Sectional curvature

Let (M,g) be a Riemannian manifold. Fix a point $p \in M$ and Σ_1 be a 2-dimensional vector subspace in T_pM. Let $\{e_1, e_2\}$ be an orthonormal basis of Σ_1 (w.r. to $g|_{\Sigma_1}$).

Now it is easy to verify that

$$(SC) \quad K(\Sigma_1) = R_p(e_1, e_2, e_1, e_2) \in \mathbb{R}$$

does not depend on the choice of an orthonormal basis of Σ_1.

Here $R \in \mathcal{T}^{4,0}(M)$ is the Riemannian curvature of g.

<u>Definition</u> The number defined in (SC) is called the sectional curvature of Σ_1.

Remark: Sectional curvature is of great interest in Riemannian geometry, in particular, geometers are interested in spaces with constant sectional curvature.

<u>Theorem</u> The Riemannian curvature $R \in \mathcal{T}^{4,0}(M)$ of g is completely determined by $K(\Sigma_1)$ as Σ_1 varies over all 2-dimensional subspaces of T_pM and $p \in M$.

Proof: We claim the following: if $p \in M$ is fixed and $R_1, R_2 \in T_p^{4,0}(M)$ have symmetries (RS1)–(RS4)

and $(*)$ $R_1(e_1, e_2, e_1, e_2) = R_2(e_1, e_2, e_1, e_2)$ for every orthonormal sequence $\{e_1, e_2\}$ of vectors from T_pM, then we claim that $R_1 = R_2$.

From $(*)$ it follows that we may even assume that

$$R_1(u,v,u,v) = R_2(u,v,u,v) \quad \text{for every } u,v \in T_pM.$$

Put $T = R_1 - R_2 \in T_p^{4,0}(M)$. We have to show that $T = 0$. Let $u, v, w \in T_pM$, then

$$0 = T(u+w, v, u+w, v) = T(u,v,u+w,v) + T(w,v,u+w,v)$$

$$= \cancel{T(u,v,u,v)} + T(u,v,w,v) + T(w,v,u,v) + \cancel{T(w,v,w,v)}$$

$$= T(u,v,w,v) + T(w,v,u,v) = 2T(u,v,w,v) \quad \text{by (RS 4)}.$$

Let $u, v, w, z \in T_pM$. Then we have shown that

$$\begin{aligned}
0 = T(u, v+w, z, v+w) &= T(u, v, z, v+w) + T(u, w, z, v+w) \\
&= \cancel{T(u, v, z, v)} + T(u, v, z, w) + T(u, w, z, v) + \cancel{T(u, w, z, w)} \\
&= T(u, v, z, w) + T(u, w, z, v) \qquad \text{by (RS4) \& (RS1)} \\
&= T(z, w, u, v) - T(w, u, z, v)
\end{aligned}$$

$$\Rightarrow \quad T(z, w, u, v) = T(w, u, z, v).$$

We see that T is invariant under a cyclic permutation in the first three entries. By (RS2)

$$3T(z, w, u, v) = T(z, w, u, v) + T(w, u, z, v) + T(u, z, w, v) = 0.$$

We proved that $T = 0$ and hence $R_1 = R_2 \in T_p^{4,0}(M)$. $\square$

<u>Remark</u> If $\dim M = 2$, then $K(\Sigma)$ for $\Sigma = T_pM$, $p \in M$, agrees with the Gauss curvature and it also agrees with the scalar curvature at p.

<u>Applications of Riemannian geometry</u>

I. <u>Topology of manifolds</u>

Let (M, g) be a Riemannian manifold and $B_p(r) \subseteq T_pM$ the open ball of radius $r > 0$ centered at $0 \in T_pM$. We know that if $r > 0$ is small enough then $B_p(r)$ belongs to the domain of the exponential map

$$\exp_p : D_p \longrightarrow M, \quad \exp_p(v) = \gamma_v(1)$$

where γ_v is the unique geodesic for the Levi-Civita connection of g determined by $\gamma_v(0) = p$, $\gamma_v'(0) = v$. (Recall that $D_p \subseteq T_pM$.) Moreover we know that if r is small enough, then the map

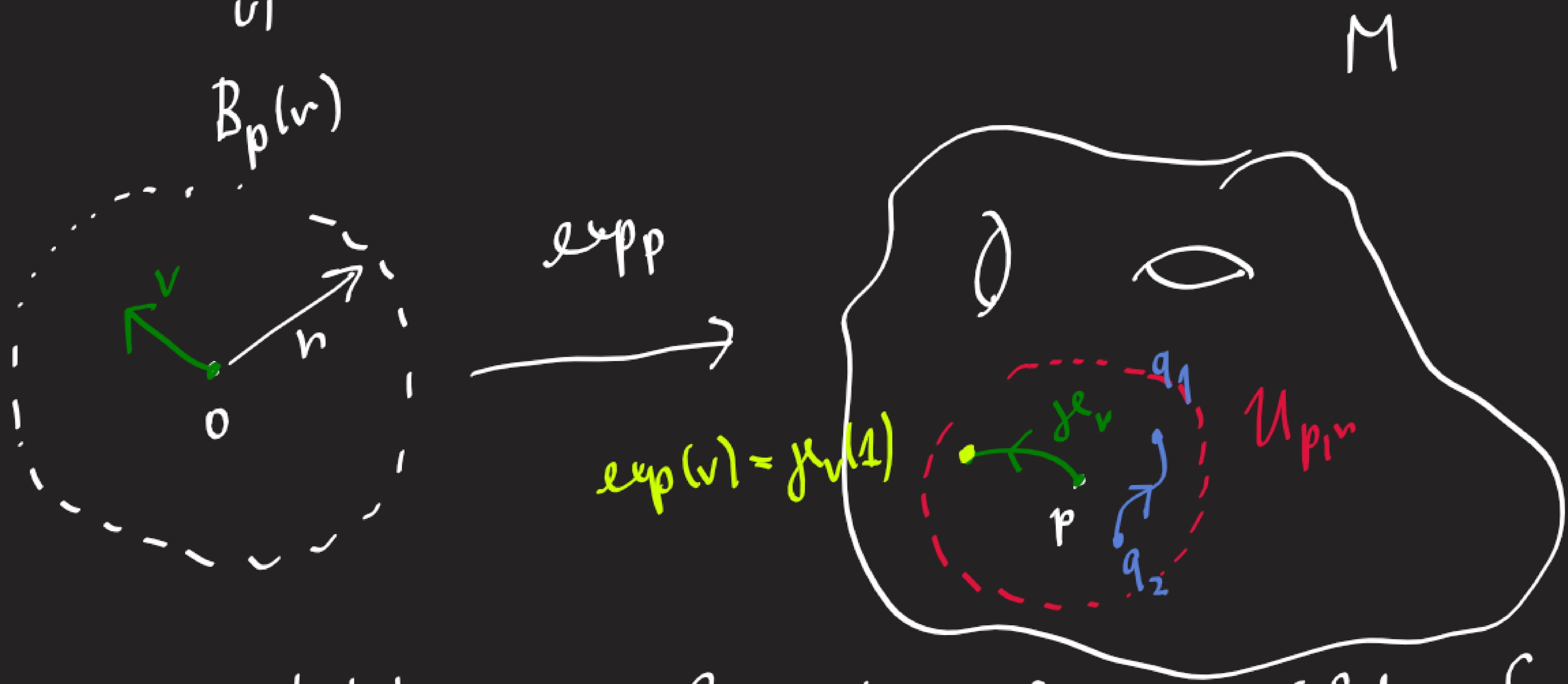
$$\exp_p|_{B_p(r)} : B_p(r) \to M \quad \text{is a diffeomorphism onto its}$$

image $U_{p,r}$. It can be proved even a stronger result:

<u>Theorem</u> If p, $B_p(r)$, $U_{p,r}$ are as above, then $U_{p,r}$ is geodesically convex if $r > 0$ is small enough. More precisely, any two points, say $q_1, q_2 \in U_{p,r}$, can be connected by a unique

shortest geodesic and this geodesic lies entirely in $\mathcal{U}_{p,r}$.

$$\begin{array}{c} T_pM \\ \cup\!\!\!\mid \\ \exp_p : D_p \longrightarrow M \\ \cup\!\!\!\mid \\ B_p(r) \end{array}$$



See Section 4.1.4 in Geometry of Manifolds for the proof.

If $r>0$ is small enough then we know that $\mathcal{U}_{p,r}$ is

(P1) diffeomorphic to an open ball in $\mathbb{R}^m$ (if $\dim M = m$) and

(P2) geodesically convex.

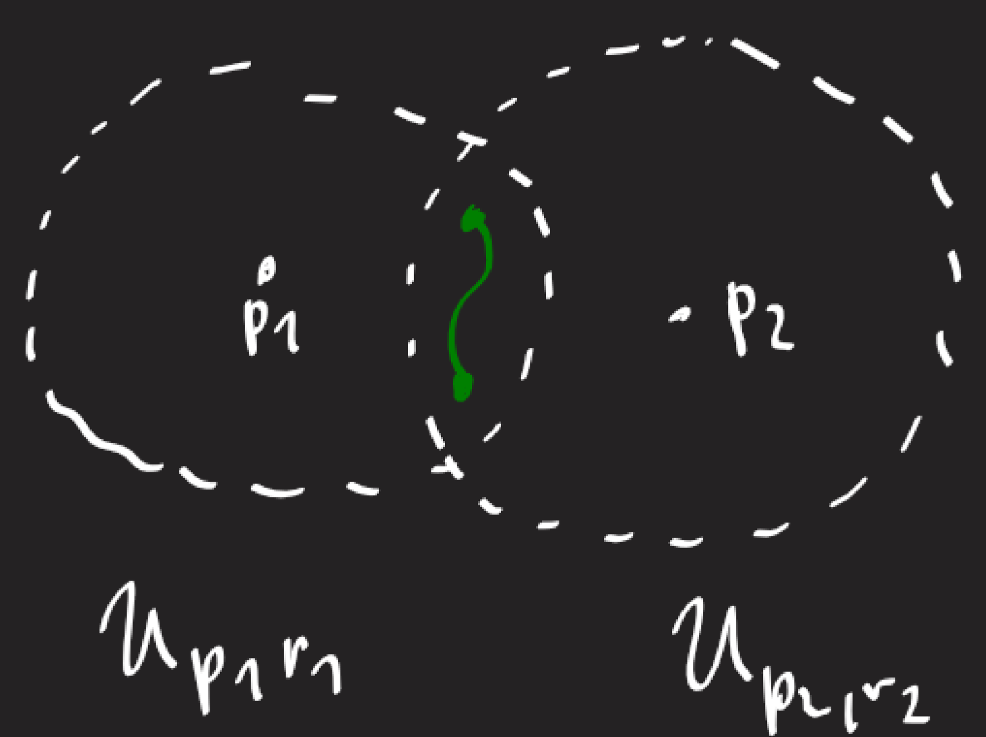
From (P1) it follows $\quad H^k(\mathcal{U}_{p,r}, \mathbb{Z}) = 0 \quad$ if $\quad k>0$.

$\uparrow$ singular cohomology

From (P2) it follows also that

$$H^k(\mathcal{U}_{p_1,r_1} \cap \mathcal{U}_{p_2,r_2}, \mathbb{Z}) = 0 \quad \text{if } k>0$$

if $\quad \mathcal{U}_{p_1,r_1}$ and $\mathcal{U}_{p_2,r_2}$ have properties (P1) and (P2).



Let $\mathcal{U} = \{ \mathcal{U}_\alpha : \alpha \in A \}$ be an open covering of the manifold M (with the fixed Riemannian metric g on M) which consists of geodesically convex sets. Repeating the argument as above, it follows that also an intersection of finitely geodesically convex sets is geodesically convex. Now we can use $\mathcal{U}$ to compare the de Rham cohomology of M with the singular cohomology of M.

Recall that the k-th de Rham cohomology group of M is the cohomology of the de Rham complex

$$C^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \dots \to \Omega^m(M)$$

where
- $\Omega^k(M)$ is the vector space of k-forms on M,
- $m = \dim M$
- d is the de Rham differential or the exterior derivative.

$$H^k_{DR}(M) = \operatorname{Ker} d_k / \operatorname{Im} d_{k-1}$$

$$\dots \xrightarrow{d_{k-1}} \Omega^k(M) \xrightarrow{d_k} \Omega^{k+1}(M) \to \dots$$

spectral sequences

$$H^k_{DR}(M) \cong \mathcal{H}^k(M, \mathbb{R}) \cong \mathcal{H}^k(\mathcal{U}, \mathbb{R}) \cong H^k(M, \mathbb{R})$$

$\mathcal{H}^k(M,\mathbb{R})$: the k-th sheaf cohomology of M with the constant sheaf $\mathbb{R}$

$\mathcal{H}^k(\mathcal{U},\mathbb{R})$: the k-th sheaf cohomology group of the constant sheaf $\mathbb{R}$ w.r. to the open covering $\mathcal{U}$

$H^k(M,\mathbb{R})$: singular cohomology

a standard result in the theory of sheaves which uses that the de Rham complex is a resolution of the constant sheaf $\mathbb{R}$ by fine sheaves

Leray Theorem
uses that $\mathcal{H}^k(U_{\alpha_1} \cap U_{\alpha_2} \cap \dots \cap U_{\alpha_k}, \mathbb{R}) = 0$
if $U_{\alpha_1}, \dots, U_{\alpha_k} \in \mathcal{U}$ (i.e. these are geodesically convex sets)

This leads to

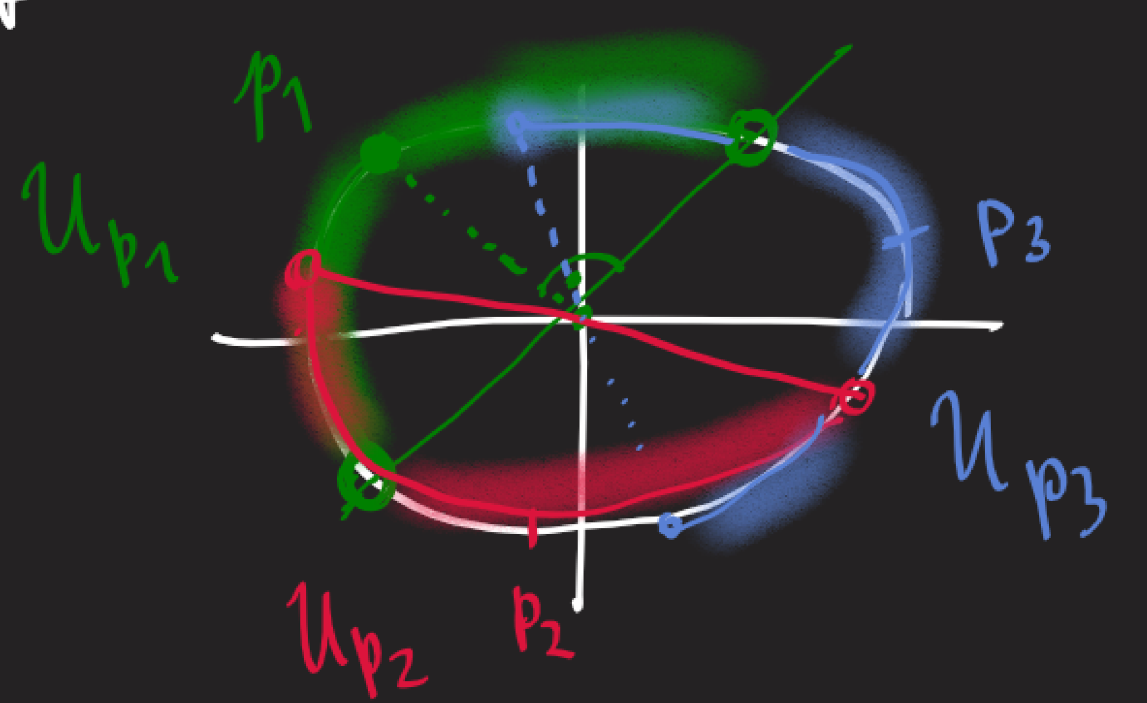
<u>de Rham Theorem</u>: $H^k_{DR}(M) \cong H^k(M, \mathbb{R})$

The next application is that $H^k(M, \mathbb{R})$ and $H^k_{DR}(M)$ can be computed on nice spaces using the Leray theorem as $\mathcal{H}^k(\mathcal{U}, \mathbb{R})$.

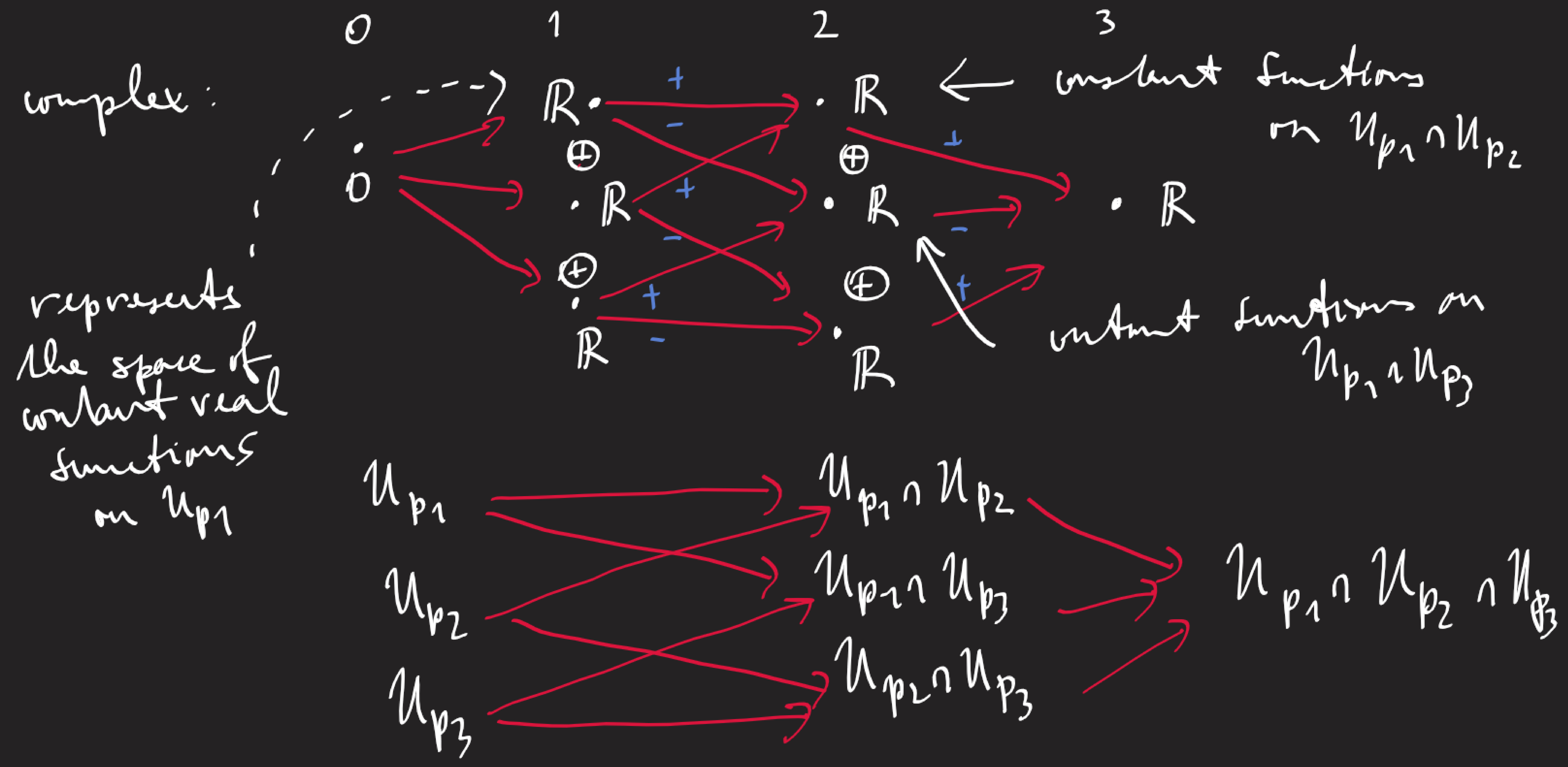
<u>Example</u> $M = S^1 \subseteq \mathbb{R}^2$

$M = U_{p_1} \cup U_{p_2} \cup U_{p_3}$

$\mathcal{U} = \{U_{p_1}, U_{p_2}, U_{p_3}\}$



Čech complex:



Question: Does every smooth manifold posses a Riemannian metric?

Answer: YES. The existence of a Riemannian metric on a smooth manifold M can be proved as follows:

- Start with any atlas $\mathcal{A} = \{ \varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^m \mid \alpha \in A \}$ on M. Since M is second countable (even paracompactness is enough), there is a locally finite covering
$$\mathcal{B} = \{ \varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^m \mid \alpha \in B \subseteq A \} \quad \text{of } A.$$
Local finiteness means that for every $x \in M$ there only finitely many $\mathcal{U}_\alpha$ with $\alpha \in B$ such that $x \in \mathcal{U}_\alpha$.
- Partition of unity subordinated to the open covering $\mathcal{B}$, this is a collection of smooth maps
$$\chi_\alpha : \mathcal{U}_\alpha \longrightarrow \mathbb{R}, \alpha \in B, \text{ (meaning that}$$
$$\chi_\alpha \circ \varphi_\alpha^{-1} : \underset{\substack{\cap \\ \mathbb{R}^m}}{\varphi_\alpha(\mathcal{U}_\alpha)} \longrightarrow \mathbb{R} \text{ is smooth)}$$
such that

(PU1) the image of χ_α is contained in $[0,1]$,

(PU2) the support of χ_α ($\operatorname{supp} \chi_\alpha := \overline{\{ x \in \mathcal{U}_\alpha : \chi_\alpha(x) \neq 0 \}}$) is a relatively compact of $\mathcal{U}_\alpha$, i.e. the closure is a compact subset of $\mathcal{U}_\alpha$ and

(PU3) $\sum_{\alpha \in B} \chi_\alpha = 1$ ... constant function 1 on M

here we extend each χ_α by zero outside $\mathcal{U}_\alpha$ so that we obtain a smooth function on M.

Note that in (P43) we have that for each $x \in M$

$\sum_{\alpha \in B} \chi_\alpha(x)$ there are only finitely many α's in B for which $\chi_\alpha(x) \neq 0$. There are no issues with convergence.

·) Now for each $\alpha \in B$ choose a Riemannian metric g_α on $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^m$. For example one can choose the restriction $g|_{\varphi_\alpha(U_\alpha)}$ of the Euclidean metric on $\varphi_\alpha(U_\alpha)$.

·) Then it is easy to verify that

$$\sum_{\alpha \in B} \chi_\alpha\left(\left(\varphi_\alpha^{-1}\right)^* g_\alpha\right)$$

is a Riemanian metric on M.

From this construction it also follows that are many Riemannian metrics on M.

II. General relativity

In the end of the 19'th century people believed that the fondations physics were almost complete and there was only part of physics, namely optics, that physicists had troubles to describe using the laws of physics from other branches. It turned eventually (Michelson) that the laws of optics had to completely redone.

1905 Special general relativity, no absolute time,
Einstein no absolute space

The model for this theory is the Minkowski space $\mathbb{R}^4$ with a non-degenerate bilinear form of signature $(3,1)$ or $(1,3)$, 3 directions and 1 direction for time, the group of symmetries $O(1,3)$ or $O(3,1)$ together with translations

Disadvantage: is that it describes only particles moving with constant velocity. Einstein seeked a theory of gravity in which he can describe particles or objects moving with varying velocity.

(1912) General relativity

Pseudo-Riemannian geometry on a 4-dimensional manifold $\mathbb{R}^4$ with pseudo-Riemannian metric of signature (3,1) or (1,3). As in the Riemannian case one can show that there is a unique affine connection which is metric or torsion free for any pseudo-Riemannian metric of signature (3,1) or (1,3). We call this connection the Levi-Civita connection. The presence of a gravitational field is then equivalent to the non-vanishing of the curvature of the Levi-Civita connection.

Fundamental equation of GR:

$$\boxed{R_{ik} - \frac{1}{2} S g_{ik} - \Lambda g_{ik} = T_{ik}}$$ Einstein field equations

where:

1) R_{ik}.. Ricci curvature of the pseudo-Riemannian metric g,
2) S.. scalar curvature,
3) g_{ik}.. pseudo-Riemannian metric,
4) Λ.. cosmological constant and
4) T_{ik}.. stress-momentum tensor field of type (2,0) which describes the matter, in particular $T_{ik} = 0$ in vacuum

Schwartzschield metric

describes rotationally symmetric gravitational field in vacuum ($T_{ik} = 0$)

$$g = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 (\sin^2\theta \, d\varphi^2 + d\theta^2)$$

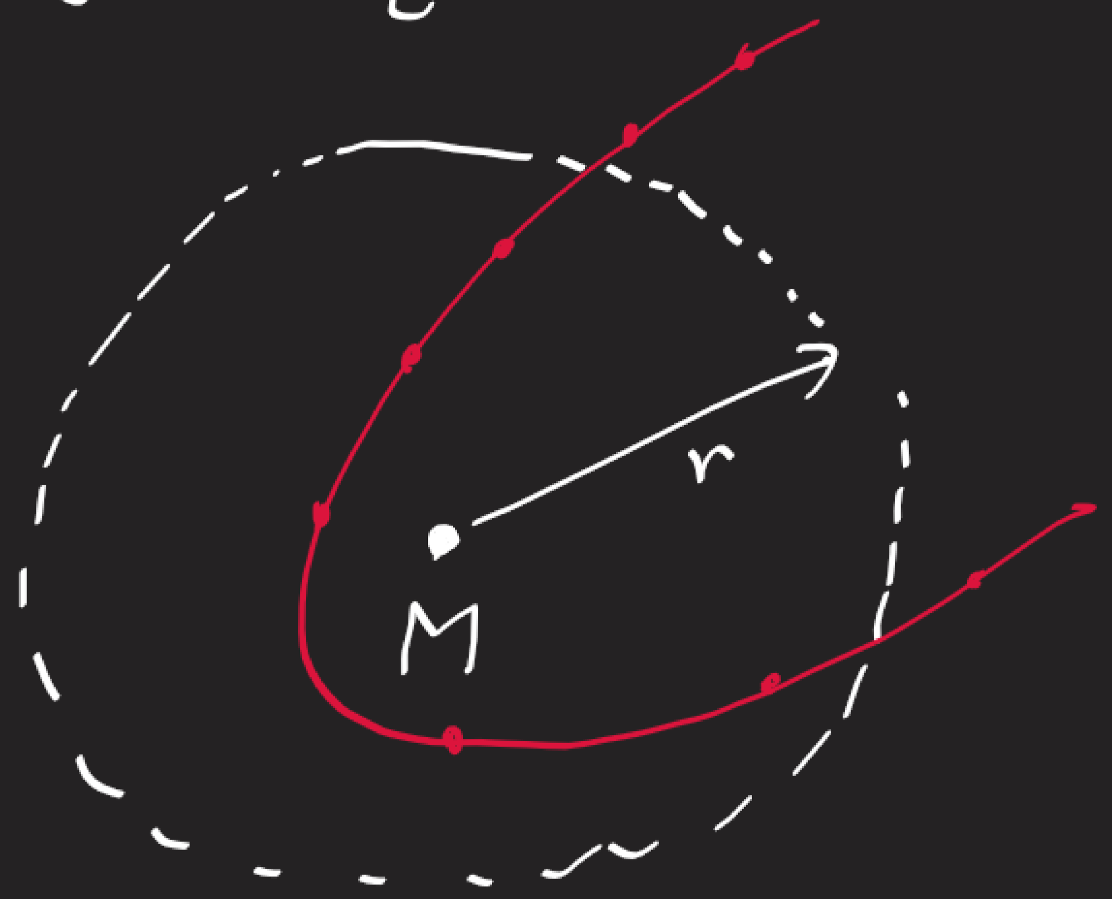
$dt^2 = dt \otimes dt$, c.. speed of light, $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$, (t, r, φ, θ)

(r, φ, θ).. spherical coordinates on $\mathbb{R}^3$, the Euclidean metric

$g = dr^2 + r^2 (\sin^2\theta \, d\varphi^2 + d\theta^2)$, $x = r \sin\theta \sin\varphi$, $y = r \sin\theta \sin\varphi$, $z = r\cos\theta$

k .. gravitational constant

$$r_g = \frac{2kM}{c^2}$$ M is the mass of the center



the particles are moving in the gravitational field as along geodesics

in GE geodesics are generalizations of linear motion

Geodesic equations in the Schwartzshild metric are:

$$\frac{d^2u}{d\varphi^2} + u = \frac{kM}{h_0^2} + \boxed{\frac{3kMu^2}{c^2}}$$

GE relativity correction term

$$r^2 \frac{d\varphi}{ds} = \frac{h_0}{c}, \quad \theta = \frac{\pi}{2}$$

where $h_0 \in \mathbb{R}$, $u = \frac{1}{r}$.

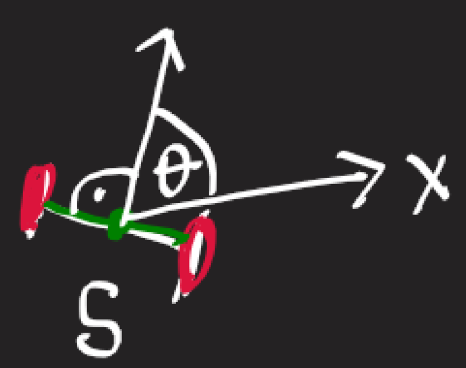
These equations are the classical equations of motion in central field already known to Kepler with the relativistic correction term $\frac{3kMu^2}{c^2}$

1. Kepler law: objects are moving on an ellipsis in a central force

III. Robotics and sub-Riemannian geometry

Example Consider a segway in $\mathbb{R}^2$ and let M be the configuration space of segway

$\mathbb{R}^2$



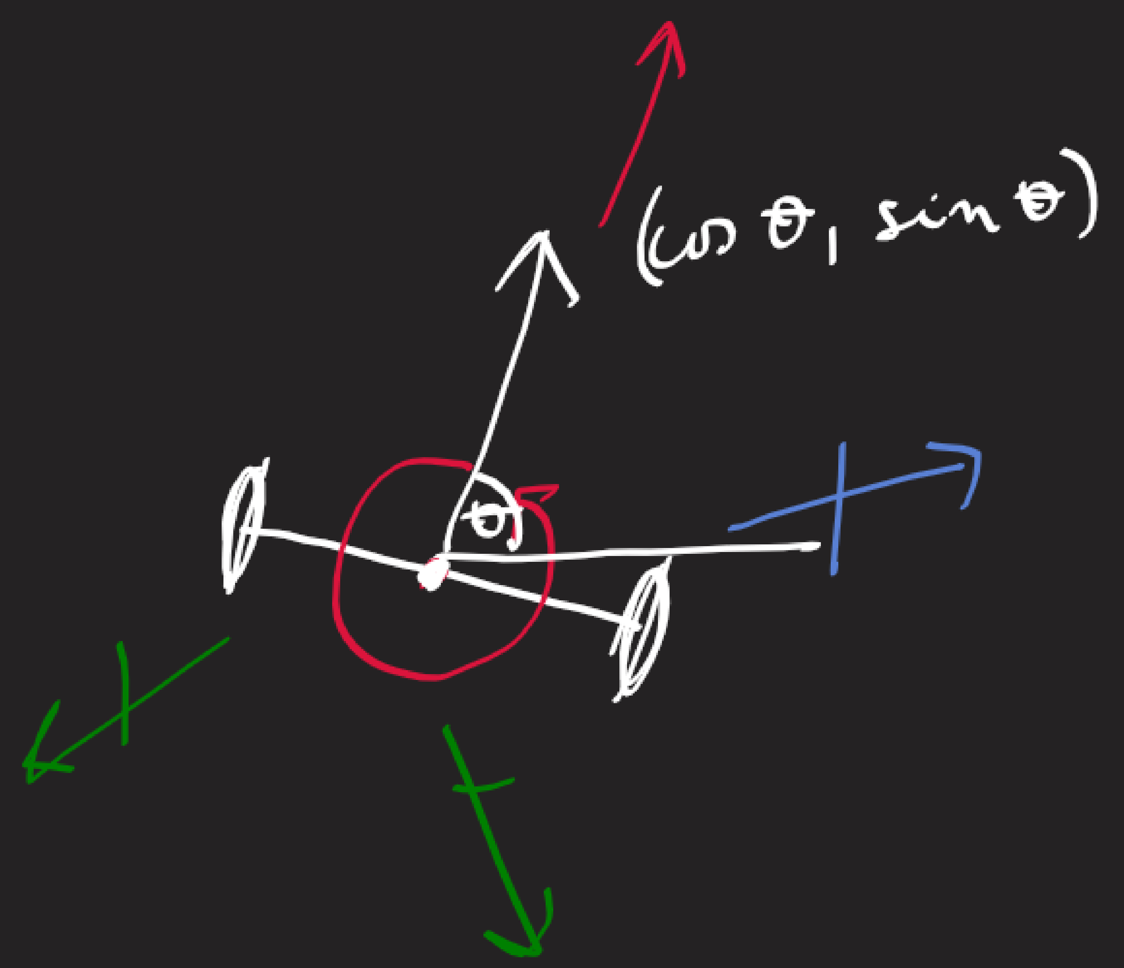
S... the middle point of the axis of segway

$S = (x, y)$

the position of segway is completely determined by (x,y) and the angle θ

Configuration space M of segway is $\underset{(x,y)}{\mathbb{R}^2} \times \underset{(\cos\theta, \sin\theta)}{S^1}$.

We are mainly interested in motion of segway, here we are assuming that the segway moves without slipping. In other words, there are only some allowed directions in which the segway can move. These allowed directions at a fixed point $p \in M$ form a 2-dimensional vector supspace in $T_p M$.



This 2-dimensional subspace E_p is the linear span of

$$\frac{\partial}{\partial\theta}, \quad \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}.$$

We see that at each point of $p \in M$ there is a 2-dimensional vector subspace of $T_p M$ of all allowed motions.

<u>Definition</u> Let M be an m-dimensional manifold. Then a k-dimensional distribution on M is a family $E = \bigcup_{p \in M} E_p$ where for every $p \in M$:

(D1) E_p is a k-dim. subspace of $T_p M$ and

(D2) for every $p \in M$ there is a chart $\varphi: \mathcal{U} \to M$ around p such that $E_p = \operatorname{span}\{X_1(p), \dots, X_k(p)\}$ where $X_1, \dots, X_k$ are smooth vector fields on $\mathcal{U}$.

Motivation: is to study curves γ on M that have the property that $\gamma'(t) \in E_{\gamma(t)}$. It is natural to study such curves that minimize the distance with respect to a Riemannian structure on E.

Sub-Riemannian structure.