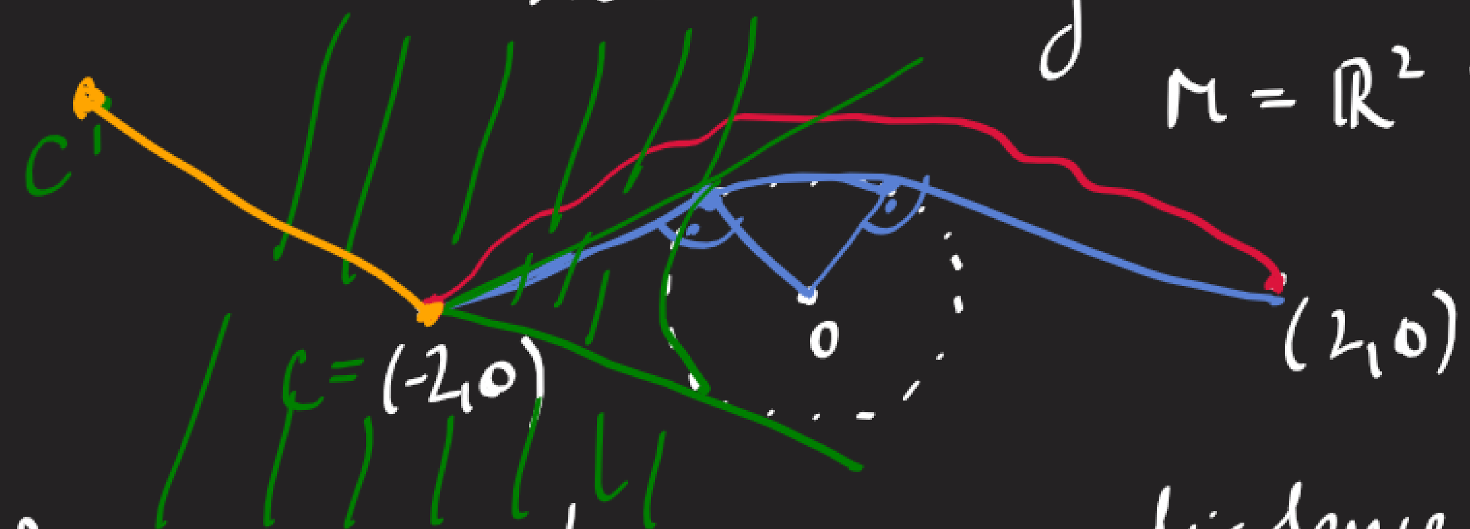
Short summary from last week

(M, g_M) .. a Riemannian manifold

assume that γ_1 is a distance minimizing curve between fixed points C, D $\Longrightarrow$ (*)

(•) in general there need be any distance minimizing curve between C, D

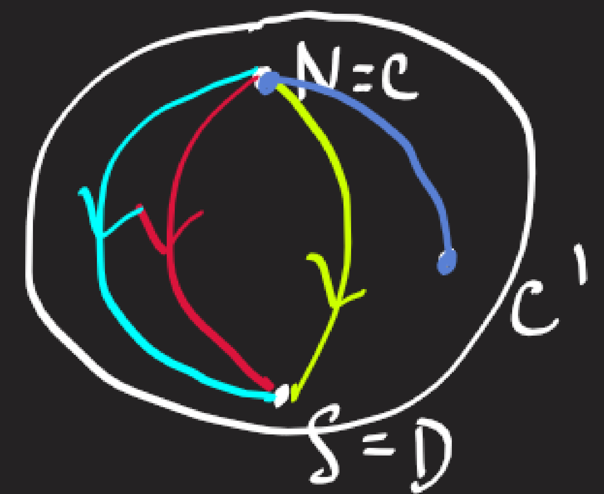
$M = \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2, x^2+y^2 \leq 1\}$



•) there may be many distance minimizing curves between C, D

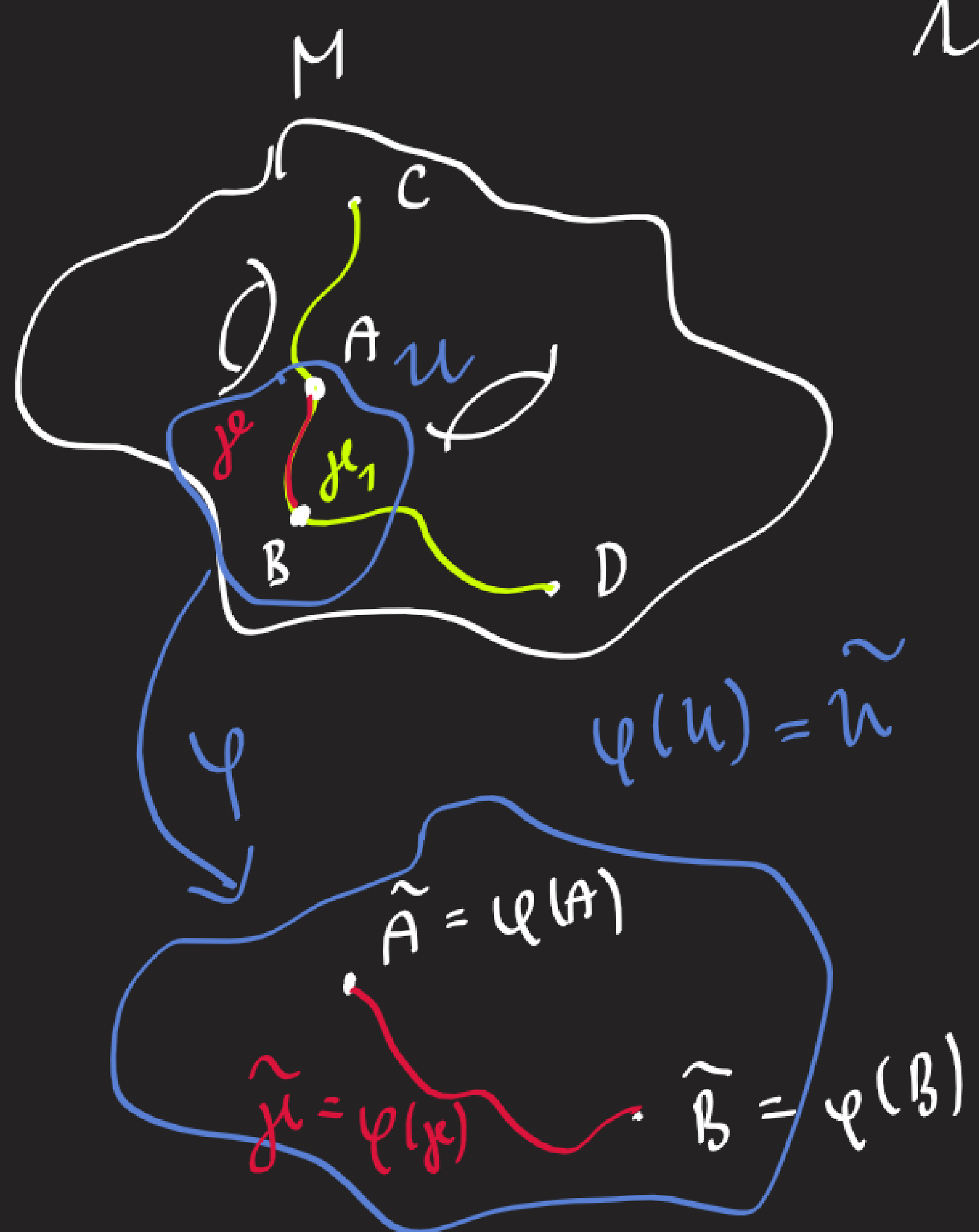
$M = S^2 \subseteq \mathbb{R}^3$ (S^2, g_{S^2})

$d(N, S) = \pi$



•) if D is close to C, then there is a unique distance minimizing curve between C and D, more precisely, there is an open neighbourhood $\mathcal{U}$ of C such that given any point $C' \in \mathcal{U}$ there is a unique distance minimizing curve betwee C and C')

(*) $\Rightarrow$ γ_1 is a distance minimizing curve between any pair of points, say A and B, that lie on γ_1, let γ be the part γ_1 between A, B, assume that γ is contained in the domain of a chart $\varphi: \mathcal{U} \to \mathbb{R}^m$ on M



$\Rightarrow$ put $\tilde{\gamma} := \varphi \circ \gamma$, $\tilde{A} = \varphi(A)$, $\tilde{B} = \varphi(B)$, $\tilde{\mathcal{U}} = \varphi(\mathcal{U})$ and $\tilde{g} = (\varphi^{-1}) g_M$ be the pullbach of g_M to $\tilde{\mathcal{U}}$, then $\tilde{\gamma}$ is distance w.r. to $\tilde{g}$ between $\tilde{A}$ and $\tilde{B}$

$\Rightarrow$ $\tilde{\gamma}$ is local minimizer of the length functional

$$L(\tilde{\gamma}) = \int_a^b \| \tilde{\gamma}'(t) \|_{\tilde{\gamma}(t)} dt$$

among all C^1 curves that start at A and end at B

we noticed that if we reparametrize $\tilde{\gamma}$ so that $\|\gamma'(t)\|_{\tilde{\gamma}(t)}$ is constant $\Rightarrow$

$\tilde{\gamma}$ is a local minimizer of the energy functional

$$E(\tilde{\gamma}) = \int_a^b \|\tilde{\gamma}'(t)\|^2_{\tilde{\gamma}(t)}\, dt$$

among all $\mathcal{C}^1$ curves that start at $\tilde{A}$ and end at $\tilde{B}$

$\Rightarrow$ $\tilde{\gamma}$ solves the Euler-Lagrange equations for the functional E, these are precisely the geodesic equations for g_M

Motivation: the geodesic equation is a system of m second order non-linear ODE's, such system is in general extremely difficult to solve, difficulty of this system it heavily depends on the choice of coordinates,

in case that we can find a coordinates in which the Riemannian metric g_M on M looks as the Euclidean metric

$$g = \sum_{i=1}^{m} dx_i \otimes dx_i \quad \text{on } \mathbb{R}^m,$$

then the geodesic equations are very simple:

$$\gamma_i''(t) = 0, \quad i = 1, \dots, m.$$

Then the solutions are just line segments with parametrization with constant velocity. The goal of today lecture is to show that this is not always possible as the existence of the chart map with the prescribed properties as above is obstructed by nonvanishing of the Riemannian curvature.

Curvature of affine connection

Theorem Let ∇ be an affine connection on a manifold M. Then the map

$$R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$$R(X, Y) Z \equiv R(X, Y, Z) :=$$

$$\nabla_X \nabla_Y Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z,$$

where $X, Y, Z \in \mathfrak{X}(M)$, is a tensor field of type $(3,1)$ on M.

Proof: We have to verify that R is linear in each argument over $C^\infty(M)$. By definition, it is clear that R is linear over constants, that is, for example we have

$$R(c_1 X_1 + c_2 X_2, Y, Z) = c_1 R(X_1, Y, Z) + c_2 R(X_2, Y, Z)$$

where $X_1, X_2, Y, Z \in \mathfrak{X}(M)$ and $c_1, c_2 \in \mathbb{R}$.

Hence, it remains to verify that

$$R(fX, Y, Z) = R(X, fY, Z) = R(X, Y, fZ) = f R(X, Y, Z)$$

where $f \in C^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$.

•) $R(fX, Y, Z) = D_{fX} D_Y Z - D_Y D_{fX} Z - D_{[fX, Y]} Z$

$$= f(D_X D_Y Z) - D_Y(f D_X Z) - D_{(f[X,Y] - (Yf)X)} Z$$

$$= f(D_X D_Y Z) - (Yf) D_X Z - f D_Y D_X Z$$
$$- D_{f[X,Y]} Z + D_{(Yf)X} Z$$

$$= f(D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z) - (Yf) D_X Z + (Yf) D_X Z$$

$$= f R(X, Y, Z).$$

•) $R(X, fY, Z) = D_X D_{fY} Z - D_{fY} D_X Z - D_{[X, fY]} Z$

$$= D_X(f D_Y Z) - f D_Y D_X Z - D_{f[X,Y] + (Xf)Y} Z$$

$$= \cancel{(Xf)(D_Y Z)} + f D_X D_Y Z - f D_Y D_X Z$$
$$- f D_{[X,Y]} Z + \cancel{(Xf) D_Y Z} = f R(X, Y, Z).$$

•) $R(X, Y, fZ) = D_X D_Y(fZ) - D_Y D_X(fZ) - D_{[X,Y]} fZ$

$$= D_X(Yf \cdot Z + f D_Y Z) - D_Y(Xf \cdot Z + f D_X Z) - ([X,Y]f) Z$$
$$- f D_{[X,Y]} Z = \cancel{(X(Yf))} \cdot Z + \cancel{(Yf) D_X Z} + \cancel{(Xf) D_Y Z} + f D_X D_Y Z$$
$$- \cancel{(Y(Xf))} \cdot Z - \cancel{(Xf) \cdot D_Y Z} - \cancel{(Yf) D_X Z} - f D_Y D_X Z$$
$$- \cancel{([X,Y]f)} Z - f D_{[X,Y]} Z = f R(X, Y, Z). \quad \square$$

Definition The tensor field $R \in \mathcal{T}^{3,1}(M)$ is called the curvature of the affine connection ∇.

Theorem (Symmetries of R)
Let ∇ be a torsion-free affine connection on M with curvature R. Then

① $R(X,Y,Z) = -R(Y,X,Z)$ and

② $R(X,Y,Z) + R(Y,Z,X) + R(Z,X,Y) = 0$

(the first Bianchi identity) where $X,Y,Z \in \mathcal{X}(M)$.

Proof: ad ① follows at once from the definition of R.

ad ② follows from the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Let us compute

$$R(X,Y,Z) + R(Y,Z,X) + R(Z,X,Y) =$$

$$= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$+ \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X$$
$$+ \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y$$

$\nabla_X Y - \nabla_Y X = [X,Y]$

$$= \nabla_X([Y,Z]) + \nabla_Y([Z,X]) + \nabla_Z([X,Y])$$
$$- \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y - \nabla_{[X,Y]} Z$$

$$= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \qquad \square$$

Local formula for curvature

Let $\varphi: U \to \mathbb{R}^m$ be a chart on M with coordinate functions $x_1, \dots, x_m$ and coordinate vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$. Assume that
$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^m \Gamma_{ij}^k(x) \frac{\partial}{\partial x_k}.$$
Then we may also write
$$R(x) = \sum_{i,j,k,l=1}^m R_{ijk}^l(x)\, dx_i \otimes dx_j \otimes dx_k \otimes \frac{\partial}{\partial x_l}$$

where $R_{ijk}^{\ell}(x)$ are again smooth functions of $x_1, \dots, x_m$.

By definition,

$$R_x\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = \sum_{\ell=1}^{m} R_{ijk}^{\ell}(x) \frac{\partial}{\partial x_\ell} \quad \text{for } i,j,k \text{ fixed.}$$

Let us first recall $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ for every $i,j = 1, \dots, m$. Thus,

$$R_x\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k} =$$

$$= \nabla_{\frac{\partial}{\partial x_i}} \left(\sum_{\ell=1}^{m} \Gamma_{jk}^{\ell}(x) \frac{\partial}{\partial x_\ell} \right) - \nabla_{\frac{\partial}{\partial x_j}} \left(\sum_{\ell=1}^{m} \Gamma_{ik}^{\ell}(x) \frac{\partial}{\partial x_\ell} \right)$$

$$= \sum_{\ell=1}^{m} \left(\frac{\partial}{\partial x_i} \Gamma_{jk}^{\ell} \right)(x) \frac{\partial}{\partial x_\ell} + \sum_{\ell=1}^{m} \Gamma_{jk}^{\ell}(x) \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_\ell}$$

$$- \sum_{\ell=1}^{m} \left(\frac{\partial}{\partial x_j} \Gamma_{ik}^{\ell} \right)(x) \frac{\partial}{\partial x_\ell} - \sum_{\ell=1}^{m} \Gamma_{ik}^{\ell}(x) \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_\ell}$$

$$= \sum_{\ell=1}^{m} \left[\left(\frac{\partial}{\partial x_i} \Gamma_{jk}^{\ell} \right)(x) - \left(\frac{\partial}{\partial x_j} \Gamma_{ik}^{\ell} \right)(x) \right] \frac{\partial}{\partial x_\ell}$$

$$+ \sum_{u,\ell=1}^{m} \left[\Gamma_{jk}^{\ell}(x) \Gamma_{i\ell}^{u}(x) - \Gamma_{ik}^{\ell}(x) \Gamma_{j\ell}^{u}(x) \right] \frac{\partial}{\partial x_u}$$

$$= \sum_{u,\ell=1}^{m} \Big[\left(\frac{\partial}{\partial x_i} \Gamma_{jk}^{\ell} \right)(x) - \left(\frac{\partial}{\partial x_j} \Gamma_{ik}^{\ell} \right)(x)$$

$$+ \Gamma_{jk}^{u}(x) \Gamma_{iu}^{\ell}(x) - \Gamma_{ik}^{u}(x) \Gamma_{ju}^{\ell}(x) \Big] \frac{\partial}{\partial x_\ell}$$

<u>Example</u> For the flat connection ∇ on $\mathbb{R}^m$ we know that

$$0 = \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{m} \Gamma_{ij}^{k}(x) \frac{\partial}{\partial x_k} \Rightarrow \Gamma_{ij}^{k}(x) = 0, \ i,j,k = 0$$

$\Rightarrow$ the curvature R of ∇ is vanishing everywhere.

Riemannian curvature

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and let $R \in \mathcal{T}^{3,1}(M)$ be the curvature of ∇. We are going to define a new tensor field $R \in \mathcal{T}^{4,0}(M)$

$$R(X, Y, Z, W) = g(R(X, Y, Z), W)$$

where $X, Y, Z, W \in \mathfrak{X}(M)$. Note that $R \in \mathcal{T}^{3,1}(M)$ can be recovered from $R \in \mathcal{T}^{4,0}(M)$ and so these two tensor fields can be thought of as equivalent objects. It should be always clear from context what is the type of R and so there is no risk of confusion.

Definition We call the tensor field $R \in \mathcal{T}^{4,0}(M)$ the Riemannian curvature of g.

Theorem (Symmetries of Riemannian curvature)
Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and Riemannian curvature $R \in \mathcal{T}^{4,0}(M)$. Then

(RS1) $R(X, Y, Z, W) = -R(Y, X, Z, W)$,

(RS2) $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W)$ (the first Bianchi identity)

(RS3) $R(X, Y, Z, W) = -R(X, Y, W, Z)$ and

(RS4) $R(X, Y, Z, W) = R(Z, W, X, Y)$

where $X, Y, Z, W \in \mathfrak{X}(M)$.

Proof: ad (RS1) is the well known symmetry of curvature.
ad (RS2) is the first Bianchi identity, here we need that the Levi-Civita connection is torsion-free.

ad (RS3) $R(X, Y, Z, W) = g(R(X, Y, Z), W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W) = g(\nabla_X \nabla_Y Z, W) - g(\nabla_Y \nabla_X Z, W) - g(\nabla_{[X,Y]} Z, W)$
$= X\, g(\nabla_Y Z, W) - g(\nabla_Y Z, \nabla_X W) - Y g(\nabla_X Z, W) + g(\nabla_X Z, \nabla_Y W) - [X, Y] g(Z, W) + g(Z, \nabla_{[X,Y]} W) = g(Z, \nabla_{[X,Y]} W) - \cancel{Y g(Z, \nabla_X W)}$
$+ g(Z, \nabla_Y \nabla_X W) - Y[\cancel{X g(Z, W)} - \cancel{g(Z, \nabla_X W)}] + \cancel{X g(Z, \nabla_Y W)} -$
$X[\cancel{Y g(Z, W)} - \cancel{g(Z, \nabla_Y W)}] - g(Z, \nabla_X \nabla_Y W) - \cancel{[X, Y] g(Z, W)} =$

$$= g(Z, -R(X,Y)W) = -g(R(X,Y)W, Z) = -R(X,Y,W,Z).$$

ad (RS4) We have

$$\begin{aligned} 0 = {} & \cancel{R(X,Y,Z,W)} + \cancel{R(Y,Z,X,W)} + \underline{R(Z,X,Y,W)} + \\ & \cancel{R(Y,Z,W,X)} + \cancel{R(Z,W,Y,X)} + \underline{R(W,Y,Z,X)} + \\ & \cancel{R(Z,W,X,Y)} + \cancel{R(W,X,Z,Y)} + \underline{R(X,Z,W,Y)} + \\ & \cancel{R(W,X,Y,Z)} + \cancel{R(X,Y,W,Z)} + \underline{R(Y,W,X,Z)}. \end{aligned}$$

$$2\big(R(Z,X,Y,W) + R(W,Y,Z,X)\big) = 0$$

$$R(Z,X,Y,W) = R(Y,W,Z,X). \qquad \square$$

We see that eventhough the Riemannian curvature $R \in \mathcal{T}^{4,0}(M)$ has m^4 components

$$\Big(R(x) = \sum_{i,j,k,l=1}^{m} R_{ijkl}(x)\, dx_i \otimes dx_j \otimes dx_k \otimes dx_l \Big),$$

only some of them are really indipendendent. For example,

$$R_{1212}(x) = -R_{2112}(x) = -R_{1221}(x) = R_{2121}(x).$$

<u>Example</u> Riemannian curvature on S^2

$$\Phi : (-\pi,\pi) \times (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \longrightarrow \mathbb{R}^3$$

$$\Phi(\varphi,\theta) = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta)$$

is a parametrization of $S^2 \setminus \{\text{meridian } 180°\}$

$$\Phi^* g_{S^2} = \cos^2\theta\, d\varphi \otimes d\varphi + d\theta \otimes d\theta, \qquad \begin{pmatrix} \cos^2\theta & 0 \\ 0 & 1 \end{pmatrix}$$

$$g_{\varphi\varphi} = \cos^2\theta, \quad g_{\varphi\theta} = g_{\theta\varphi} = 0, \quad g_{\theta\theta} = 1$$

$$\Gamma^{\varphi}_{\varphi\varphi} = \Gamma^{\theta}_{\theta\varphi} = \Gamma^{\theta}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\theta} = \Gamma^{\theta}_{\theta\theta} = 0$$

$$\Gamma^{\theta}_{\varphi\varphi} = \cos\theta\sin\theta, \qquad \Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = -\tan\theta.$$

From the symmetries of the Riemannian curvature R of g_{S^2} we know that

$R_{\varphi\theta\varphi\theta} = -R_{\varphi\theta\varphi\theta} = -R_{\varphi\theta\theta\varphi} = R_{\theta\varphi\theta\varphi}$ and all other components of R are vanishing.

$$R^{\theta}_{\varphi\theta\varphi} = \frac{\partial}{\partial\varphi}\Gamma^{\theta}_{\theta\varphi} - \frac{\partial}{\partial\theta}\Gamma^{\theta}_{\varphi\varphi} + \sum_{u=1}^{2}\left(\Gamma^{u}_{\theta\varphi}\Gamma^{\theta}_{\varphi u} - \Gamma^{u}_{\varphi\varphi}\Gamma^{\theta}_{\theta u}\right) =$$

$$= 0 - \frac{\partial}{\partial\theta}(\cos\theta\sin\theta) + \Gamma^{\varphi}_{\theta\varphi}\Gamma^{\theta}_{\varphi\varphi} - \Gamma^{\theta}_{\varphi\varphi}\underbrace{\Gamma^{\theta}_{\theta\varphi}}_{=0}$$

$$= 0 - (-\sin^2\theta + \cos^2\theta) - \tan\theta\cos\theta\sin\theta - 0$$

$$= \sin^2\theta - \cos^2\theta - \sin^2\theta = \underline{\underline{-\cos^2\theta}}$$

$$R_{\varphi\theta\varphi\theta} = g_{\theta\theta}R^{\theta}_{\varphi\theta\varphi} + \underbrace{g_{\varphi\theta}}_{=0}R^{\varphi}_{\varphi\theta\varphi} = -\cos^2\theta$$

$$R_{\varphi\theta\theta\varphi} = \cos^2\theta = g_{\varphi\varphi}\cdot g_{\theta\theta} - g_{\varphi\theta}\,g_{\theta\varphi}$$

Note that

$$(*) \quad \boxed{R_{abcd} = g_{ad}\,g_{bc} - g_{ac}\,g_{bd}}$$

where $a,b,c,d \in \{\varphi,\theta\}$. Observe that the tensor field on S^2 on the right hand side in $(*)$ is a tensor field of type $(4,0)$ on S^2 that has the symmetries (RS1 – RS4) of the Riemannian curvature. We also noticed that there is at most one independent component (or coefficient function) of Riemannian curvature tensor in dimension $m=2$. It follows in dimension $m=2$, there is a function $K \in \mathcal{C}^\infty(S^2)$ such that

$$R_{ijkl}(x) = K(x)\left(g_{il}(x)\,g_{jk}(x) - g_{ik}(x)\,g_{jl}(x)\right).$$

<u>Theorem</u> Let M be a 2-dimensional surface in $\mathbb{R}^3$ and g_M be the canonical Riemannian metric on M given by the inclusion $\iota: M \hookrightarrow \mathbb{R}^3$ (so that $g_M = \iota^* g$ where g is the Euclidean metric on $\mathbb{R}^3$). Then there is a unique function $K \in \mathcal{C}^\infty(M)$ such that the Riemannian curvature $R \in \mathcal{T}^{4,0}(M)$ of g_M is equal to

$$R_{ijkl}(x) = K(x)\left(g_{il}(x)\,g_{jk}(x) - g_{ik}(x)\,g_{jl}(x)\right)$$

or equivalently

$$R_x(X,Y,Z,W) = K(x)\left(g(X,W)\,g(Y,Z) - g(X,Z)\,g(Y,Z)\right)$$

and K is the Gauss curvature.

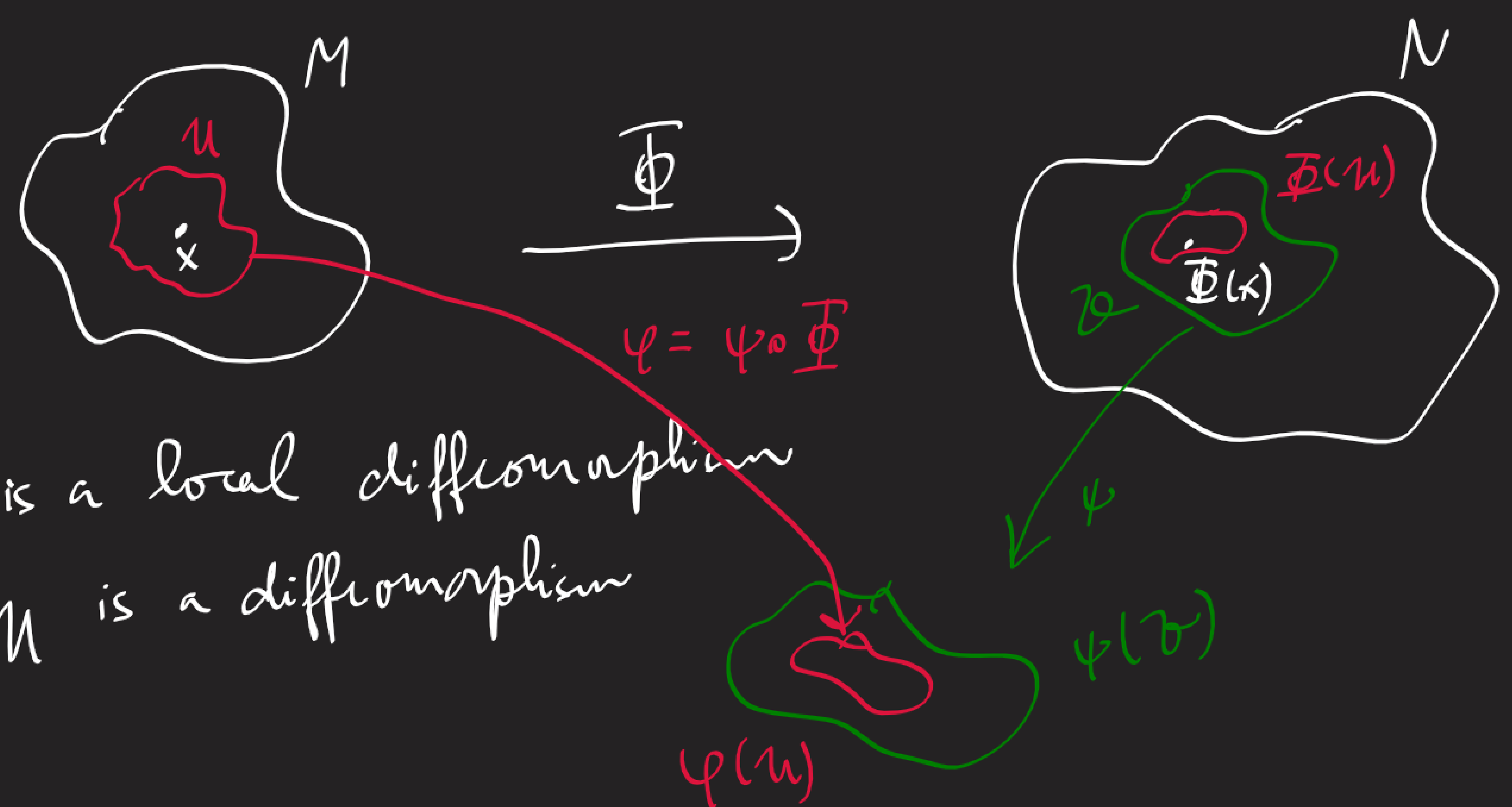
Corollary The Gauss curvature of M depends only on g_M (and not on the embedding $M \hookrightarrow \mathbb{R}^3$).

Theorem Let (M, g_M) and (N, g_N) be Riemannian manifolds of the same dimension m. Assume that

$$\Phi : M \to N$$

is a local isometry so that $\Phi^* g_N = g_M$. Then $\Phi^* R_N = R_M$ where R_M and R_N are the Riemannian curvatures of g_M and g_N, respectively.

Sketch of proof: It uses the fact that R_M depends on the first partial derivatives of the Christoffel symbols of g_M and the Christoffel symbols itself. We know that the Christoffel symbols of g_M depend on the first partial derivatives of the coefficients of g_M. All in all, the components of R_M depend locally of the first and the second partial derivatives of components of g_M. If we assume that $\Phi^* g_N = g_M$, then in some suitable charts on N and M the coefficients functions of g_M and g_N agree.



Φ is a local diffeomorphism

$\Phi|_{\mathcal{U}}$ is a diffeomorphism

Then it is straightforward to verify that the coefficients of R_M in the chart φ coincide with the coefficients of $\Phi^* R_N$ in the same chart φ. □

Corollary There is no chart on S^2 in which g_{S^2} looks as the Euclidean metric on $\mathbb{R}^2$, more precisely, there is no chart $\varphi : \mathcal{U} \to \mathbb{R}^2$ on S^2 such that $(\varphi^{-1})^* g_{S^2} = g|_{\varphi(\mathcal{U})}$.

Proof: We know that the Riemannian curvature R of g is zero. On the other hand, we know that the Riemannian curvature R_{S^2} of g_{S^2} satisfies

$$(R_{S^2})_{ijkl} = (g_{S^2})_{il}(g_{S^2})_{jk} - (g_{S^2})_{ik}(g_{S^2})_{jl}.$$

If we assume that $(\varphi^{-1})^* g_{S^2} = g$, then by the previous Theorem:

$$\begin{aligned}(\varphi^{-1})^*(R_{S^2})_{ijkl} &= ((\varphi^{-1})^* g_{S^2})_{il}((\varphi^{-1})^* g_{S^2})_{jk} - \\ &\quad ((\varphi^{-1})^* g_{S^2})_{ik}((\varphi^{-1})^* g_{S^2})_{jl} = \\ &= \underbrace{g_{il}g_{jk} - g_{ik}g_{jl}}_{\text{non-zero}} \neq \underset{\substack{\| \\ 0}}{R}. \quad \lightning \quad \square\end{aligned}$$