Plan: 1) review of gen. topology
2) operations on top. spaces
3) top. and smooth manifolds

1) $P(X)$.. the power set of set X, the set all subsets of X

Def A topological space is (X,τ) where X is a set and $\tau \subseteq P(X)$ s.t.:

(T1) $\emptyset, X \in \tau$,

(T2) $U_1, \dots, U_m \in \tau \Rightarrow \bigcap_{i=1}^{m} U_i \in \tau$,

(T3) $U_\alpha \in \tau$ for $\forall \alpha \in A \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \tau$.

τ is called a topology on X and any $U \in \tau$ is called an open subset of X.

Example $U \subseteq \mathbb{R}^n$ is open $\Leftrightarrow$

$$(O)\ \forall x \in U\ \exists \varepsilon > 0 : U_\varepsilon(x) \subseteq U$$

$$U_\varepsilon(x) := \{ y \in \mathbb{R}^n : \|y - x\|_2 < \varepsilon \}$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}, \quad x = (x_1, \dots, x_n)$$

Claim: $\tau^n := \{ U \subseteq \mathbb{R}^n : U \text{ satisfies } (O) \}$ is a topology on $\mathbb{R}^n$.

$\mathcal{B} := \{ U_\varepsilon(x) : \varepsilon > 0,\ x \in \mathbb{R}^n \}$

$U \subseteq \mathbb{R}^n$ is open $\Leftrightarrow$ $U = \bigcup_{V \in \mathcal{B},\, V \subseteq U} V$

$\mathcal{B}$ is a basis of τ^n or that $\mathcal{B}$ generates τ^n

Definition Let (X,τ) be a top. space. Then $\mathcal{B} \subseteq P(X)$ is called a basis for τ if $\mathcal{B} \subseteq \tau$ and

$$(G)\ \forall U \in \tau\ \forall x \in U\ \exists V \in \mathcal{B} : x \in V \subseteq U.$$

Question Let X be a set and $\mathcal{B} \subseteq P(X)$. When is

$$(*)\ \tau := \{ U \subseteq X : U \text{ satisfies } (G) \}$$

a topology on X?

Theorem Assume that

(B1) $\forall x \in X\ \exists V \in \mathcal{B} : x \in V$ and

(B2) $\forall V_1, V_2 \in \mathcal{B}\ \forall x \in V_1 \cap V_2\ \exists W \in \mathcal{B} : x \in W \subseteq V_1 \cap V_2$.

Then τ defined in $(*)$ is a topology on X.

Def We say that τ is generated by $\mathcal{B}$.

Remark The condition $(*)$ is equivalent to the fact that τ is the smallest topology (w.r.t. $\subseteq$) such that $\mathcal{B} \subseteq \tau$.

Definition Let (X, τ_X) and (Y, τ_Y) be top. spaces. $f: X \to Y$ is called continuous if $U \in \tau_Y \Rightarrow f^{-1}(U) \in \tau_X$. In case that f has inverse which is also continuous, then f is called to be a homeomorphism.

Example $f: \mathbb{R}^n \to \mathbb{R}^m$

(AD) $U \in \tau^m \Rightarrow f^{-1}(U) \in \tau^n$

$(\varepsilon - \delta\ D)$ $\forall x \in \mathbb{R}^m\ \forall \varepsilon > 0\ \exists \delta > 0 : y \in U_\delta(x) \Rightarrow f(y) \in U_\varepsilon(f(x))$.

$(AD) \Leftrightarrow (\varepsilon - \delta\ D)$

2) Operations on top. spaces

A. Topological product

(X_i, τ_i), $i = 1, \dots, n$, question: is there a natural topology on $\prod_{i=1}^{n} X_i = X_1 \times \dots \times X_n$.

$\mathcal{B}_\Pi := \{ U_1 \times \dots \times U_n : U_i \in \tau_i,\ i = 1, \dots, n \}$ satisfies (B1) and (B2). So it generates topology

$\tau_1 \times \dots \times \tau_n = \prod_{i=1}^{n} \tau_i$, called the product topology.

We call $\left(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} \tau_i\right)$ the topological product.

Example $\mathbb{R}^n = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{n\text{-copies}}$ $\quad \tau^1 \times \dots \times \tau^1$ on $\mathbb{R}^n$?

$\|x\|_{+\infty} = \max_{i=1,\dots,n} |x_i|$, $U_{\varepsilon,+\infty}(x) := \{ y \in \mathbb{R}^n : \|x - y\|_{+\infty} < \varepsilon \}$

$\|x\|_{+\infty}$ is equivalent to $\|x\|_2 \Rightarrow$

$\|\cdot\|_{+\infty}$ and $\|\cdot\|_2$ generate the same topology $\tau^n \Rightarrow$

$U \subseteq \mathbb{R}^n \Leftrightarrow$

$(O_{+\infty})$ $\forall x \in U\ \exists \varepsilon > 0 : U_{\varepsilon,+\infty}(x) \subseteq U$.

$U_{\varepsilon,+\infty}(x) = U_\varepsilon(x_1) \times \dots \times U_\varepsilon(x_n) \in \mathcal{B}_\Pi$

$x = (x_1, \dots, x_n)$

$U \in \tau^n \Rightarrow U \text{ sat. } (O_{+\infty}) \Rightarrow U \in \tau^1 \times \dots \times \tau^1$

any $V \in \mathcal{B}_\Pi$ is open in $\mathbb{R}^n$, that is $V \in \tau^n$, since $\tau^1 \times \dots \times \tau^1$ is the smallest topology that contains $\mathcal{B}_\Pi$ and $\mathcal{B}_\Pi \subseteq \tau^n$

$\Rightarrow \tau^1 \times \dots \times \tau^1 \subseteq \tau^n$

$\tau^1 \times \dots \times \tau^1 = \tau^n$

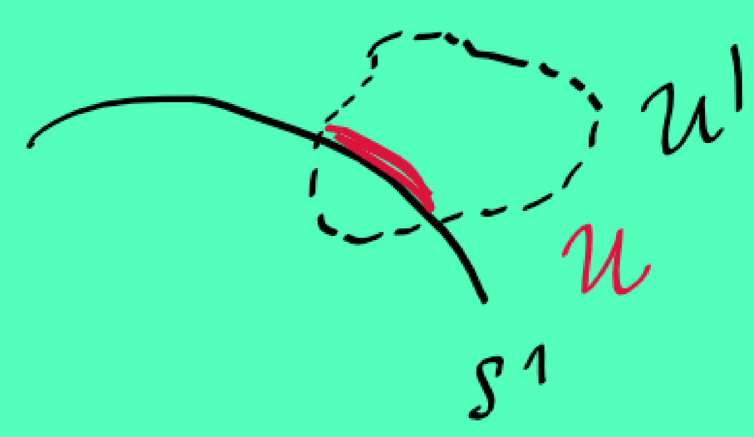
B. Induced topology

(Y,τ) .. top-space, X .. be a set, $f: X \to Y$

- then $f^{-1}\tau = \{f^{-1}(\mathcal{U}) : \mathcal{U} \in \tau\}$ is a topology on X, called the induced topology
- if $\mathcal{B}$ is a basis for τ, then $f^{-1}\mathcal{B} = \{f^{-1}(V) : V \in \mathcal{B}\}$ is a basis for $f^{-1}\tau$
- $X \subseteq Y$, $f: X \hookrightarrow Y$ inclusion, then $f^{-1}\tau$ is called the subspace topology

<u>Example</u> $X = S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$, $Y = \mathbb{R}^2$, τ^2

$f^{-1}\tau^2$ on S^1



$\mathcal{U} \subseteq S^1$ is open iff $\exists\, \mathcal{U}' \in \tau^2$: $\mathcal{U} = S^1 \cap \mathcal{U}'$

<u>C. Quotient topology</u>

(X,τ) .. top. space, equivalence $\sim$ on X, $\overline{X} := X/\sim$, $p: X \to \overline{X}$,

$[x]$ .. eq. class of $x \in X$

$\overline{\tau} := \{\mathcal{U} \subseteq \overline{X} : p^{-1}(\mathcal{U}) \in \tau\}$ ... quotient topology

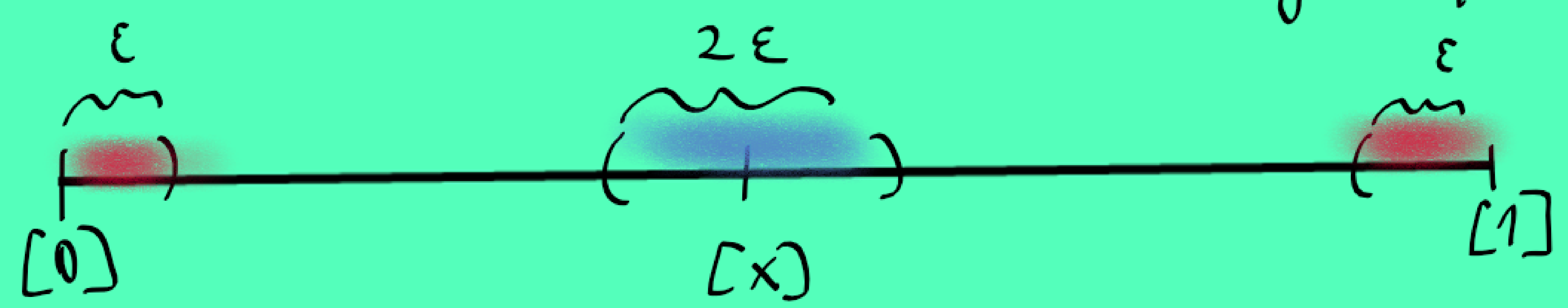
<u>Examples</u> (i) $X = I = [0,1] \subset \mathbb{R}$ subspace topology τ_I

$$\mathcal{U} \in \tau_I \iff \forall x \in \mathcal{U}\ \exists \varepsilon > 0 : \begin{cases} \mathcal{U}_\varepsilon(x) \subseteq \mathcal{U}, & x \in (0,1) \\ [0,+\varepsilon) \subseteq \mathcal{U}, & x = 0 \\ (1-\varepsilon, 1] \subseteq \mathcal{U}, & x = 1 \end{cases}$$

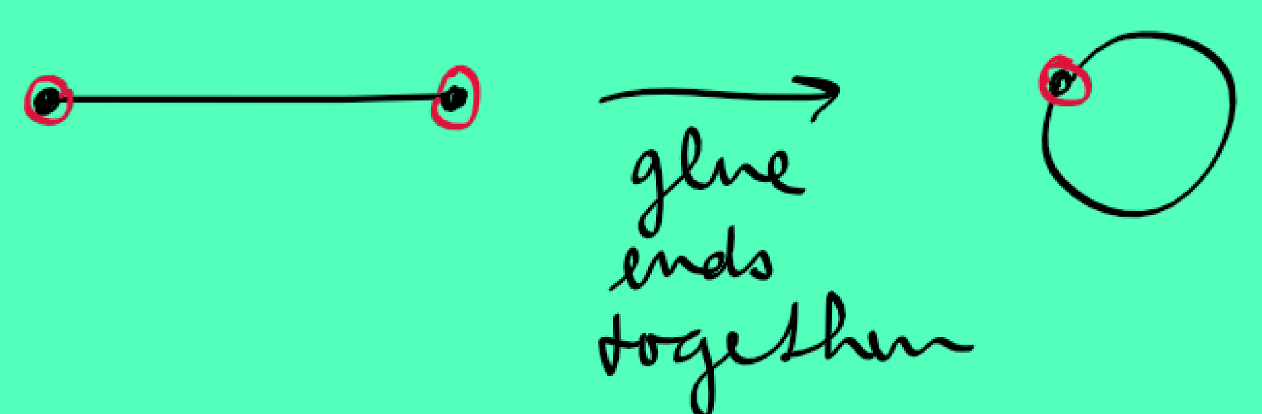
on X consider equivalence gen. by $0 \sim 1$

$\overline{X} := X/\sim$, $\overline{\tau}_I = ?$, $[x]$.. eq. class of $x \in I$

$$\mathcal{U} \in \overline{\tau}_I \iff \forall [x] \in \mathcal{U}\ \exists\, \varepsilon > 0 \begin{cases} \forall y \in \mathcal{U}_\varepsilon(x) : [y] \in \mathcal{U}, & x \in (0,1) \\ \forall y \in [0,+\varepsilon) \cup (1-\varepsilon, 0] \in \mathcal{U}, & x = 0 \text{ or } x = 1 \end{cases}$$



$\overline{I}$

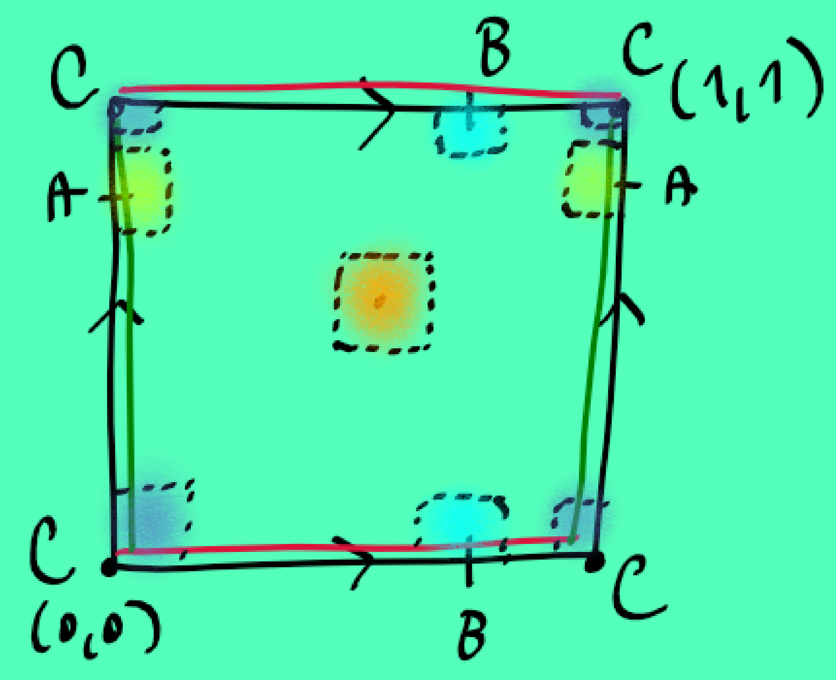


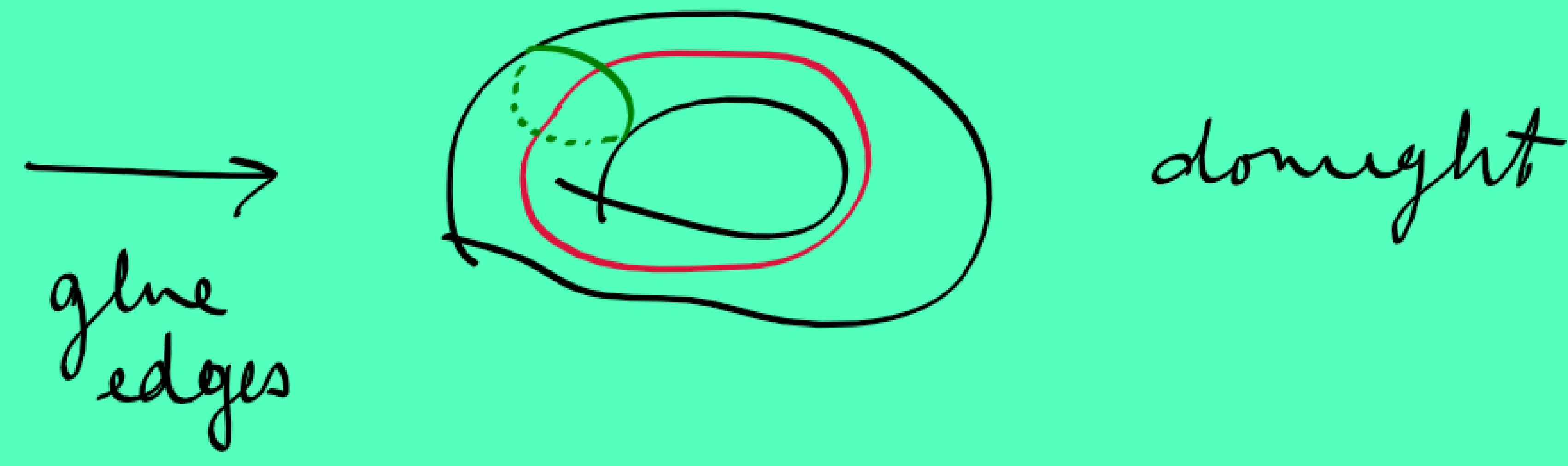
$\varphi: \overline{I} \to S^1$, $\varphi([x]) = (\cos 2\pi x, \sin 2\pi x)$

is well defined, bijective, homeomorphism

(2) Torus T^2 $\quad (\bar{I} \times \bar{I}, \overline{\tau_I \times \tau_I})$

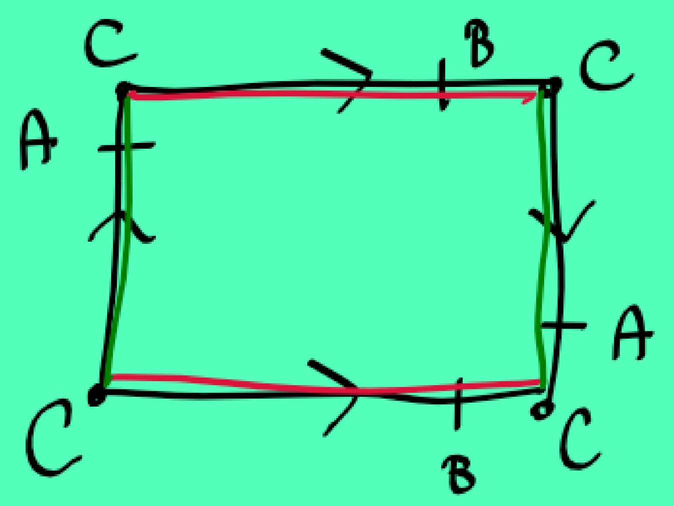
$(x,0) \sim (x,1) \quad x \in [0,1]$
$(0,y) \sim (1,y) \quad y \in [0,1]$

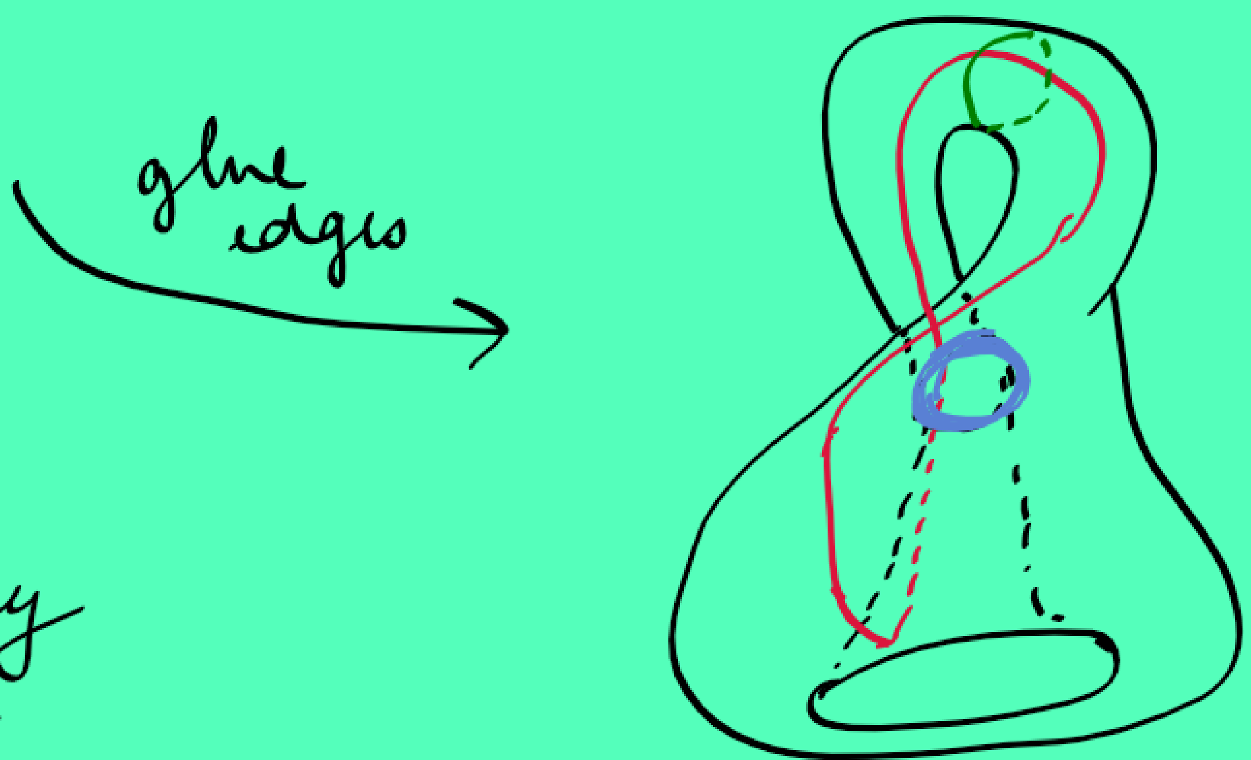




(3) Klein bottle

$(x,0) \sim (x,1) \quad x \in [0,1]$
$(0,y) \sim (1,1-y) \quad y \in [0,1]$





not homeomorphic to the original quotient space

Theorem
There is no embedding of the Klein bottle into $\mathbb{R}^3$ which is a homeomorphism onto its image.

Remark Torus and the Klein bottle are topo. spaces that locally look like $\mathbb{R}^2$, they are examples of 2-dim. top. manifolds.

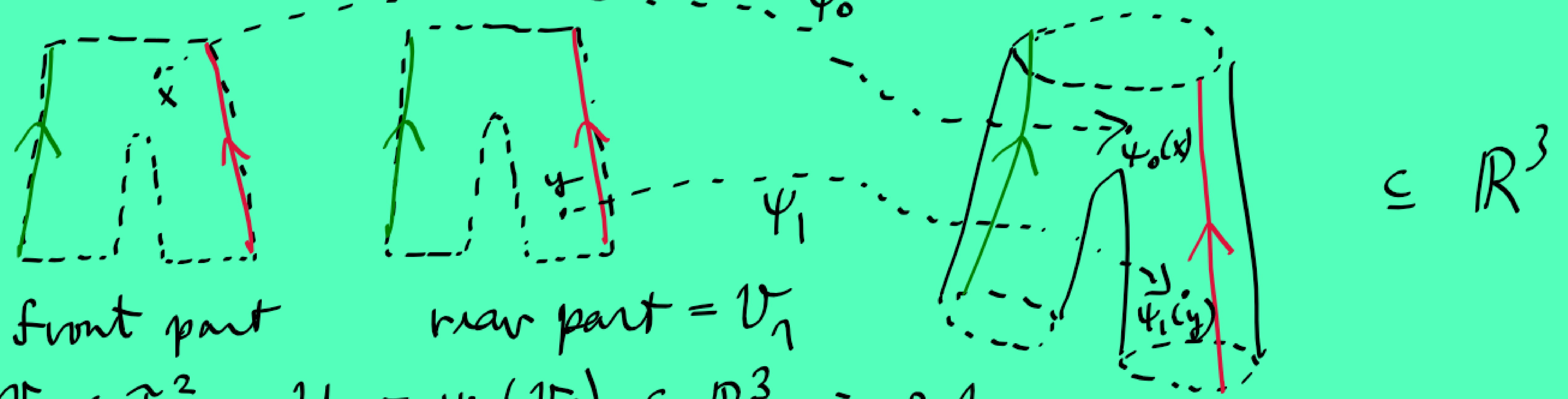
3) Topological and smooth manifolds

Definition A topological manifold of dimension n is topological space (X,τ) with atlas $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^n : \alpha \in A\}$ such that:

(TM1) $\forall \alpha \in A : U_\alpha \in \tau$ and $\bigcup_{\alpha \in A} U_\alpha = X$,
(TM2) $\forall \alpha \in A$: φ_α is a homeomorphism onto its image and $\varphi_\alpha(U_\alpha) \in \tau^n$,
(TM3) (X,τ) is Hausdorff and
(TM4) (X,τ) is second countable.

Example Pants as 2-dim. top. manifold

$\mathbb{R}^2$

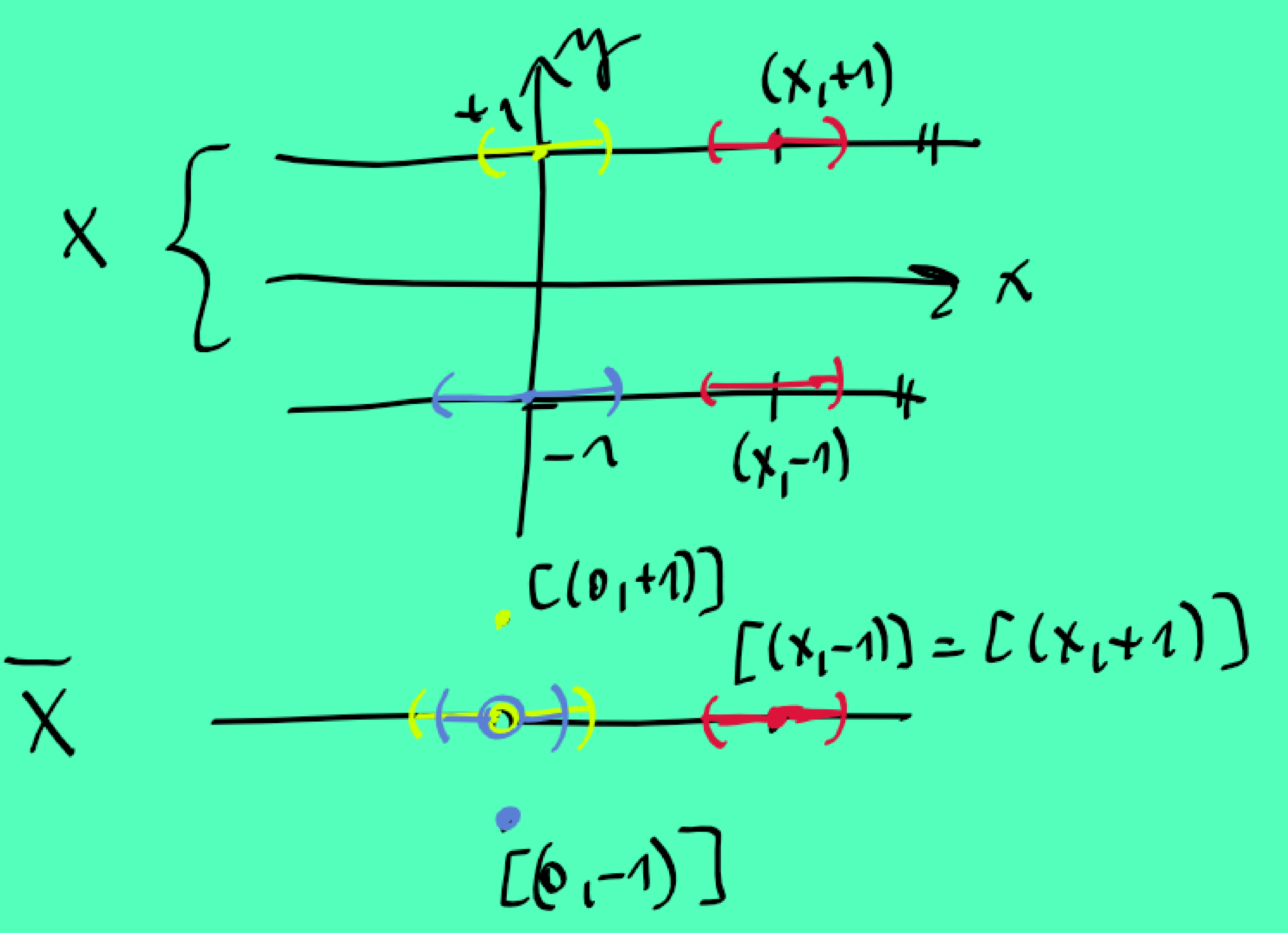


$\subseteq \mathbb{R}^3$

V_0 = front part $\qquad$ rear part = V_1
$V_0, V_1 \in \tau^2$, $\quad U_i = \psi_i(V_i) \subseteq \mathbb{R}^3$, $i = 0,1$
$X = U_0 \cup U_1 \subseteq \mathbb{R}^3 \quad \tau_X$ the subspace topology
ψ_i are homeomorphisms, $\varphi_i = \psi_i^{-1}$, $i=0,1$
$\mathcal{A} = \{\varphi_i : U_i \to \mathbb{R}^2 : i = 0,1\}$

Example Top. space that has the property that every point has an open neigh. which is homeom. to an open subset of $\mathbb{R}$ but which is not a top. manifold as it is not Hausdorff.

$X = \{(x, \pm 1) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ $\quad \tau_X$.. subspace top.



$\sim$ on X gen. by $(x,-1) \sim (x,+1)$ for $x \neq 0$

$\bar{X} := X/\sim \qquad p : X \to \bar{X}$

$[x]$.. eq. class of $x \in X$

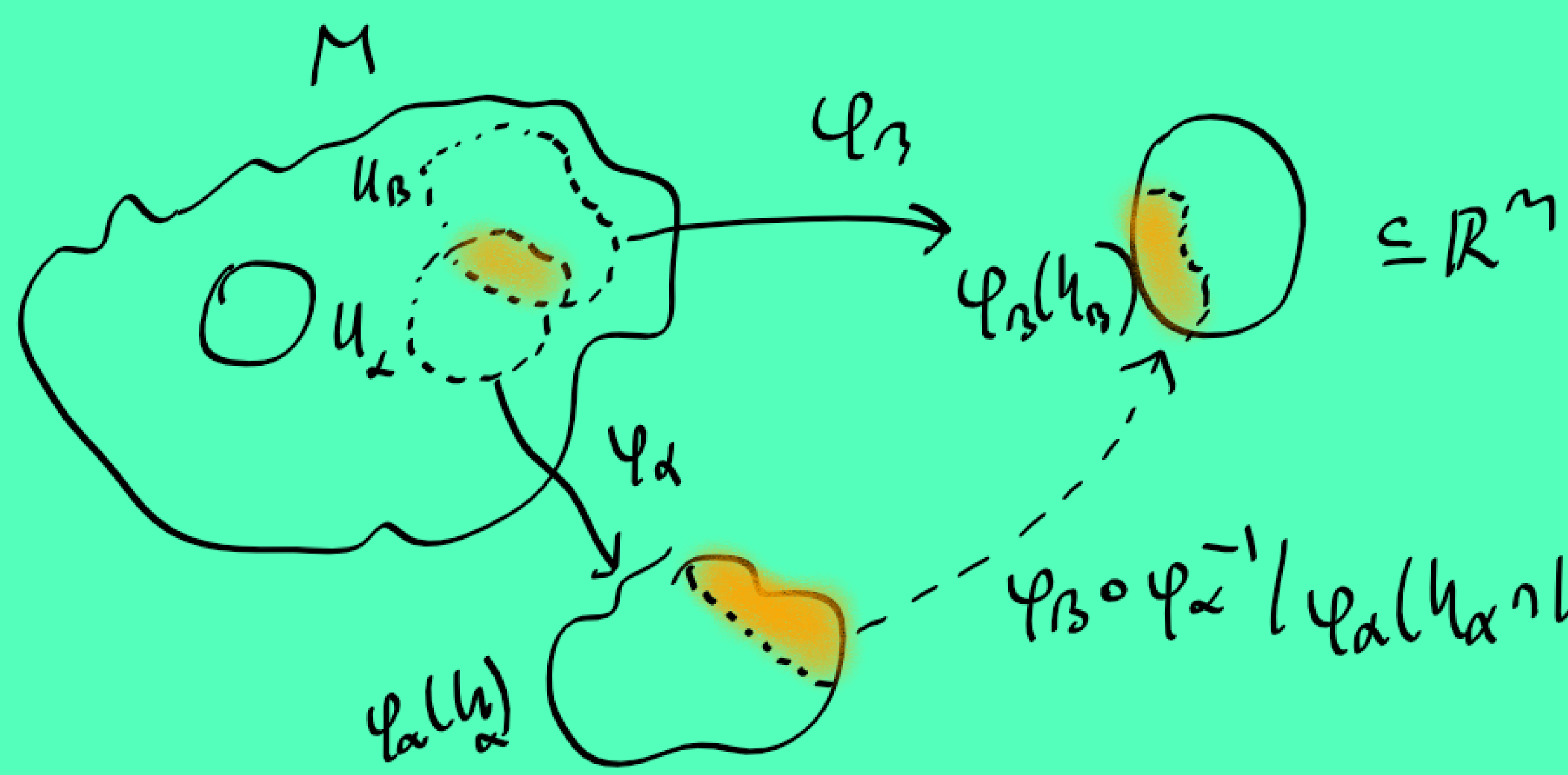
any open neighbourhood of $[(0,+1)]$ meets any open neigh. of $[(0,-1)]$

$\Rightarrow$ $\bar{X}$ is not Hausdorff

Definition Smooth manifold of dimension n is a top. manifold (M, τ) of dim. n with atlas $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^n : \alpha \in A\}$ such that $\forall \alpha, \beta \in A$ s.t. $U_\alpha \cap U_\beta \neq \emptyset$ we have that

(TF) $$\varphi_\beta \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap U_\beta)} \quad \underset{\mathbb{R}^n}{\underset{\cap}{\varphi_\alpha(U_\alpha \cap U_\beta)}} \to \underset{\mathbb{R}^n}{\underset{\cap}{\varphi_\beta(U_\alpha \cap U_\beta)}}$$

is smooth. We call each φ_α a chart and $\varphi_\beta \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap U_\beta)}$ a transition function.



Example Standard smooth atlas on $S^n = \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$

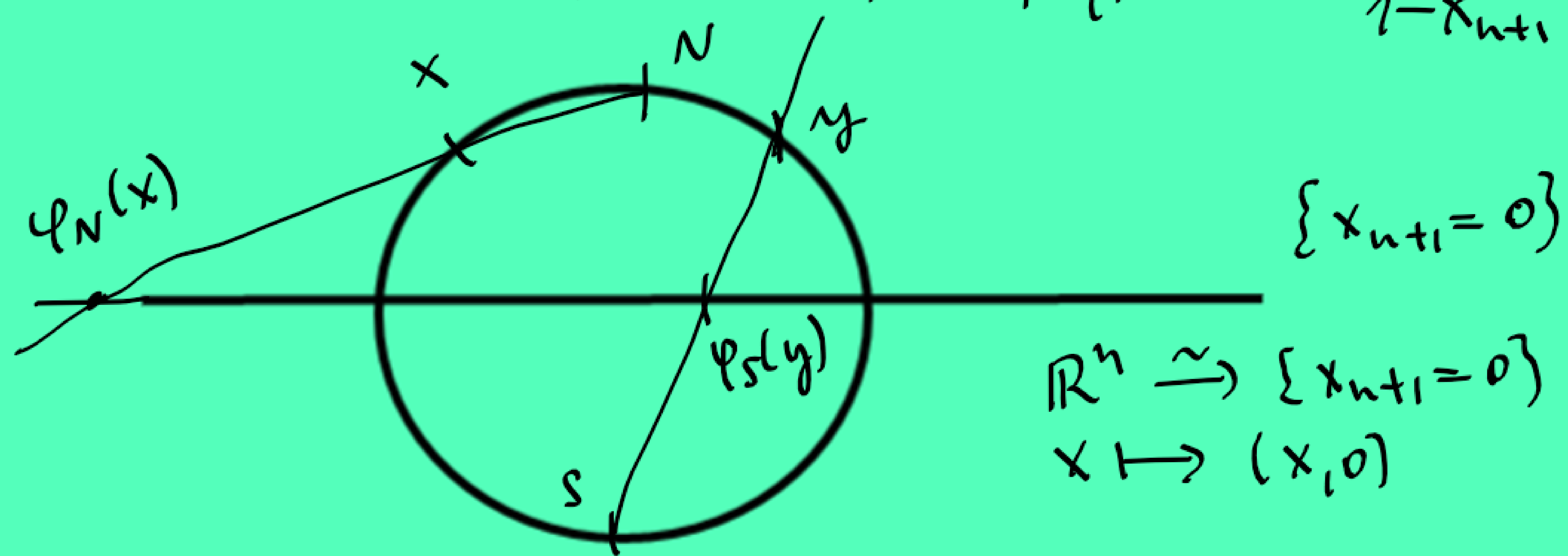
$S^n \subseteq \mathbb{R}^{n+1}$ subspace topology

$N = (0, .., 0, 1) \quad S = (0, .., 0, -1) \qquad U_N = S^n \setminus \{N\},\ U_S = S^n \setminus \{S\}$

U_N, U_S are open in S^n, $\quad U_N \cup U_S = S^n$

Let $\varphi_S : U_S \to \mathbb{R}^n$, $\varphi_S(x) = \frac{1}{1+x_{n+1}}(x_1, .., x_n)$

$\varphi_N : U_N \to \mathbb{R}^n$, $\varphi_N(x) = \frac{1}{1-x_{n+1}}(x_1, .., x_n)$



$\mathbb{R}^n \xrightarrow{\sim} \{x_{n+1} = 0\}$

$x \mapsto (x, 0)$

φ_S and φ_N are homeomorphisms, $\varphi_S(U_N \cap U_S) = \varphi_N(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\}$

$$\varphi_S \circ \varphi_N^{-1}\big|_{\mathbb{R}^n \setminus \{0\}} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$$

$$\varphi_S \circ \varphi_N^{-1}(x) = \frac{x}{\|x\|}$$ is smooth

$\mathcal{A} = \{\varphi_S : U_S \to \mathbb{R}^n, \ \varphi_N : U_N \to \mathbb{R}^n\}$ is a smooth