \otimes dr + r^2\, d\varphi \otimes d\varphi.
\end{aligned}$$

Note that $\Phi^* g_C$ looks as the Euclidean metric g on $\mathbb{R}^2$ in polar coordinates. Since the Riemannian curvature tensor of g vanishes, it follows that the Riemannian curvature of $\Phi^* g_C$ also vanishes. (But try to verify this by direct calculation.) Thus it should be be feasible that there is a parametrization $\Psi: \mathcal{V} \to \mathbb{R}^3$ on C such that $\Psi^* g_C = g|_{\mathcal{V}}$. Then it would be easy to find geodesics on C.
(How is this related to the bonus question in HW 7?)

Geometric interpretation of the curvature of affine connection

Let D be an affine connection on manifold M with curvature tensor $R \in \mathcal{T}^{3,1}(M)$. Fix $x \in M$ and $u, v, w \in T_x M$. Now we are interested in computing $R_x(u, v, w) \in T_x M$ in terms of D. Let $\varphi: \mathcal{U} \to \mathbb{R}^m$ be a chart around x and assume that $\varphi(x) = 0$. If u, v are linearly dependant, then $R_x(u, v, w) = 0$ and so we may assume that u, v are linearly independent. Without loss of generality we may assume that $T_x\varphi(u) = e_1$ and $T_x\varphi(v) = e_2$ where $\{e_1, \ldots, e_m\}$ is the standard basis of $T_0\mathbb{R}^m = \mathbb{R}^m$. In other words

$u = \frac{\partial}{\partial x_1}$ and $v = \frac{\partial}{\partial x_2}$. Consider the following parallelogram Γ_t in U which is the concatenation of the following curves

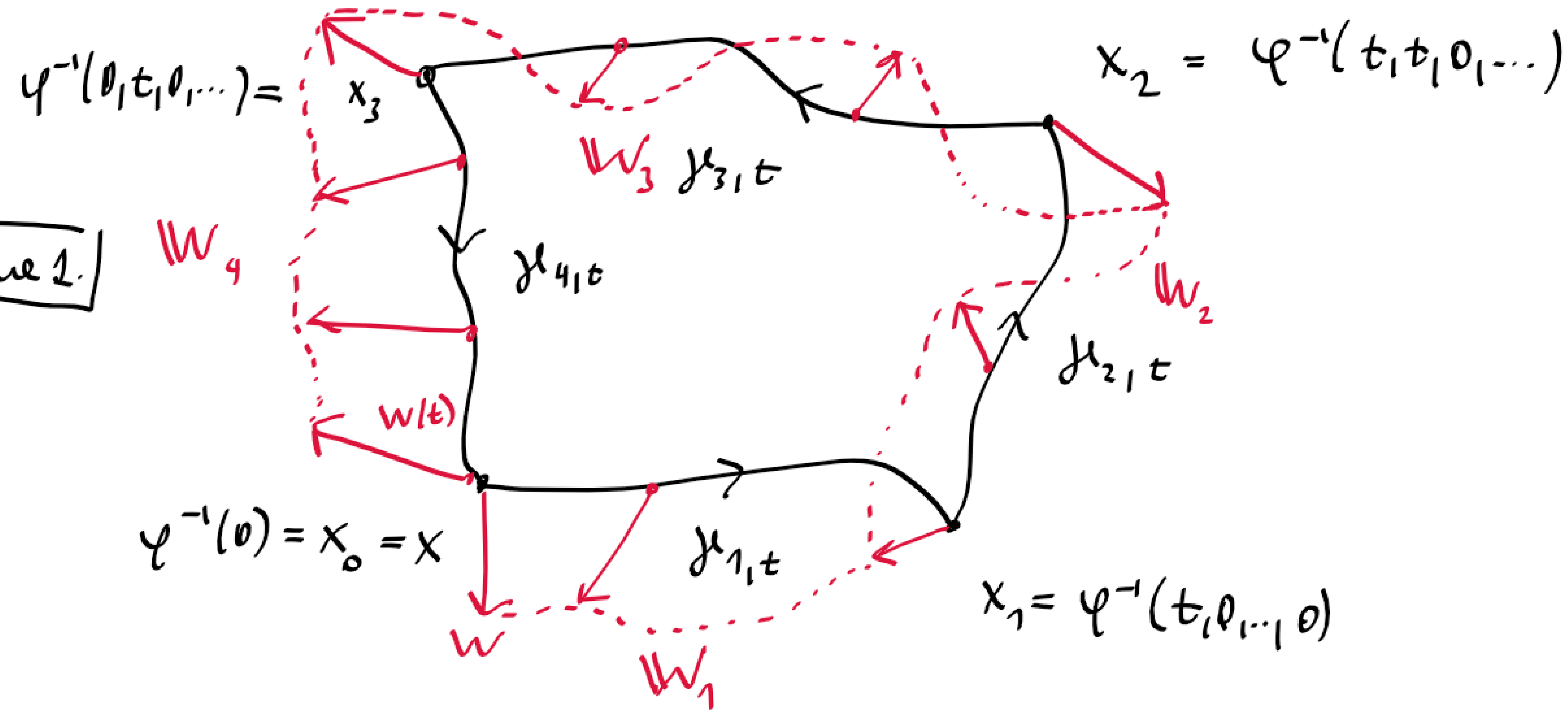
- $\gamma_{1,t}(s) = \varphi^{-1}(s, 0, \dots, 0)$
- $\gamma_{2,t}(s) = \varphi^{-1}(t, s, 0, \dots, 0)$
- $\gamma_{3,t}(s) = \varphi^{-1}(t, t-s, 0, \dots, 0)$
- $\gamma_{4,t}(s) = \varphi^{-1}(t-s, 0, \dots, 0)$

where $s \in [0, t]$ and $t > 0$ is small enough. Moreover let $\mathbb{W}_{i,t}$ be the parallel vector field along $\gamma_{i,t}$ determined by:

- $\mathbb{W}_{1,t}(0) = w$
- $\mathbb{W}_{i+1,t}(0) = \mathbb{W}_{i,t}(t)$, $i = 1, 2, 3$.

Finally put $w(t) = \mathbb{W}_{4,t}(t) \in T_x M$. (See Picture 1.)

Picture 1.



Now it can shown that w is a smooth curve in the tangent space $T_x M$ and

$$w'(0) = 0, \quad w''(0) = 2R_x(u, v, w).$$