Exponential map

Let (M, g) be a Riemannian manifold of dimension m. Fix $p \in M$. The pair (T_pM, g_p) is a real vector space of dimension m together with an inner product. Let $B_p(r)$ be the open ball inside T_pM of radius $r > 0$ centered at $0 \in T_pM$.

If $\varphi : \mathcal{U} \to M$ is a chart on M with $\varphi(x) = 0$, coordinate functions $x_1, \dots, x_m$ and coordinate vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$, then we know that the geodesic equations are

$$\tilde{\gamma}''_k(t) + \sum_{i,j=1}^{k} \Gamma^k_{ij}(\tilde{\gamma}(t))\, \tilde{\gamma}'_i(t)\, \tilde{\gamma}'_j(t) = 0, \quad k = 1, \dots, m$$

where $\tilde{\gamma} = \varphi \circ \gamma$ and $\Gamma^k_{ij}(x)$ are the Christoffel symbols of the Levi-Civita connection in the chart φ. This is a system of k non-linear second order ODE's.

Lemma A Let $v \in T_pM$ and $p \in M$ be as above. Then there is $\varepsilon > 0$ and a unique geodesic $\gamma_v : (-\varepsilon, \varepsilon) \to M$ for the Levi-Civita connection of g that satisfies the initial conditions

$$\gamma_v(0) = p, \quad \gamma_v'(0) = v.$$

(Moreover, there is a unique solution defined on a maximal interval containing 0.)

Proof follows from the existence and uniqueness theorem on ODE's. $\square$

With notation of Lemma A, note that $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$ where $v \in T_pM$, $\lambda \in \mathbb{R}$ and t is small enough so that $\gamma_v(\lambda t)$ is defined.

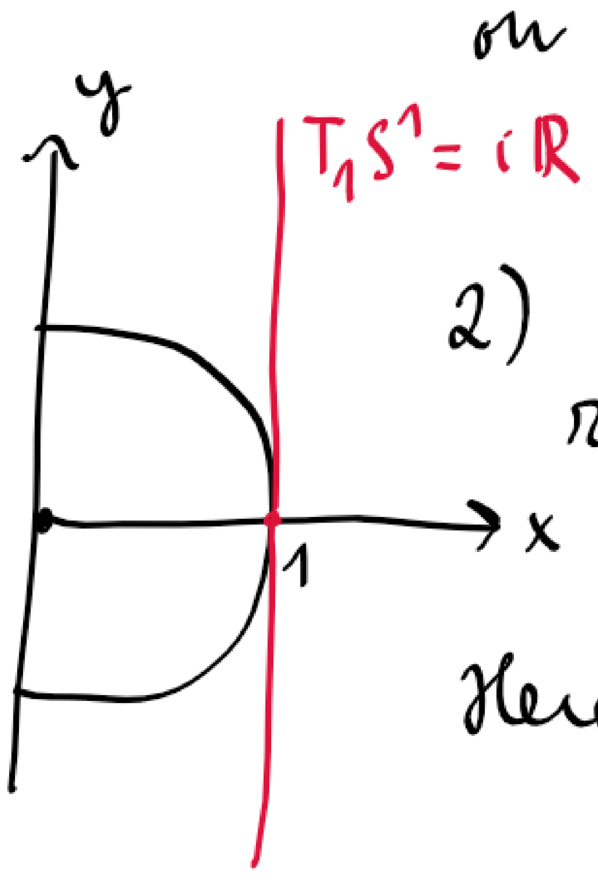
Definition With the notation set above, let $D_p \subset T_pM$ be the domain of the map

$$T_pM \ni v \longmapsto \exp(v) = \gamma_v(1).$$

The map $\exp_p : D_p \to M$ is called the exponential map at p.

Examples: 1) Let $M = \mathbb{R}^m$ and g be the Euclidean metric on $\mathbb{R}^m$. If we view $T_p\mathbb{R}^m$ as $\mathbb{R}^m$ in the usual way, then $\exp_p$ is defined on the whole of $\mathbb{R}^m$ and

$$\exp_p : \mathbb{R}^m \to \mathbb{R}^m, \quad \exp_p(v) = p + v.$$



2) Let $M = S^1$ be the unit sphere in $\mathbb{R}^2 = \mathbb{C}$ with its canonical round metric g_{S^1}. Then the exponential map at $p = 1$ is

$$\exp : T_1S^1 \to S^1, \quad \exp_1(iy) = e^{iy} = \cos y + i \sin y.$$

Here we view T_1S^1 as the orthogonal compement $1^\perp = i\mathbb{R} = \{iy : y \in \mathbb{R}\}$ to 1.

3) Let S^3 be the unit sphere in the vector space of quaternions $\mathbb{H} = \{x_0 + ix_1 + jx_2 + kx_3 : x_i \in \mathbb{R},\ i = 0,1,2,3\}$. On S^3 we consider the standard round metric g_{S^3} given by the inclusion $S^3 \subseteq \mathbb{H} = \mathbb{R}^4$. The tangent space T_1S^3 ~~to~~ $1 \in S^3$ can be identified with the orthogonal complement to 1 which is

$$1^\perp = \operatorname{Im} \mathbb{H} = \{x_1 i + x_2 j + x_3 k : x_1, x_2, x_3 \in \mathbb{R}\}.$$

(Here we work with the standard inner product $g_{\mathbb{H}}$ on $\mathbb{H}$
$g_{\mathbb{H}}(x_0+ix_1+jx_2+kx_3, y_0+iy_1+jy_3+ky_3) = \sum_{i=0}^{3} x_i y_i$.)

If $q \in T_1 S^3 = \operatorname{Im}\mathbb{H}$, $g_{\mathbb{H}}(q,q) = 1$ and $t>0$, then
$$\exp_1 T_1 S^3 \to S^3, \quad \exp_1(q) = e^{tq} = \cos t + q \sin t.$$

4) Let $G = GL(n,\mathbb{R})$ and g_G be the left invariant metric on G such that
$$(g_G)_e(A,B) = \operatorname{Tr}(A^T B), \quad A,B \in M_{n\times n}(\mathbb{R}),$$
where $e\in G$ is the identity element, we view the tangent space at e as $M_{n\times n}(\mathbb{R})$ and Tr denotes the trace of $n\times n$ matrix.
Then
$$\exp_e : T_e G = M_{n\times n}(\mathbb{R}) \to G, \quad \exp_e(A) = \exp(A).$$
Here $\exp(A) = 1 + A + \frac{A^2}{2!} + \dots = \sum_{k=0}^{+\infty} \frac{A^k}{k!}$, $A \in M_{n\times n}(\mathbb{R})$.

Normal coordinates

Theorem There exists $r>0$ such that the map
(EB) $$B_p(r) \to M, \quad \exp_p(v) = \gamma_v(1)$$
is well defined and a diffeomorphism onto its image (which is an open subset of M).

Proof: See Section 4.6 in [10]. $\square$

Let $U_{p,r}$ be the image of (EB) and $B = \{e_1,\dots,e_m\}$ be an orthonormal basis of (T_pM, g_p). Then we get a linear isomorphism
$$\phi_B : T_pM \to \mathbb{R}^m, \quad v \mapsto [v]_B.$$
Then $B_p(r)$ gets identified with the open ball $U_r(0)$ of radius r centered at 0 in $\mathbb{R}^m$. The composition
(NC) $$\varphi : U_{p,r} \xrightarrow{\exp^{-1}} B_p(r) \xrightarrow{\phi_B} U_r(0) \subseteq \mathbb{R}^m$$
is then a chart on M with $\varphi(x) = 0$. If $x_1,\dots,x_m$ are the associated coordinate functions, then
$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ijkl}(x)\, x^k x^l + O(|x|^3), \text{ as } x \to p,$$
where $R_{ijkl}(x)$ are the components of the Riemannian curvature $R \in \mathcal{T}^{4,0}(M)$ of g in the chart φ. Moreover, one has

Definition The chart $\varphi : U_{p,r} \to \mathbb{R}^m$ from (NC) is called the Riemannian normal coordinates (around p).

Theorem Assume that the Riemannian curvature R of g vanishes on some open neighbourhood of p. Then there is $r>0$ such that
$$(\varphi^{-1})^* g(x) = \sum_{i=1}^{m} dx_i \otimes dx_i \quad \text{on } U_{p,r}.$$

Definition Riemannian manifold (M,g) is called (locally) flat if the Riemannian curvature R vanishes on M.

Examples $\mathbb{R}^m$, $M = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1\}$ and $C = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = z^2\}$ with their canonical Riemannian metrics are examples with locally flat Riemannian manifolds.

Ricci and scalar curvature

Lemma B Let V be a real vector space of dimension m and $T \in T^{k+1,\ell+1}(V)$. If $M = \{e_1, \dots, e_m\}$ is a basis of V and $M^* = \{\varepsilon^1, \dots, \varepsilon^m\}$ is the dual basis and $r = 1, \dots, k+1$, $s = 1, \dots, \ell+1$, $C_r^s T \in T^{k,\ell}(V)$ where

$$(CT) \qquad C_r^s T(v_1, \dots, v_k, \alpha_1, \dots, \alpha_\ell) = \sum_{l=1}^{m} T(v_1, \dots, v_{r-1}, e_l, v_r, \dots, v_k, \alpha_1, \dots, \alpha_{s-1}, \varepsilon^l, \alpha_s, \dots, \alpha_\ell)$$

and $v_1, \dots, v_k \in V$, $\alpha_1, \dots, \alpha_\ell \in V^*$. Moreover, $C_r^s T$ does not depend on the choice of the basis of V.

Proof: The fact that $C_r^s T$ is a tensor is obvious, it immediately follows from definition. It only remains to show the second claim. If $M' = \{e'_1, \dots, e'_m\}$ is a different basis of V with dual basis $M'^* = \{\varepsilon'^1, \dots, \varepsilon'^m\}$ and

$$T = \sum_{\substack{i_1, \dots, i_{k+1}, \\ j_1, \dots, j_{\ell+1} = 1}}^{m} T^{j_1 \dots j_{\ell+1}}_{i_1 \dots i_{k+1}} \varepsilon^{i_1} \otimes \dots \otimes \varepsilon^{i_{k+1}} \otimes e_{j_1} \otimes \dots \otimes e_{j_{\ell+1}},$$

then we know that (from week 5)

$$(CCT) \quad T = \sum_{\substack{p_1, \dots, p_{k+1} \\ q_1, \dots, q_{\ell+1} = 1}}^{m} (T')^{q_1 \dots q_{\ell+1}}_{p_1 \dots p_{k+1}} \varepsilon'^{p_1} \otimes \dots \otimes \varepsilon'^{p_{k+1}} \otimes e'_{q_1} \otimes \dots \otimes e'_{q_{\ell+1}}$$

where

$$(T')^{q_1 \dots q_{\ell+1}}_{p_1 \dots p_{k+1}} = \sum_{\substack{i_1, \dots, i_{k+1} \\ j_1, \dots, j_{\ell+1}}=1}^{m} T^{j_1 \dots j_{\ell+1}}_{i_1 \dots i_{k+1}} (\Lambda^{-1})^{i_1}_{p_1} \dots (\Lambda^{-1})^{i_{k+1}}_{p_{k+1}} \Lambda^{q_1}_{j_1} \dots \Lambda^{p_{\ell+1}}_{j_{\ell+1}}$$

and $(\Lambda^i_j)^{j=1,\dots,m}_{i=1,\dots,m} = [\mathrm{Id}]^{M}_{M'}$, $\left((\Lambda^{-1})^i_j\right)^{j=1,\dots,m}_{i=1,\dots,m} = [\mathrm{Id}]^{M'}_{M}$.

The we have that

$$C_r^s T = \sum_{\substack{i_1, \dots, i_k \\ j_1, \dots, j_\ell = 1}}^{m} \sum_{u=1}^{m} T^{j_1 \dots j_{s-1} u j_{s+1} \dots j_\ell}_{i_1 \dots i_{r-1} u i_{r+1} \dots i_k} \varepsilon^{i_1} \otimes \dots \otimes \varepsilon^{i_k} \otimes e_{j_1} \otimes \dots \otimes e_{j_\ell}$$

Using (CCT) and the fact that $\sum_{l=1}^{m} \Lambda^i_l (\Lambda^{-1})^l_j = \sum_{l=1}^{m} (\Lambda^{-1})^i_l \Lambda^l_j = \delta_{ij}$, we conclude that also

$$C_r^s T = \sum_{p_1, \dots, p_k} \sum_{u=1}^{m} (T')^{q_1 \dots q_{s-1} u q_{s+1} \dots q_\ell}_{p_1 \dots p_{r-1} u p_{r+1} \dots p_k} (\varepsilon')^{p_1} \otimes \dots \otimes (\varepsilon')^{p_k} \otimes e'_{q_1} \otimes \dots \otimes e'_{q_\ell}. \qquad \square$$

Let M be a manifold of dimension m and $T \in \mathcal{T}^{k+1,\ell+1}(M)$. If r, s are as above and $\varphi: \mathcal{U} \to \mathbb{R}^m$ is a chart on M with coordinate vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$ and dual forms $\{dx_1, \dots, dx_m\}$, then $C_r^s T$ defined in the chart by

$$C_r^s T(X_1, \dots, X_{r-1}, \tfrac{\partial}{\partial x_l}, X_r, \dots, X_k, \theta_1, \dots, \theta_{s-1}, dx_l, \theta_s, \dots, \theta_\ell),$$

where $X_1, \dots, X_k \in \mathfrak{X}(M)$ and $\theta_1, \dots, \theta_\ell \in \Omega^1(M)$, is a smooth tensor field of type (k,ℓ) on M. Indeed, if

$$T(x) = \sum_{\substack{i_1,\dots,i_{k+1} \\ j_1,\dots,j_{\ell+1}}}^{m} T_{i_1 \dots i_{k+1}}^{j_1 \dots j_{\ell+1}}(x)\, dx_{i_1} \otimes \dots \otimes dx_{i_{k+1}} \otimes \frac{\partial}{\partial x_{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{j_{\ell+1}}}$$

in the chart φ, then by the proof of Lemma B,

$$\text{(LCT)}\quad C_r^s\, T(x) = \sum_{\substack{i_1,\dots,i_k \\ j_1,\dots,j_\ell}} \sum_{u=1}^{m} T_{i_1 \dots i_{r-1} u\, i_r \dots i_k}^{j_1 \dots j_{s-1} u\, j_s \dots j_\ell}\, dx_{i_1} \otimes \dots \otimes dx_{i_k} \otimes \frac{\partial}{\partial x_{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{j_\ell}}.$$

This shows that the coefficients of $C_{r,s}T$ are smooth. Moreover, by Lemma B, $C_r^s\, T(x)$ does not depend on the choice of basis of T_xM. This proves the claim.

Definition

Let (M,g) be a Riemannian manifold with Levi-Civita connection D and curvature tensor $R \in \mathcal{T}^{3,1}(M)$. Then

$$Ric := C_2^1\, R \in \mathcal{T}^{2,0}(M)$$

is called the Ricci curvature tensor (or Ricci curvature for short) of g.

Theorem i) The Ricci curvature of g is symmetric, that is,

$$Ric(X,Y) = Ric(Y,X), \quad X, Y \in \mathfrak{X}(M).$$

ii) If $\varphi: U \to \mathbb{R}^m$ is a chart on M and

$$R(x) = \sum_{i,j,k,\ell=1}^{m} R_{ijk}^{\ell}(x)\, dx_i \otimes dx_j \otimes dx_k \otimes \frac{\partial}{\partial x_\ell},$$

then

$$Ric(x) = \sum_{i,j=1}^{m} \sum_{\ell=1}^{m} R_{i\ell j}^{\ell}(x)\, dx_i \otimes dx_j.$$

Proof: ad ii) Was proved above in (LCT).

ad i) If $g(x) = \sum_{i,j=1}^{m} g_{ij}(x)\, dx_i \otimes dx_j$, then we have that

$$Ric_{ij}(x) = \sum_{\ell=1}^{m} R_{i\ell j}^{\ell}(x) = \sum_{\ell,u,v=1}^{m} R_{i\ell j}^{v}(x)\, g_{vu}(x)\, g^{u\ell}(x)$$

$$= \sum_{\ell,u=1}^{m} R_{i\ell j u}(x)\, g^{u\ell}(x)$$

$$= \sum_{\ell,u,v=1}^{m} R_{jui}^{v}(x)\, g_{v\ell}(x)\, g^{u\ell}(x)$$

by the symmetry (RS4) of the Riemannian curvature and $g^{\ell u}(x) = g^{u\ell}(x)$

$$= \sum_{\ell=1}^{m} R_{j\ell i}^{\ell}(x) = Ric_{ji}(x).$$

$\square$

Lemma B Let V be a real vector space of dimension m with inner product g. If $M = \{e_1, \dots, e_m\}$ is an orthonormal basis of V, $T \in T^{k+2,\ell}(V)$ and $1 \le r < s \le k+1$, then $C_{r,s}T \in T^{k,\ell}(V)$ where

$$C_{r,s}T(v_1, \dots, v_k, \alpha_1, \dots, \alpha_\ell) = \sum_{l=1}^{m} T(v_1, \dots, v_{r-1}, e_l, v_r, \dots, v_{s-1}, e_l, v_s, \dots, v_k, \alpha_1, \dots, \alpha_\ell) \tag{CT}$$

and $v_1, \dots, v_k \in V$, $\alpha_1, \dots, \alpha_\ell \in V^*$. Moreover, $C_{r,s}T$ does not depend on the choice of the basis of V.

Proof is left as an exercise. $\square$

Note that if $M = \{\varepsilon^1, \dots, \varepsilon^m\}$ is the dual basis and

$$T = \sum_{\substack{i_1, \dots, i_{k+2}, \\ j_1, \dots, j_\ell = 1}}^{m} T_{i_1 \dots i_{k+2}}^{j_1 \dots j_\ell} \, \varepsilon^{i_1} \otimes \dots \otimes \varepsilon^{i_{k+2}} \otimes e_{j_1} \otimes \dots \otimes e_{j_\ell},$$

then

$$C_{r,s}T = \sum_{\substack{i_1, \dots, i_k, \\ j_1, \dots, j_\ell = 1}}^{m} T_{i_1 \dots i_{r-1} l i_r \dots i_{s-1} l i_s \dots i_k}^{j_1 \dots j_\ell} \, \varepsilon^{i_1} \otimes \dots \otimes \varepsilon^{i_k} \otimes e_{j_1} \otimes \dots \otimes e_{j_\ell}.$$

This operation extends in an obvious way also to tensor fields of type $(k+2, \ell)$ on a manifold.

Definition Let (M, g) be a Riemannian manifold with Ricci curvature $\mathrm{Ric} \in \mathcal{T}^{2,0}(M)$. We call

$$S := C_{1,1}\,\mathrm{Ric} \in \mathcal{T}^{0,0}(M) = \mathcal{C}^\infty(M)$$

the scalar curvature of M.

Note that if

$$\mathrm{Ric}(x) = \sum_{i,j=1}^{m} R_{ij}(x)\, dx_i \otimes dx_j \quad \text{and} \quad g(x) = \sum_{i,j=1}^{m} g_{ij}(x)\, dx_i \otimes dx_j$$

in the chart $\varphi: U \to \mathbb{R}^m$ on U. Then

$$S(x) = \sum_{i,j=1}^{m} g^{ij}(x) R_{ij}(x) \quad \text{on } U.$$

Geometric meaning of the Ricci and scalar curvature

Let (M, g) be a Riemannian manifold. The volume form dV_g associated to g is in any chart $\varphi: U \to \mathbb{R}^m$ on M defined as

$$dV_g(x)\, dx_1 \wedge \dots \wedge dx_m = \sqrt{|\det g(x)|}\; dx_1 \wedge \dots \wedge dx_m \tag{VF}$$

where

$$\det g(x) = \det\left(g_{ij}(x)\right)_{i=1,\dots,m}^{j=1,\dots,m} \quad \text{and} \quad g(x) = \sum_{i,j=1}^{m} g_{ij}(x)\, dx_i \otimes dx_j.$$

Be aware that eventhough the formula (VF) might suggests that $dV_g \in \mathcal{T}^{(m,0)}(M)$, it is not in so. The volume form dV_g is an example of a density on M.

Now it can be shown that in normal coordinates around p we have that

$$dV_g(x) = \left(1 - \frac{1}{6} R_{ij}(x)\, x_i x_j + \mathcal{O}(|x|^3)\right) \quad \text{as } x \to p.$$

The scalar curvature of g then measures the difference between the Riemannian volume and Euclidean volume. More precisely, if $U_{p,r}$ is the image of the open ball $B_p(0)$ under the exponential map, then

$$RVol(U_{p,r}) = (1 - S(p)\, C_m r^2 + \mathcal{O}(r^4))\, EucVol(U_r(0)), \quad r \to 0,$$

where

- $RVol(U_{p,r})$ is the volume of $U_{p,r} \subseteq M$ w.r. to dV_g,
- C_m is a constant which depends only on dimension m and
- $EucVol(U_r(0)) = (\pi^{m/2} r^m)/(\Gamma(m + \frac{1}{2}))$ is the Euclidean volume of an open ball of radius $r > 0$ in $\mathbb{R}^m$.

Sectional curvature

Let (M, g) be a Riemannian manifold of dimension $m \geq 2$ and $R \in \mathcal{T}^{4,0}$ be the Riemannian curvature of g so that by definition $R_x \in T_x^{4,0}M$ for any $x \in M$. Now let Σ_1 be a 2-dimensional subspace of T_pM for some fixed $p \in M$. Then

$$R_p|_{\Sigma_1} \in T^{4,0}(M) = \{f : \Sigma_1 \times \Sigma_1 \times \Sigma_1 \times \Sigma_1 \to \mathbb{R} \mid f \text{ multilinear}\}.$$

Now if $\{e_1, e_2\}$ is an orthonormal basis of $(\Sigma_1, g_p|_{\Sigma_1})$, then

$$\text{(Sec C)} \qquad K(\Sigma_1) := R_p(e_1, e_2, e_1, e_2) \in \mathbb{R}$$

does not depend on the choice of the particular basis of T_pM. (again left as an exercise)

Definition The value $K(\Sigma_1)$ is called the sectional curvature of Σ_1.

Theorem The Riemannian curvature R of g at a fixed point $p \in M$ is uniquely determined by $K(\Sigma_1)$ over all 2-dimensional subspaces of T_pM.

Proof: Let R_1, R_2 be two tensors of type $(4,0)$ on T_pM that have the symmetries (RS1)—(RS4) of Riemannian curvature and that the associated sectional curvatures agree on every $\Sigma_1 \subseteq T_pM$, $\dim \Sigma_1 = 2$. Put $T = R_1 - R_2$. We need to show that $T = 0$.
Then by assumption, $T(u, v, u, v) = 0$ for any $u, v \in T_pM$.
Let $u, v, w, z \in T_pM$ be arbitrary then we have:

$$0 = T(u+w, v, u+w, v) = T(u, v, w, v) + T(w, v, u, v) = 2T(u, v, w, v),$$

$$\begin{aligned} 0 &= T(u, v+w, z, v+w) = T(u, v, z, w) + T(u, w, z, v) \\ &= T(z, w, u, v) - T(w, u, z, v). \end{aligned}$$

The last equality shows that T is invariant cyclic permutation (321). Thus by the first Bianchi identity:

$$3\,T(z, w, u, v) = 0.$$

This completes the proof. $\square$

Example If M is a 2-dimensional surface in $\mathbb{R}^3$, then

$$K(T_x M) = S(x) = K(x), \quad x \in M.$$

Here $K(x)$ is the Gauss curvature.