Review of topology

Definition A topological space is a pair (X,τ) where X is a set and τ is a set of subsets of X such that:

(T1) $\emptyset, X \in \tau$,

(T2) if $U_1,\dots,U_m \in \tau$, then also $\bigcap_{i=1}^{m} U_i \in \tau$ and

(T3) if $U_\alpha \in \tau$ for every $\alpha \in A$, then also $\bigcup_{\alpha\in A} U_\alpha \in \tau$.

We call τ a topology on X and any $U \in \tau$ an open subset of X.

Examples: (i) Standard topology on $\mathbb{R}^n$.

Let $\|x\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2}$, $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $U_\varepsilon(x) := \{y \in \mathbb{R}^n : \|x-y\|_2 < \varepsilon\}$.

A subset $U \subseteq \mathbb{R}^n$ is called open if

(O) $\forall x \in U\ \exists \varepsilon > 0 : U_\varepsilon(x) \subseteq U$.

Put $\tau^n := \{U \subseteq \mathbb{R}^n : U \text{ satisfies } (O)\}$.

Claim: τ^n is a topology on $\mathbb{R}^n$.

Proof: ad (T1) Clearly $\emptyset$ and $\mathbb{R}^n$ satisfy (O).

ad (T2) Let $U_1,\dots,U_m$ satisfy (O) and $x \in \bigcap_{i=1}^m U_i$.
Then $\forall i = 1,\dots,m\ \exists \varepsilon_i > 0 : U_{\varepsilon_i}(x) \subseteq U_i$.
Hence $\varepsilon = \min\{\varepsilon_1,\dots,\varepsilon_m\} > 0$ and $U_\varepsilon(x) \subseteq \bigcap_{i=1}^m U_i$.

ad (T3) Let U_α, $\alpha \in B$, satisfy (O).
If $x \in \bigcup_{\alpha\in B} U_\alpha$, then $\exists \alpha \in B$ with $x \in U_\alpha$ and so $\exists \varepsilon > 0$ with $U_\varepsilon(x) \subseteq U_\alpha$. Thus $U_\varepsilon(x) \subseteq \bigcup_{\alpha\in B} U_\alpha$. $\square$

(ii) $\tau_{min} = \{\emptyset, \mathbb{R}^n\}$ and $\tau_{max} = \mathcal{P}(\mathbb{R}^n)$ ($\equiv$ the set of all subsets of $\mathbb{R}^n$) are noninteresting topologies on $\mathbb{R}^n$. $\tau_{min} \subsetneq \tau^n \subsetneq \tau_{max}$.

Definition Let (X,τ) be a topological space. A collection $\mathcal{B}$ of subsets of X is called a basis for τ if $\mathcal{B} \subseteq \tau$ and

(G) $\forall U \in \tau\ \forall x \in U\ \exists V \in \mathcal{B} : x \in V \subseteq U$.

Example: $\mathcal{B}_1 = \{U_\varepsilon(x) : \varepsilon > 0, x \in \mathbb{R}^n\}$ is a basis for τ^n. Also $\mathcal{B}_2 = \{U_\varepsilon(x) : \varepsilon > 0, \varepsilon \in \mathbb{Q}, x \in \mathbb{Q}^n\}$ is a basis of τ^n which is countable.

Question: Let $\mathcal{B}$ be a system of subsets of X. Define τ' as the collection of all subsets of X that satisfy (G). When is τ' a topology?

Theorem A Let X be a set and $\mathcal{B}$ be a collection of subsets of X. Assume that:

(B1) $\forall x \in X\ \exists V \in \mathcal{B} : x \in V$ and

(B2) $\forall V_1, V_2 \in \mathcal{B}\ \forall x \in V_1 \cap V_2\ \exists W \in \mathcal{B} : x \in W \subseteq V_1 \cap V_2$.

Then $\tau := \{U \subseteq X : U \text{ satisfies } (G)\}$ is a topology on X.

Proof: ad(T1) Clearly $\emptyset$ satisfies (G) and by (B1) also X satisfies (G).

ad(T3) If $U_\alpha, \alpha \in C$, satisfy (G) and $x \in \bigcup_{\alpha\in C} U_\alpha$, then $\exists \alpha \in C$: $x \in U_\alpha$. Then $\exists V \in \mathcal{B} : x \in V \subseteq U_\alpha \subseteq \bigcup_{\alpha\in C} U_\alpha$.

ad(T2) If $U_1,\dots,U_n$ satisfy (G) and $x \in \bigcap_{i=1}^n U_i$, then for every $i = 1,\dots,n$ $\exists V_i \in \mathcal{B} : x \in V_i \subseteq U_i$. By induction on n it holds that: if $V_1,\dots,V_n \in \mathcal{B}$ and $x \in \bigcap_{i=1}^n V_i$, then $\exists W \in \mathcal{B}$: $x \in W \subseteq \bigcap_{i=1}^n V_i$. Thus the intersection satisfies (G) as well. $\square$

Definition In the situation of Theorem A we say that $\mathcal{B}$ generates τ or that τ is generated by $\mathcal{B}$.

Remark: In Theorem A, τ can be equivalently defined as the smallest topology on X (w.r.to $\subseteq$) that contains $\mathcal{B}$.

Definition Let (X,τ_X) and (Y,τ_Y) be topological spaces. A map $f: X \to Y$ is called continuous if

$$\mathcal{U} \in \tau_Y \implies f^{-1}(\mathcal{U}) \in \tau_X .$$

If f is bijective and also f^{-1} is continuous, then f is called a homeomorphism.

Claim A map $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous (w.r.to τ^n and τ^m)
iff $\forall x \in \mathbb{R}^m\ \forall \varepsilon > 0\ \exists \delta > 0 : y \in \mathcal{U}_\delta(x) \implies f(y) \in \mathcal{U}_\varepsilon(f(x))$
or equivalently $\quad$ —"— $\quad : f(\mathcal{U}_\delta(x)) \subseteq \mathcal{U}_\varepsilon(f(x))$.

Proof: "$\Rightarrow$" Pick $x \in \mathbb{R}^m$ and $\varepsilon > 0$. As $\mathcal{U} := \mathcal{U}_\varepsilon(f(x)) \in \tau^m$, then $f^{-1}(\mathcal{U}) \in \tau^n$. Since $x \in f^{-1}(\mathcal{U})$, then $\exists \delta > 0 : \mathcal{U}_\delta(x) \subseteq f^{-1}(\mathcal{U})$ and so $f(\mathcal{U}_\delta(x)) \subseteq \mathcal{U}$.

"$\Leftarrow$" Let $\mathcal{U} \in \tau^m$ and $x \in f^{-1}(\mathcal{U})$. Then $f(x) \in \mathcal{U}$ and so $\exists \varepsilon > 0$ $\mathcal{U}_\varepsilon(f(x)) \subseteq \mathcal{U}$. By assumption, $\exists \delta > 0$ s.t. $f(\mathcal{U}_\delta(x)) \subseteq \mathcal{U}_\varepsilon(f(x))$. Thus $\mathcal{U}_\delta(x) \subseteq f^{-1}(\mathcal{U}_\varepsilon(f(x))) \subseteq f^{-1}(\mathcal{U})$ which shows $f^{-1}(\mathcal{U}) \in \tau^n$. $\square$

Operations on topological spaces

A. Topological product

Given topological spaces $(X_1,\tau_1),\dots,(X_m,\tau_m)$, it is easy to see that the collection

$$\mathcal{B}_{\tau_1 \times \dots \times \tau_n} = \{ \mathcal{U}_1 \times \dots \times \mathcal{U}_m : \mathcal{U}_i \in \tau_i,\ i = 1,..,m \}$$

of subsets of $\prod_{i=1}^m X_i := X_1 \times \dots \times X_m$ satisfies (B1) and (B2). We denote by $\prod_{i=1}^m \tau_i = \tau_1 \times \dots \times \tau_n$ the topology generated by $\mathcal{B}_{\tau_1 \times \dots \times \tau_m}$.

Definition The topological space $\left(\prod_{i=1}^m X_i, \prod_{i=1}^m \tau_i\right)$ is called the topological product of $(X_1,\tau_1),\dots,(X_m,\tau_m)$.

Example The product topology on $\mathbb{R}^n = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{n \text{ copies}}$.

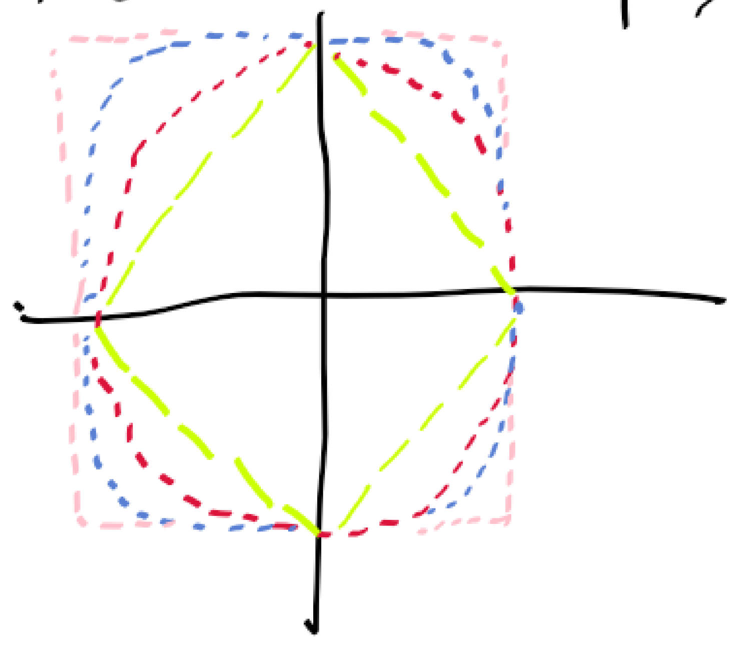
Question: what is the product topology $\tau^1 \times \dots \times \tau^1$ on $\mathbb{R}^n$?

Put $\|x\|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$, $\|x\|_{+\infty} = \max_{i=1,..,n} |x_i|$ and $\mathcal{U}_{\varepsilon,p}(x) = \{ y \in \mathbb{R}^n : \|x-y\|_p < \varepsilon \}$.
$x = (x_1,..,x_n) \in \mathbb{R}$

We know that $\|\cdot\|_p$ is a norm when $p \geq 1$.

Unit balls:

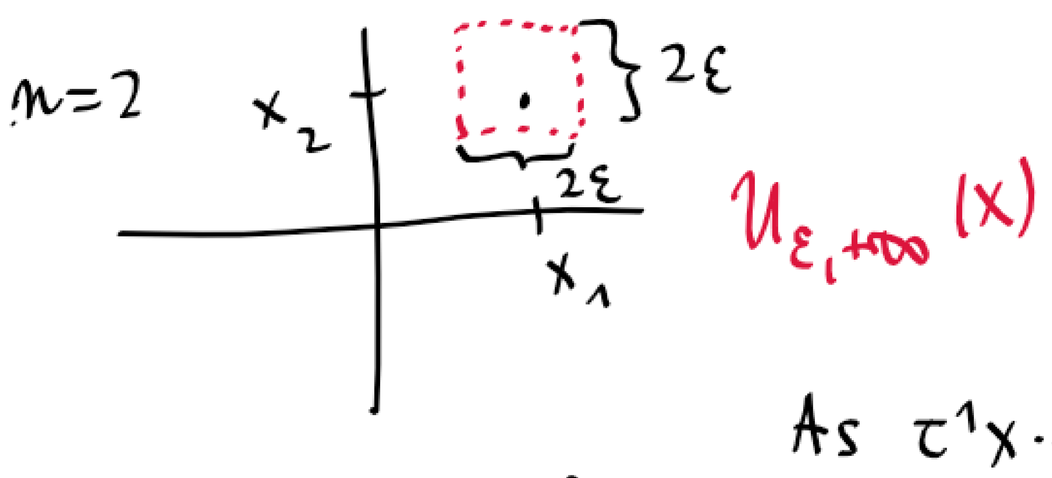
$p=1$
$p=2$
$p=3$
$p=+\infty$



We know that on $\mathbb{R}^n$ all norms equivalent. Thus all norms generate the same topology τ^n on $\mathbb{R}^n$, that is $\mathcal{U} \subseteq \mathbb{R}^n$ is open iff

$(O_p)'$ $\forall x \in \mathcal{U}\ \exists\, \varepsilon > 0 : \mathcal{U}_{\varepsilon,p}(x) \subseteq \mathcal{U}$. (Here $p \geq 1$ is arbitrary.)

$\mathcal{U}_{\varepsilon,+\infty}(x) = \mathcal{U}_\varepsilon(x_1) \times \mathcal{U}_\varepsilon(x_2) \times \dots\dots\dots \times \mathcal{U}_\varepsilon(x_n) \in \mathcal{B}_{\tau^1 \times \dots \times \tau^1}$.



We see that if $\mathcal{U}$ satisfies $(O_p)'$ with $p = +\infty$ then $\mathcal{U} \in \tau^1 \times \dots \times \tau^1$. Thus $\tau^n \subseteq \tau^1 \times \dots \times \tau^1$. On the other hand, any subset from $\mathcal{B}_{\tau^1 \times \dots \times \tau^1}$ satisfies $(O_p)'$ with $p = +\infty$ and so it belongs to τ^n. As $\tau^1 \times \dots \times \tau^1$ is the smallest topology that contains $\mathcal{B}_{\tau^1 \times \dots \times \tau^1}$, it holds $\tau^1 \times \dots \times \tau^1 \subseteq \tau^n$. Thus $\tau^n = \tau^1 \times \dots \times \tau^1$.

B. Induced topology

Theorem B Let X be a set, (Y, τ) be a topological space and $f : X \to Y$ be a map (of sets). Then $f^{-1}\tau = \{ f^{-1}(\mathcal{U}) : \mathcal{U} \in \tau \}$ is a topology on X.
If $\mathcal{B}$ is a basis for τ, then $f^{-1}\mathcal{B} = \{ f^{-1}(\mathcal{V}) : \mathcal{V} \in \mathcal{B} \}$ is a basis for $f^{-1}\tau$.

In the proof of Theorem B we will need

Lemma Let $f : X \to Y$ be a map and $\mathcal{U}_\alpha$, $\alpha \in A$, be a collection of subsets of Y. Then

(i) $f^{-1}\left(\bigcap_{\alpha \in A} \mathcal{U}_\alpha\right) = \bigcap_{\alpha \in A} f^{-1}(\mathcal{U}_\alpha)$ and (ii) $f^{-1}\left(\bigcup_{\alpha \in A} \mathcal{U}_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(\mathcal{U}_\alpha)$.

Proof: ad(i) $x \in f^{-1}\left(\bigcap_{\alpha \in A} \mathcal{U}_\alpha\right) \Leftrightarrow f(x) \in \bigcap_{\alpha \in A} \mathcal{U}_\alpha \Leftrightarrow \forall \alpha \in A : f(x) \in \mathcal{U}_\alpha \Leftrightarrow$
$\forall \alpha \in A : x \in f^{-1}(\mathcal{U}_\alpha) \Leftrightarrow x \in \bigcap_{\alpha \in A} f^{-1}(\mathcal{U}_\alpha)$.

ad(ii) $x \in f^{-1}\left(\bigcup_{\alpha \in A} \mathcal{U}_\alpha\right) \Leftrightarrow f(x) \in \bigcup_{\alpha \in A} \mathcal{U}_\alpha \Leftrightarrow \exists\, \alpha \in A : f(x) \in \mathcal{U}_\alpha \Leftrightarrow$
$\exists\, \alpha \in A : x \in f^{-1}(\mathcal{U}_\alpha) \Leftrightarrow x \in \bigcup_{\alpha \in A} f^{-1}(\mathcal{U}_\alpha)$. $\square$

Proof of Theorem B: ad (T1) $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$.

ad (T2) Let $\mathcal{U}_1, \dots, \mathcal{U}_n \in \tau$. Then $f^{-1}(\mathcal{U}_i) \in f^{-1}\tau$ for $i = 1, \dots, n$ and we need to show that also $\bigcap_{i=1}^n f^{-1}(\mathcal{U}_i) \in f^{-1}\tau$. But $\bigcap_{i=1}^n f^{-1}(\mathcal{U}_i) = f^{-1}\left(\bigcap_{i=1}^n \mathcal{U}_i\right) \in f^{-1}\tau$ by Lemma (i).

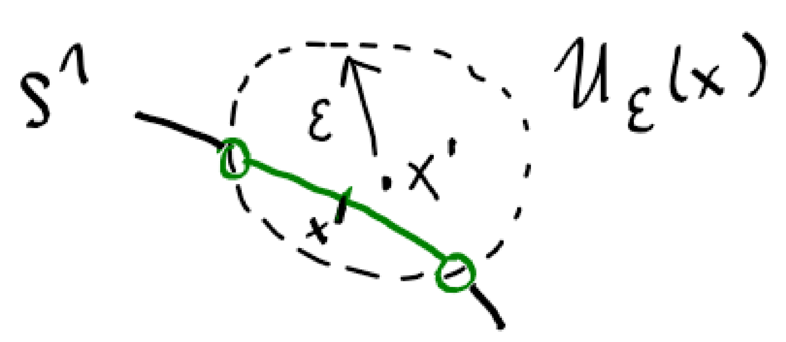
ad(T3) Let $\mathcal{U}_\alpha \in \tau$ for $\alpha \in A$. Then $f^{-1}(\mathcal{U}_\alpha) \in f^{-1}\tau$ for $\alpha \in A$ and we need to show that also $\bigcup_{\alpha \in A} f^{-1}(\mathcal{U}_\alpha) \in f^{-1}\tau$. But $\bigcup_{\alpha \in A} f^{-1}(\mathcal{U}_\alpha) = f^{-1}\left(\bigcup_{\alpha \in A} \mathcal{U}_\alpha\right) \in f^{-1}\tau$ by Lemma (ii).

Now we verify the second claim. As $\mathcal{B} \subseteq \tau$, it is clear that $f^{-1}\mathcal{B} \subseteq f^{-1}\tau$. Now let $\mathcal{U} \in \tau$ and $x \in f^{-1}(\mathcal{U}) \in f^{-1}\tau$. Then $f(x) \in \mathcal{U}$ and so there is $\mathcal{V} \in \mathcal{B}$ with $f(x) \in \mathcal{V} \subseteq \mathcal{U}$. Hence $x \in f^{-1}(\mathcal{V}) \subseteq f^{-1}(\mathcal{U})$ and $f^{-1}(\mathcal{V}) \in f^{-1}\mathcal{B}$. $\square$

Definition We call $f^{-1}\tau$ the induced topology on X (via f). If $X \subseteq Y$ and $f : X \to Y$ is the inclusion, then we call $f^{-1}\tau$ the subspace topology on X.

Remark Note that $f : X \to Y$ is continuous (w.r. to τ and $f^{-1}\tau$) and that τ is the smallest topology (w.r. to $\subseteq$) for which this is true.

Example Let $X = S^1 = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$ and $Y = \mathbb{R}^2$. Denote by $f : X \to Y$ the canonical inclusion. Then $f^{-1}\tau^2$ is the standard topology on S^1.
Now $\mathcal{U} \subseteq S^1$ is open iff there is $\mathcal{U}' \in \tau^2$ with $\mathcal{U} = S^1 \cap \mathcal{U}'$.
If $\mathcal{B}_1 = \{ \mathcal{U}_\varepsilon(x') : \varepsilon > 0,\ x' \in \mathbb{R}^2 \}$, then $f^{-1}\mathcal{B}_1$ is a basis for $f^{-1}\tau^2$.



Then $f^{-1}\mathcal{B}_1$ consists of open "intervals" on the circle (the green part of circle).

C. Quotient topology

Theorem C Let (X,τ) be a topological space, $\sim$ be an equivalence relation on X, $\overline{X} := X/\sim$ be the set of all classes of equivalence on X and $p: X \to \overline{X}$ be the canonical projection. Then
$$\overline{\tau} := \{ \mathcal{U} \subseteq \overline{X} : p^{-1}(\mathcal{U}) \in \tau \} \text{ is a topology on } \overline{X}.$$

Proof: ad (T1) Clearly $\emptyset, \overline{X} \in \overline{\tau}$.

ad (T2) If $\mathcal{U}_1, \dots, \mathcal{U}_m \in \overline{\tau}$, then $p^{-1}(\mathcal{U}_i) \in \tau$ for $i = 1, \dots, m$ and so by Lemma (i): $p^{-1}(\bigcap_{i=1}^{n} \mathcal{U}_i) = \bigcap_{i=1}^{n} p^{-1}(\mathcal{U}_i) \in \tau$. Thus $\bigcap_{i=1}^{n} \mathcal{U}_i \in \overline{\tau}$.

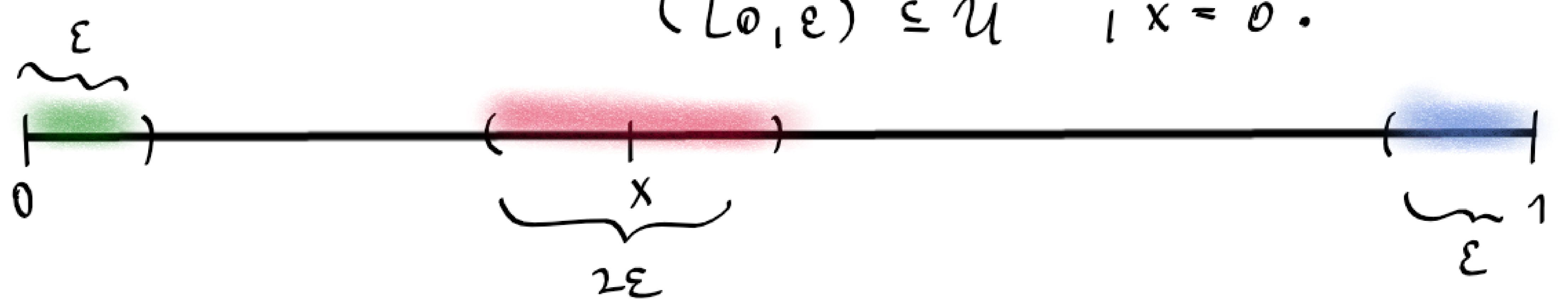
ad (T3) If $\mathcal{U}_\alpha \in \overline{\tau}$ for every $\alpha \in A$, then $p^{-1}(\mathcal{U}_\alpha) \in \tau$ for every $\alpha \in A$ and thus by Lemma (ii): $p^{-1}(\bigcup_{\alpha\in A} \mathcal{U}_\alpha) = \bigcup_{\alpha \in A} p^{-1}(\mathcal{U}_\alpha) \in \tau$.
We proved $\bigcup_{\alpha \in A} \mathcal{U}_\alpha \in \overline{\tau}$. □

Definition We call $\overline{\tau}$ the quotient topology on $\overline{X}$.

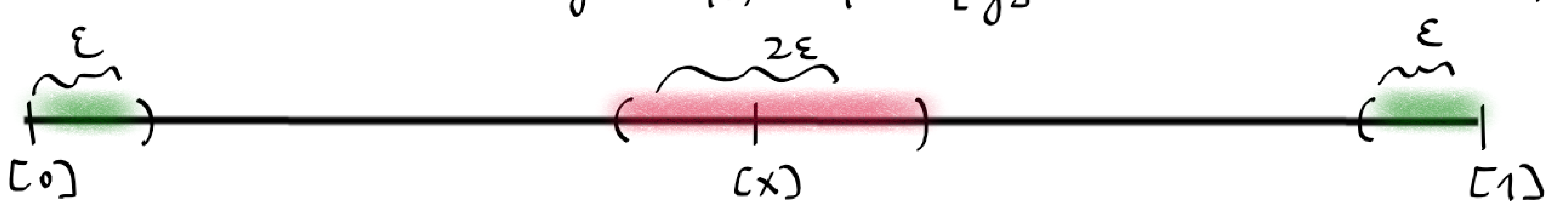
Remark: Note that $p: X \to \overline{X}$ is continuous (w.r. to τ and $\overline{\tau}$) and that $\overline{\tau}$ is the biggest topology on $\overline{X}$ for which this is true. Moreover, if (Y, τ_Y) is a topological space, then $f: \overline{X} \to Y$ is continuous iff $f \circ p$ is.

Example Standard topology on S^1 revisited.

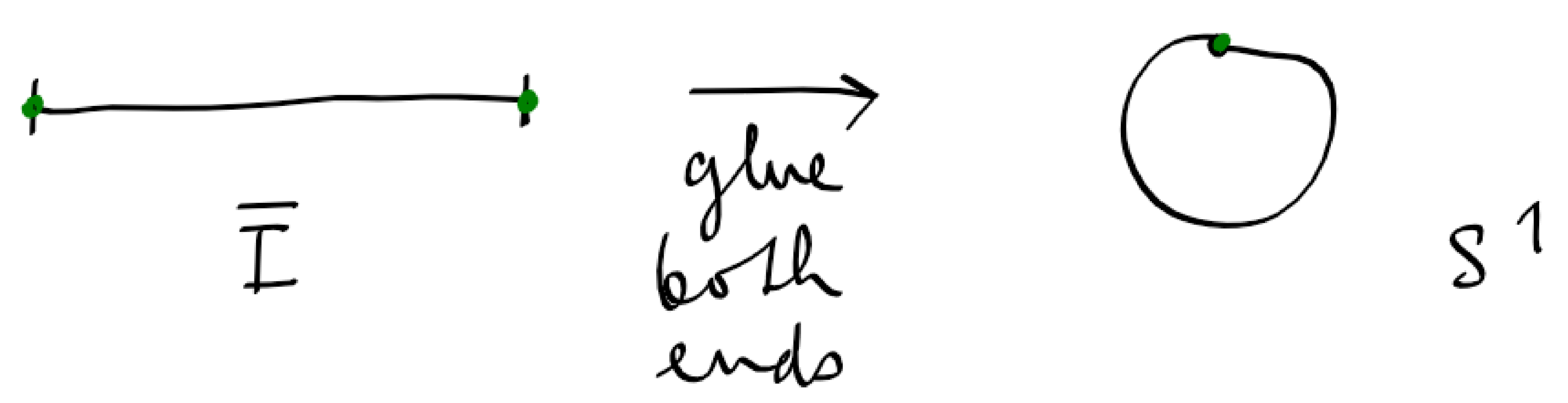
Let $I = [0,1]$, $f: I \hookrightarrow \mathbb{R}$ be the inclusion and $\tau_I := f^{-1}\tau^1$ be the subspace topology. Note that $\mathcal{U} \in \tau_I$
$$\Leftrightarrow \forall x \in \mathcal{U} \; \exists \varepsilon > 0 : \begin{cases} (x-\varepsilon, x+\varepsilon) \subseteq \mathcal{U}, & x \in (0,1), \\ (x-\varepsilon, 1] \subseteq \mathcal{U}, & x = 1, \\ [0, \varepsilon) \subseteq \mathcal{U}, & x = 0. \end{cases}$$



On I consider the equivalence $\sim$ generated by $0 \sim 1$ and $\overline{I} := I/\sim$. Let $p: I \to \overline{I}$ be the canonical projection, $[x]$ be the equivalence class of $x \in I$ and $\overline{\tau}_I$ be the quotient topology. It is easy to see that $\mathcal{U} \in \overline{\tau}_I \Leftrightarrow$
$$\forall [x] \in \mathcal{U} \; \exists \varepsilon > 0 \begin{cases} \forall y \in (x-\varepsilon, x+\varepsilon) : [y] \in \mathcal{U} & x \in (0,1), \\ \forall y \in [0,\varepsilon) \cup (\varepsilon, 1] : [y] \in \mathcal{U} & x = 0 \text{ or } x = 1. \end{cases}$$



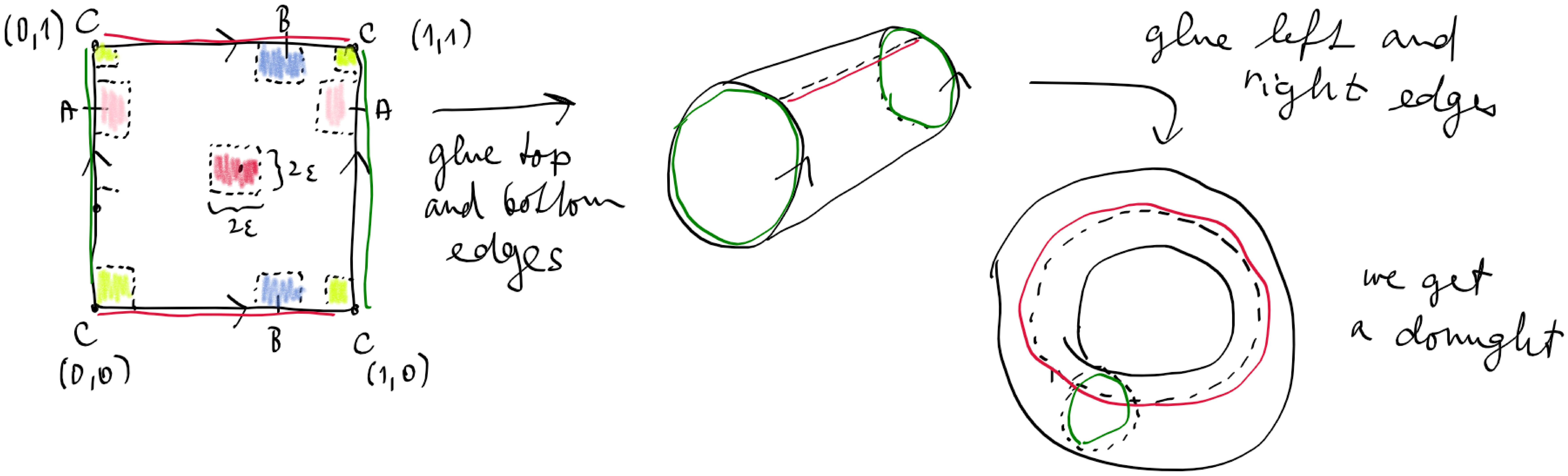
Now it is easy to see that the map
$$\phi: \overline{I} \to S^1, \; \phi([x]) = (\cos 2\pi x, \sin 2\pi x)$$
is well defined, bijective and continuous. As also ϕ^{-1} is continuous, it follows that ϕ is a homeomorphism.
Thus we can view S^1 as the closed unit interval with glued ends.



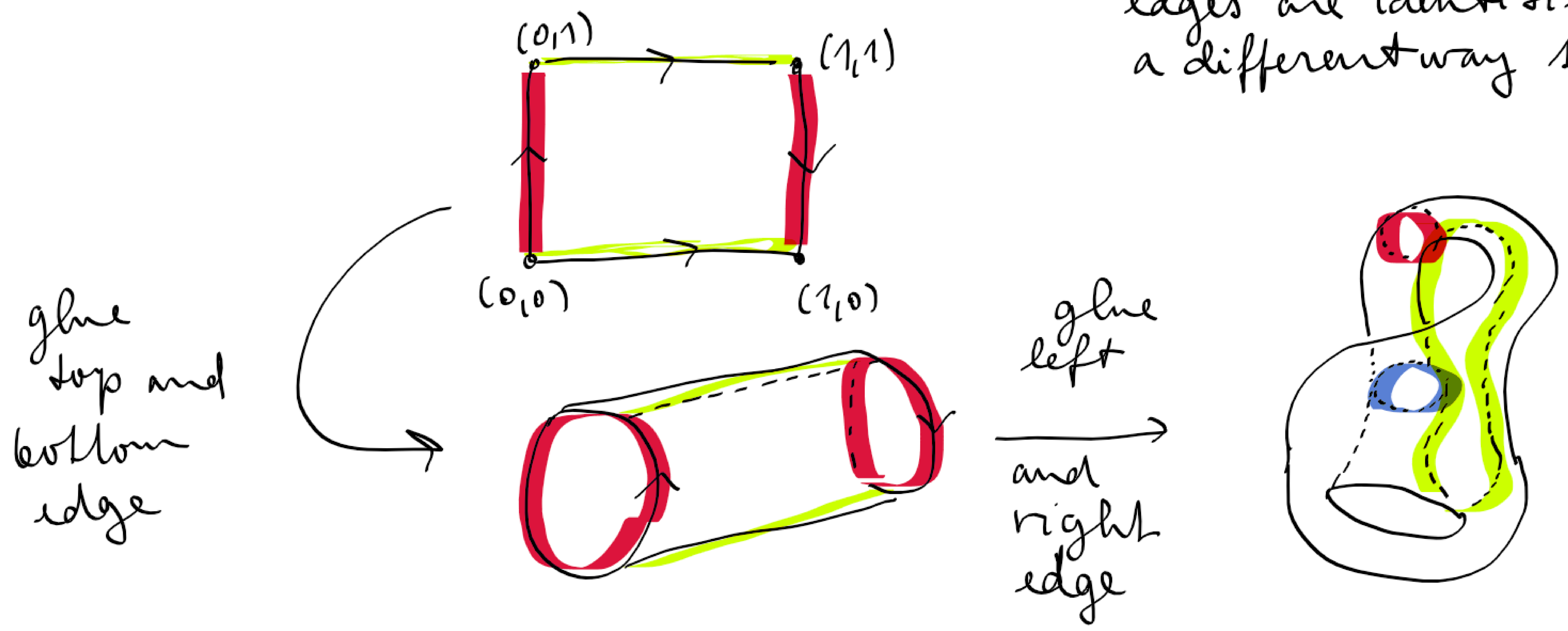
Example (Torus T^2)

Let us now consider the topological product $(\bar{I}\times\bar{I},\ \bar{\tau}_I\times\bar{\tau}_I)$. The small open squares as in Picture (P1) form basis for the topology $\bar{\tau}_I \times \bar{\tau}_I$.

(P1)



Example (Klein bottle) Let us start with the unit closed square $I^2 = I\times I$ with the subspace topology ($I^2 \subseteq \mathbb{R}^2$) or equivalently the product topology $\tau_I \times \tau_I$. On I^2 consider equivalence generated by $(x,0)\sim(x,1)$ and $(0,x)\sim(1,1-x)$ for $x\in[0,1]$. (Note that right and left edges are identified in a different way than above.)



This is only a model for the Klein bottle which is actually NOT homeomorphic to the Klein bottle problematic are those points where the bottle intersects itself (blue colour)

Theorem There is no map from the Klein bottle into $\mathbb{R}^3$ which is a homeomorphism onto its image!

Remark The torus T^2 and the Klein bottle are examples of spaces that locally look as $\mathbb{R}^2$. They are so called topological manifolds of dimension 2.

Topological manifolds

Definition A topological manifold of dimension n is a topological space (X,τ) together with atlas $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^n;\ \alpha \in A\}$ such that:

(TM1) for every $\alpha \in A$ we have that $U_\alpha \in \tau$ and $\bigcup_{\alpha\in A} U_\alpha = X$,

(TM2) for every $\alpha \in A$ the map φ_α is a homeomorphism onto its image and $\varphi_\alpha(U_\alpha)$ is open in $\mathbb{R}^n$,

(TM3) (X,τ) is Hausdorff and

(TM4) (X,τ) is second countable.

We call any map φ_α a chart.

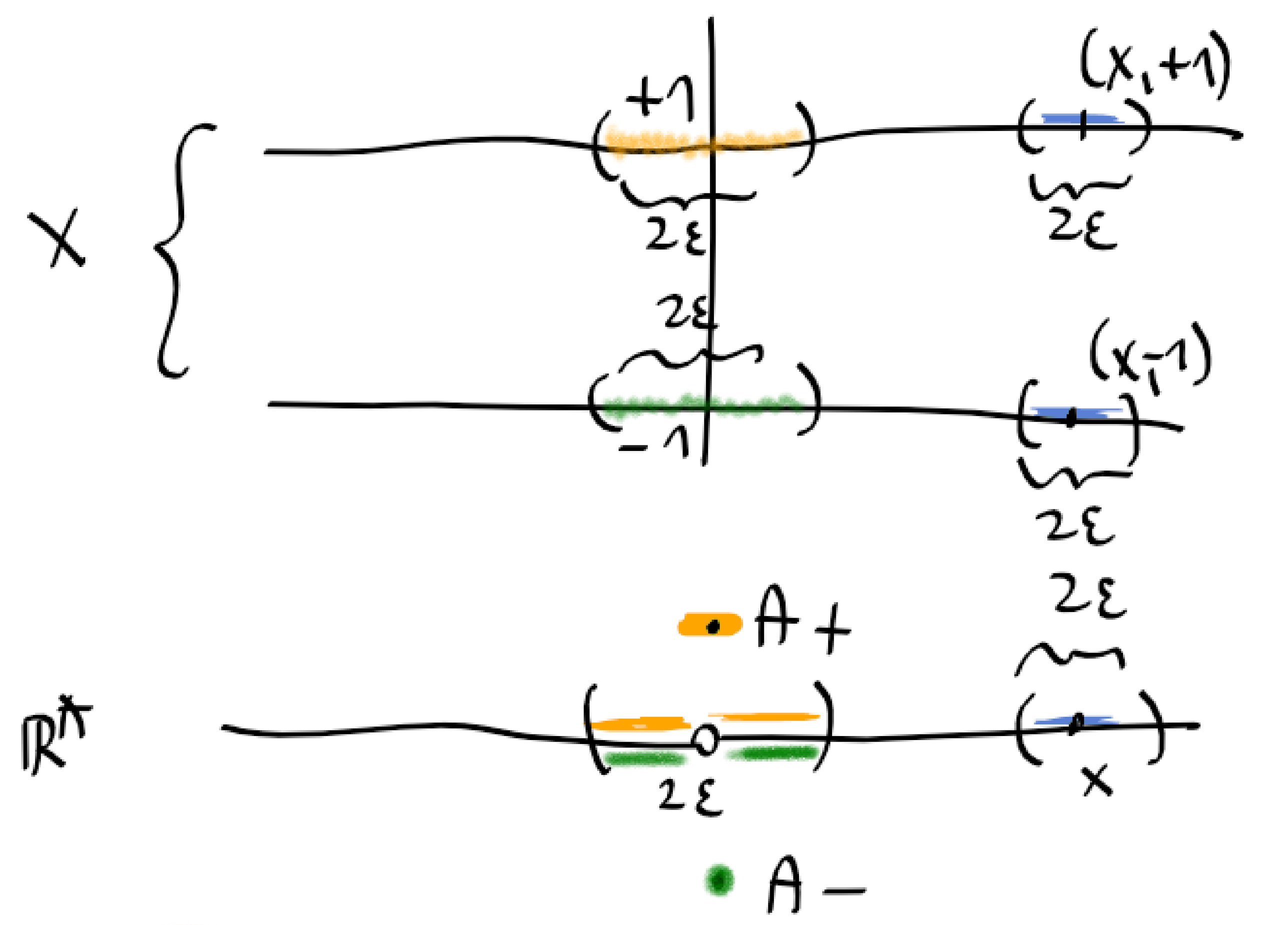
Remark $\{U_\alpha;\ \alpha \in A\}$ is only a subset of τ, need not be equal to τ.

Definition A topological space (X,τ) is called Hausdorff if for every x and $x' \in X$ there are $U_x, U_{x'} \in \tau$ with

$$U_x \cap U_{x'} = \emptyset \quad \text{and} \quad x \in U_x,\ x' \in U_{x'}.$$

Example: (i) $(\mathbb{R}^n, \tau_{MIN})$ is not Hausdorff while $(\mathbb{R}^n, \tau^n)$ is.

(ii) Let $X = \{(x,i) : x \in \mathbb{R},\ i = \pm 1\}$ be the union of two lines in $\mathbb{R}^2$ with the subspace topology τ_X.
Consider equivalence $\sim$ on X generated by $(x,-1) \sim (x,+1)$, $x \neq 0$.
Then $\overline{X} \to \mathbb{R}^* \cup \{A_-, A_+\}$, $[(x,i)] \mapsto x$, $x \neq 0$, $i = \pm 1$
$[(0,-1)] \longmapsto A_-$,
$[(0,+1)] \longmapsto A_+$,
where $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$, is a bijection of sets.
$[\ \ ]$ denotes the corresponding equivalence classes.



As for $\overline{\mathbb{R}}$, it is easy to see that
$\overline{\mathcal{B}} = \{p(B) : B \in \mathcal{B}\}$,
where $\mathcal{B}$ is a basis for τ_X, is a basis for $\overline{\tau}_X$.
Now it is easy to see that any open neighbourhood of A_+ intersects each open neighbourhood of A_-. Thus $(\overline{X}, \overline{\tau}_X)$ is not Hausdorff.

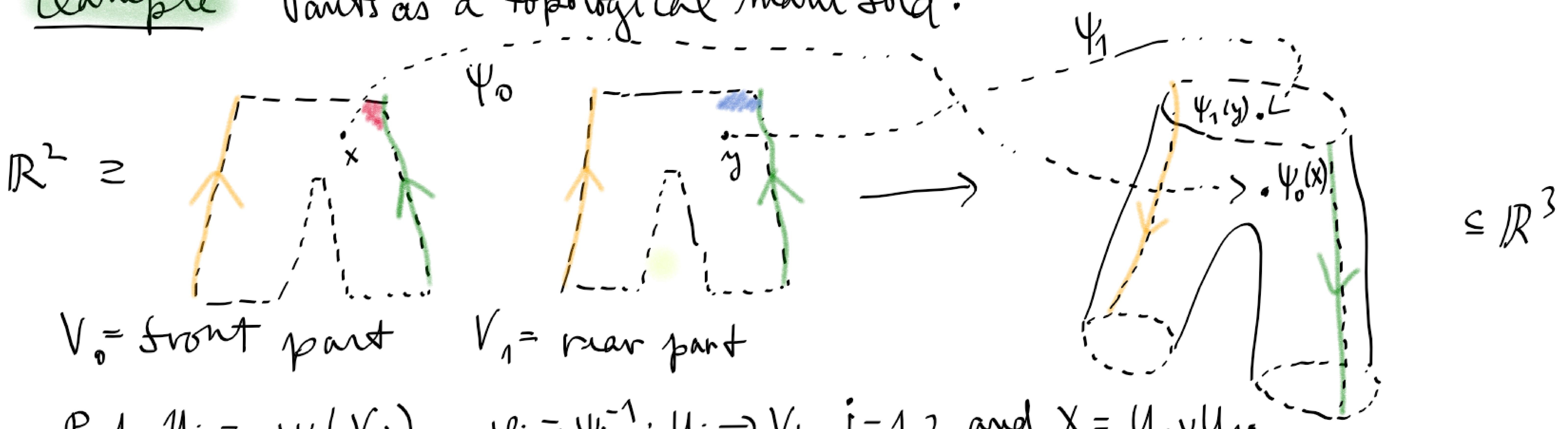
Definition A topological space (X,τ) is second countable if τ admits a countable basis.

Examples (i) $(\mathbb{R}^n, \tau^n)$ is second countable while $(\mathbb{R}^n, \tau_{MAX})$ is not.

(ii) Sorgenfrey line
Put $\mathcal{S} := \{[a,b) : a, b \in \mathbb{R}\}$. Then $\mathcal{S}$ satisfies (B1) and (B2) from Theorem A. Let τ_e be the topology on $\mathbb{R}$ generated by $\mathcal{S}$, hence $U \in \tau_e$ iff $\forall x \in U\ \exists \varepsilon > 0 : [x, x+\varepsilon) \subseteq U$.
We claim that τ_e is not second countable.
Assume that $\mathcal{S}'$ is any basis of τ_e and pick $x \in \mathbb{R}$. Then $[x, x+1) \in \tau_e$ and so $[x, x+1) = \bigcup_{S \in \mathcal{S}',\ S \subseteq [x,x+1)} S$. This implies that $x \in S_x$ for some $S_x \in \mathcal{S}'$ with $S_x \subseteq [x, x+1)$. This shows that $x = \min S_x$. Thus $\mathcal{S}'$ has at least the cardinality of $\mathbb{R}$.

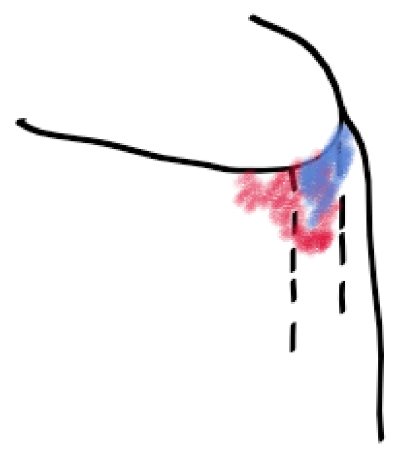
Remark: Topological variety of dimension n is a topological space that locally looks as $\mathbb{R}^n$, that is every point has neighborhood homeomorphic to $\mathbb{R}^n$.

Example Pants as a topological manifold.



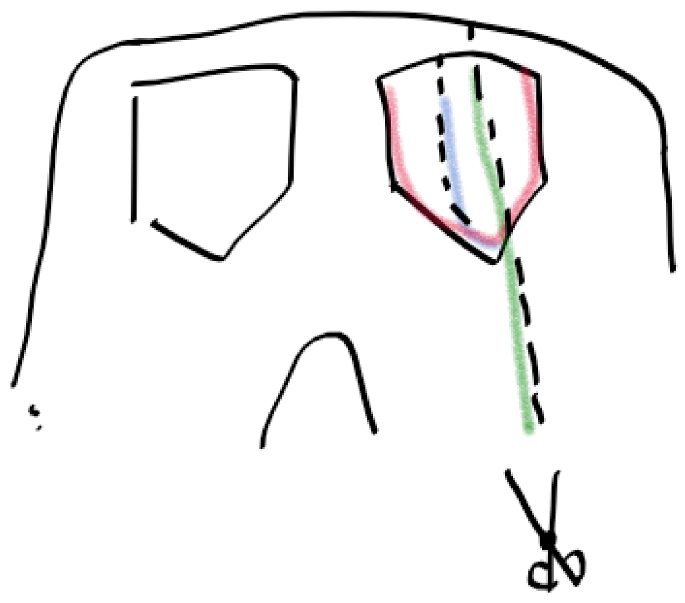
V_0 = front part $\quad V_1$ = rear part

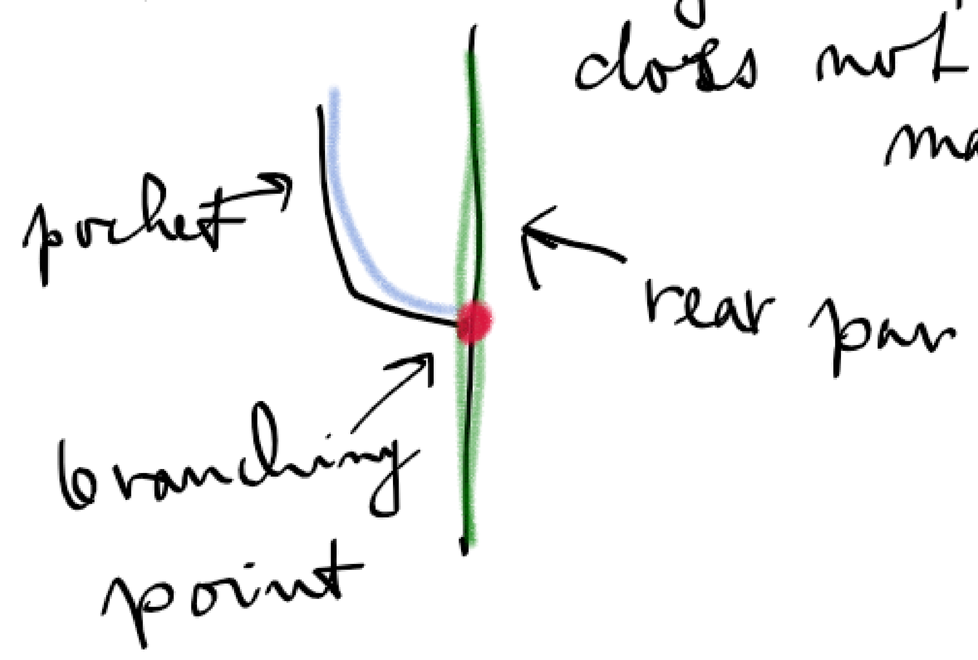
Put $U_i = \psi(V_i)$, $\varphi_i = \psi_i^{-1} : U_i \to V_i$, $i = 1,2$ and $X = U_0 \cup U_1$.
Then $X \subseteq \mathbb{R}^3$ and τ_X be the subspace topology on X. The set $\{\varphi_i : U_i \to \mathbb{R}^2 \mid i = 0,1\}$ is an atlas on X.



As $X = U_0 \cup U_1$ and U_i are open in X, then the front and the rear part overlap.

If we sew pockets to pants, then this is no longer a topological manifold. If we cut pants along one pocket, then the cut does not look as a 1-dimensional manifold.





Remarks: 1) Many important information about a given topological manifold X can be recovered from the set of continuous functions $X \to \mathbb{R}$. It is certainly more convenient to work with the class of differentiable functions which essentially carry the same information as continuous functions.
So the next step leads to the possibility of defining the notion of smooth functions on topological manifold (topological $\to$ smooth manifold).

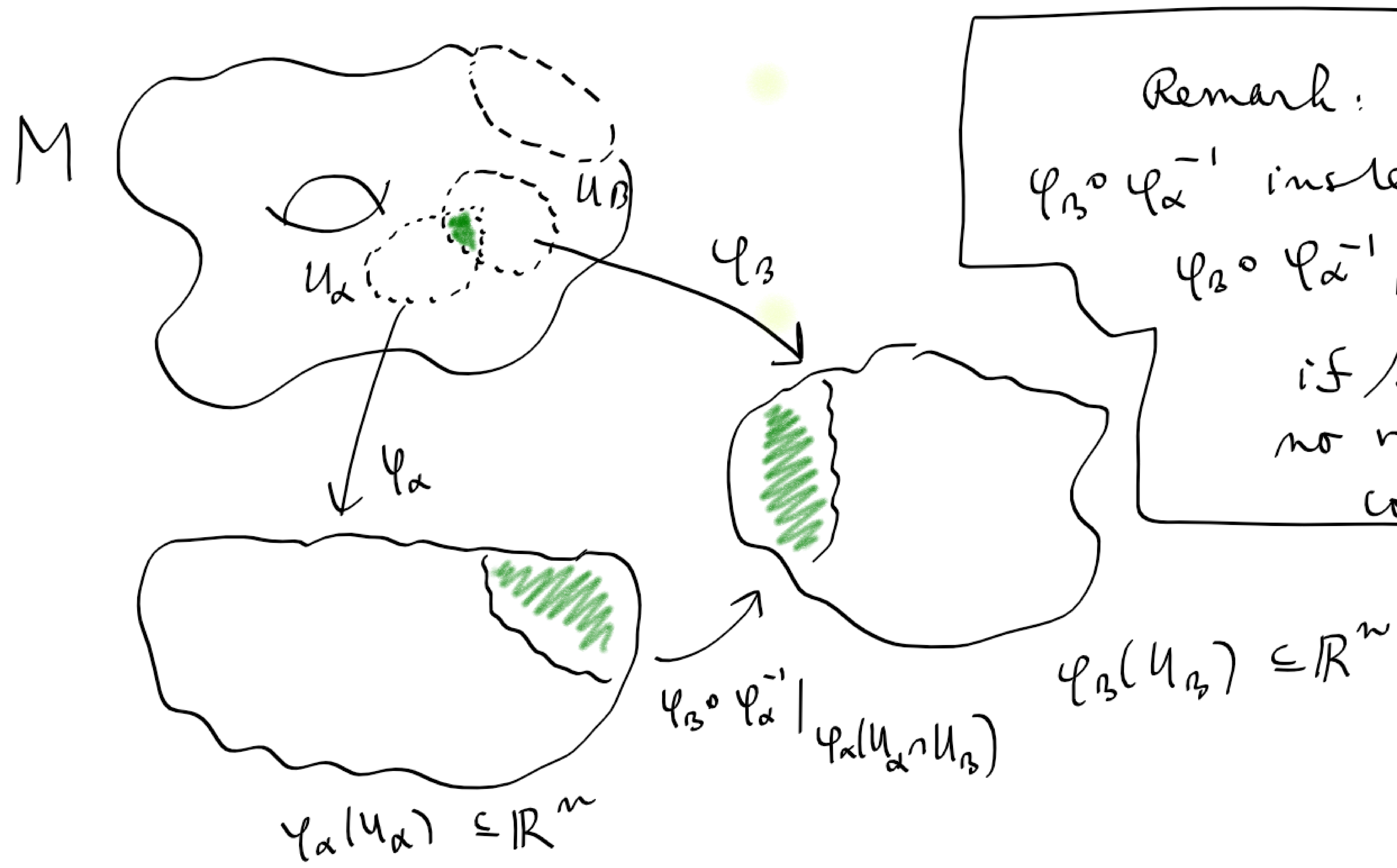
2) We will write $(X, \mathcal{A})$ to denote topological manifold (X, τ) with atlas $\mathcal{A}$.

Smooth manifolds

Definition Let $(M, \mathcal{A})$ be a topological manifold of dimension n. We say that $(M, \mathcal{A})$ is a smooth manifold of dimension n with smooth atlas $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^n : \alpha \in A\}$ if for every $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq 0$ we have that

(TF) $\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha \cap U_\beta)} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$

is smooth. We call each $\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha \cap U_\beta)}$ a transition function.



Remark: we will write $\varphi_\beta \circ \varphi_\alpha^{-1}$ instead of $\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha \cap U_\beta)}$ if there is no risk of confusion.

Examples (i) $(\mathbb{R}^n, \tau^n)$ with $\mathcal{A} = \{ Id : \mathbb{R}^n \to \mathbb{R}^n \}$ is the standard smooth atlas on $\mathbb{R}^n$.

(ii) If $(M, \mathcal{A})$ is a smooth manifold of dimension n and U is an open subset of M, then also U is a smooth manifold of dimension n with atlas

$$\mathcal{A}_U = \{ \varphi_\alpha|_{U \cap U_\alpha} : U_\alpha \cap U \to \mathbb{R}^n \mid \alpha \in A \}.$$

Here $\mathcal{A}$ is as in Definition above.

(iii) If $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ are smooth manifolds of dimensions m and n, respectively, then $(M \times N, \mathcal{A}_M \times \mathcal{A}_N)$ is a smooth manifold of dimension $m+n$ with atlas

$$\mathcal{A}_M \times \mathcal{A}_N = \{ \varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n} : (\alpha, \beta) \in A \times B \}.$$

Here $\mathcal{A}_M = \{ \varphi_\alpha : U_\alpha \to \mathbb{R}^m : \alpha \in A \}$ and $\mathcal{A}_N = \{ \psi_\beta : V_\beta \to \mathbb{R}^n : \beta \in B \}$.

(iv) Standard smooth structure on $S^n = \{ x \in \mathbb{R}^{n+1} : \|x\|_2 = 1 \}$.
On S^n we consider the subspace topology. Let
$N := (0, \dots, 0, 1)$ and $S := (0, \dots, 0, -1)$.
Then $U_N = S^n \setminus \{N\}$, $U_S = S^n \setminus \{S\}$ are open in S^n and $S^n = U_N \cup U_S$. Let

$$\varphi_S : U_S \to \mathbb{R}^n, \quad \varphi_S(x) = \frac{1}{1 + x_{n+1}} (x_1, \dots, x_n), \qquad x = (x_1, \dots, x_{n+1}).$$
$$\varphi_N : U_N \to \mathbb{R}^n, \quad \varphi_N(x) = \frac{1}{1 - x_{n+1}} (x_1, \dots, x_n)$$

Then φ_S and φ_N are homeomorphisms (we will not prove this) and $\varphi_S(U_N \cap U_S) = \varphi_N(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\}$.

and $\varphi_S \circ \varphi_N^{-1} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$

$$\varphi_S \circ \varphi_N^{-1}(x) = \frac{x}{\|x\|} \text{ is smooth.}$$

Thus $\mathcal{A} = \{ \varphi_S : U_S \to \mathbb{R}^n, \ \varphi_N : U_N \to \mathbb{R}^n \}$ is a smooth atlas on S^n.

(v) Projective space $\mathbb{RP}^n$

Let $\sim$ be the equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ given by
$$x \sim y \iff \exists \lambda \in \mathbb{R}^* : \lambda x = y. \quad (\mathbb{R}^* = \mathbb{R} \setminus \{0\}.)$$
Let $\mathbb{RP}^n := (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$ and p be the canonical projection $\mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{RP}^n$ and $[x]$ be the equivalence class of x. Note that $[x]$ can be regarded as the line in $\mathbb{R}^{n+1}$ passing through 0 and x. We endow $\mathbb{RP}^n$ with the quotient topology.

Then $U_k := \{ [x] \in \mathbb{RP}^n : x_k \neq 0 \}$ is well defined open subset of $\mathbb{RP}^n$ for $k = 1, \dots, n+1$. Consider
$$\varphi_k : U_k \longrightarrow \mathbb{R}^n, \ \varphi_k([x]) = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}) / x_k .$$

Then φ_k is well defined, it is a homeomorphism with inverse
$$\varphi_k^{-1} : \mathbb{R}^n \longrightarrow U_k,$$
$$\varphi_k^{-1}(x) = [(x_1, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_n)].$$

Moreover $U_k \cap U_\ell = \{ [x] \in \mathbb{RP}^n : x_k \neq 0, x_\ell \neq 0 \}$. If we for simplicity assume that $k < \ell$, then

$$\varphi_\ell \circ \varphi_k^{-1} : \mathbb{R}^n \setminus \{ x : x_\ell = 0 \} \longrightarrow \mathbb{R}^n \setminus \{ x : x_k \neq 0 \}$$

$$\varphi_\ell \circ \varphi_k^{-1}(x) = (x_1, \dots, x_{k-1}, x_k, \dots, x_{\ell-1}, x_{\ell+1}, \dots, x_{n+1}) / x_\ell$$

is smooth and so $\mathcal{A} = \{ \varphi_k : U_k \to \mathbb{R}^n \mid k = 1, \dots, n+1 \}$ is a smooth atlas on $\mathbb{RP}^n$.

Here x_k always denotes the k-th component of $x \in \mathbb{R}^{n+1}$!