Smooth maps on manifolds

Last time we gave a definition of smooth manifold, this is a topological manifold $(M, \mathcal{U})$ with atlas $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^n \mid \alpha \in A\}$ where for each $\alpha, \beta \in A$:

$$\varphi_\beta \circ \varphi_\alpha^{-1}\Big|_{\varphi_\alpha(U_\alpha \cap U_\beta)} : \underset{\mathbb{R}^n}{\overset{\cap}{\varphi_\alpha(U_\alpha \cap U_\beta)}} \longrightarrow \underset{\mathbb{R}^n}{\overset{\cap}{\varphi_\beta(U_\alpha \cap U_\beta)}} \quad \text{is smooth.}$$

Now we can introduce a notion of smooth function on a smooth manifold.

Remark: If there is no risk of confusion, we will write simply f instead of $f|_U$.

Definition Let $(M, \mathcal{U})$ where $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^n \mid \alpha \in A\}$ be a smooth manifold of dimension n. A function $f : M \to \mathbb{R}$ is called smooth if for every $\alpha \in A$ the composition

$$f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \xrightarrow{\varphi_\alpha^{-1}} U_\alpha \xrightarrow{f} \mathbb{R} \text{ is smooth.}$$

We denote by $\mathcal{C}^\infty_{\mathcal{A}}(M)$ the set of all smooth functions on $(M, \mathcal{A})$.

Remark: Note that the definition of a smooth function depends on $\mathcal{A}$.

Examples: 1) Let $M = \mathbb{R}$ and $\mathcal{A}_1 = \{Id : \mathbb{R} \to \mathbb{R}\}$. Then $f \in \mathcal{C}^\infty_{\mathcal{A}_1}(\mathbb{R}) \iff$

$$\mathbb{R} \xrightarrow{Id} \mathbb{R} \xrightarrow{f} \mathbb{R} \quad \text{is smooth} \iff f : \mathbb{R} \to \mathbb{R} \text{ is smooth.}$$

Hence $\mathcal{C}^\infty_{\mathcal{A}_1}(\mathbb{R})$ is the space of smooth (in the standard sense from the course of analysis) functions $\mathcal{C}^\infty(\mathbb{R})$.

2) Let $M = \mathbb{R}$ and $\mathcal{A}_2 = \{exp : \mathbb{R} \to \mathbb{R}^+\}$. Then $(M, \mathcal{A}_2)$ is again a smooth manifold. Now

$$f \in \mathcal{C}^\infty_{\mathcal{A}_2}(\mathbb{R}) \iff \mathbb{R}^+ \xrightarrow{\ln} \mathbb{R} \xrightarrow{f} \mathbb{R} \text{ is smooth} \iff$$

$\mathbb{R} \xrightarrow{exp} \mathbb{R}_+ \xrightarrow{\ln} \mathbb{R} \xrightarrow{f} \mathbb{R}$ is smooth (since $\ln$ is a smooth inverse to exp) $\iff$ $\mathbb{R} \xrightarrow{f} \mathbb{R}$ is smooth. Hence $\mathcal{C}^\infty_{\mathcal{A}_2}(\mathbb{R}) = \mathcal{C}^\infty(\mathbb{R})$.

3) Let $M = \mathbb{R}$ and $\mathcal{A}_3 = \{\varphi : \mathbb{R} \to \mathbb{R}, \varphi(x) = x^{1/3}\}$. Since the map φ is continuous, increasing and its inverse $\varphi^{-1}(x) = x^3$ is also continuous. Thus φ is a homeomorphism and so $(\mathbb{R}, \mathcal{A}_3)$ is a topological manifold. As there is only one map in $\mathcal{A}_3$, $(\mathbb{R}, \mathcal{A}_3)$ is a smooth manifold.

Now $\mathbb{R} \xrightarrow{\varphi^{-1}} \mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ is the identity map which shows that $\varphi \in \mathcal{C}^\infty_{\mathcal{A}_3}(\mathbb{R})$. Hence $\mathcal{C}^\infty_{\mathcal{A}_3}(\mathbb{R}) \neq \mathcal{C}^\infty(\mathbb{R})$.

We can now introduce the notion of smooth map of manifolds.

Definition Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be smooth manifolds of dimensions m and n, respectively. Here

$$\mathcal{A}_M = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^n \mid \alpha \in A\} \text{ and } \mathcal{A}_N = \{\psi_\beta : V_\beta \to \mathbb{R}^m \mid \beta \in B\}.$$

A function $\Phi : M \to N$ is called smooth if for every $\alpha \in A$ and $\beta \in B$ with $\Phi^{-1}(V_\beta) \cap U_\alpha \neq \emptyset$ the composition

$$\varphi_\alpha(U_\alpha \cap \Phi^{-1}(V_\beta)) \xrightarrow{\varphi_\alpha^{-1}} U_\alpha \cap \Phi^{-1}(V_\beta) \xrightarrow{\Phi} V_\beta \xrightarrow{\psi_\beta} \psi_\beta(V_\beta)$$

is smooth. We call Φ a diffeomorphism if Φ^{-1} exists and it is smooth.

Proposition Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be smooth manifolds of dimensions m and n, respectively, where

$$\mathcal{A}_M = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^m \mid \alpha \in A\} \text{ and } \mathcal{A}_N = \{\psi_\beta : V_\beta \to \mathbb{R}^m \mid \beta \in B\}.$$

Let $f: M \to \mathbb{R}$. Then TFAE (the following are equivalent):

(a) $f \in \mathcal{C}^\infty_{\mathcal{A}_M}(M)$.

(b) $\forall x \in M \ \exists \alpha \in A$: $\quad \varphi_\alpha(U_\alpha) \xrightarrow{\varphi_\alpha^{-1}} U_\alpha \xrightarrow{f} \mathbb{R}$ is smooth.

Moreover, let $\Phi: M \to N$ be a map. Then TFAE:

(a)' f is smooth.

(b)' $\forall x \in M \ \exists \alpha \in A \ \exists \beta \in B$ such that $\Phi(x) \in V_\beta$ and

$$\varphi_\alpha(U_\alpha \cap \Phi^{-1}(V_\beta)) \xrightarrow{\varphi_\alpha^{-1}} U_\alpha \cap \Phi^{-1}(V_\beta) \xrightarrow{\Phi} \Phi(U_\alpha) \cap V_\beta \xrightarrow{\psi_\beta} \psi_\beta(\Phi(U_\alpha) \cap V_\beta))$$

is smooth.

Proof: Clearly (a) $\Rightarrow$ (b). In order to show (b) $\Rightarrow$ (a), let $\beta \in A$ be arbitrary and consider the composition

$\varphi_\beta(U_\beta) \xrightarrow{\varphi_\beta^{-1}} U_\beta \xrightarrow{f} \mathbb{R}$. We have to verify that this composition is smooth. To do this, it is enough to verify that $f \circ \varphi_\beta^{-1}$ is smooth on some open neighbourhood of any point (since smoothness of map is a local property).

So let $x \in U_\beta$ be arbitrary. Then by assumption $\exists \alpha \in A$ so that (b) holds. Now consider the composition

(c) $\varphi_\beta^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\beta^{-1}} U_\alpha \cap U_\beta \xrightarrow{\varphi_\alpha} \varphi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{f} \mathbb{R}$.

Now the composition of the first two maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ in (c) is smooth since this is a transition a function. Also the composition of the last two maps $f \circ \varphi_\alpha^{-1}$ in (c) is smooth, since it is the restriction of the map from (b) to $U_\alpha \cap U_\beta$ (and we are assuming that this map is smooth). Hence the composition (c) is smooth, but this composition is clearly the restriction $f \circ \varphi_\beta^{-1}$ to $\varphi_\beta^{-1}(U_\alpha \cap U_\beta)$ which shows that $f \circ \varphi_\beta^{-1}$ is smooth on some open neighbourhood of x. As $x \in U_\beta$ was arbitrary, the proof is complete.

(a)' $\Rightarrow$ (b)' is clear. (b)' $\Rightarrow$ (a)' is left as an exercise (just follow arguments in the proof of (b) $\Rightarrow$ (a)). $\square$

Assume that $f \in \mathcal{C}^\infty_{\mathcal{A}_N}(N)$. It is straightforward to verify that $f \circ \Phi: M \to \mathbb{R} \in \mathcal{C}^\infty_{\mathcal{A}_M}(M)$.

Definition We call $\Phi^*: \mathcal{C}^\infty_{\mathcal{A}_N}(N) \to \mathcal{C}^\infty_{\mathcal{A}_M}(M)$, $\Phi^*(f) = f \circ \Phi$ the pullback associated to Φ.

Examples A: 1) The map $Id: \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of smooth manifolds $(\mathbb{R}, \mathcal{A}_1)$ and $(\mathbb{R}, \mathcal{A}_2)$, indeed the composition

$$\mathbb{R} \xrightarrow{Id} \mathbb{R} \xrightarrow{Id} \mathbb{R} \xrightarrow{\exp} \mathbb{R} \quad \text{is smooth.}$$

2) The identity map is not a diffeo. of $(\mathbb{R}, \mathcal{A}_1)$ and $(\mathbb{R}, \mathcal{A}_3)$ as

$$\mathbb{R} \xrightarrow{Id} \mathbb{R} \xrightarrow{Id} \mathbb{R} \xrightarrow{\varphi} \mathbb{R}, \text{ where } \varphi(x) = \sqrt[3]{x}, \text{ is not smooth.}$$

3) The map $\varphi^{-1}: \mathbb{R} \to \mathbb{R}$, $\varphi^{-1}(x) = x^3$ is a diffeo. of $(\mathbb{R}, \mathcal{A}_1)$ and $(\mathbb{R}, \mathcal{A}_3)$ as $\mathbb{R} \xrightarrow{Id} \mathbb{R} \xrightarrow{\varphi^{-1}} \mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ is the identity.

Remark: We see that the atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on $\mathbb{R}$ determine the same sets of smooth functions or equivalently, the identity map is a diffeomorphism of $(\mathbb{R}, \mathcal{A}_1)$ and $(\mathbb{R}, \mathcal{A}_2)$. Hence, these two atlases determine the same structure of smooth manifold on $\mathbb{R}$. This motivates the following definition.

Definition Assume that $\mathcal{A}_1 = \{\varphi_\alpha: U_\alpha \to \mathbb{R}^n \mid \alpha \in A\}$ and $\mathcal{A}_2 = \{\psi_\beta: V_\beta \to \mathbb{R}^n \mid \beta \in B\}$ are two smooth atlases on (M, τ) (so that $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$ are smooth manifolds).
We call $\mathcal{A}_1$ and $\mathcal{A}_2$ compatible if the identity map $Id: M \to M$ is a diffeomorphism of $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$ or equivalently: for each $\alpha \in A$ and $\beta \in B$ with $U_\alpha \cap V_\beta \neq \emptyset$ we have that the composition

$$\varphi_\alpha^{-1}(U_\alpha \cap V_\beta) \xrightarrow{\varphi_\alpha^{-1}} U_\alpha \cap V_\beta \xrightarrow{\psi_\beta} \psi_\beta(U_\alpha \cap V_\beta)$$

is smooth.

Remark If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two compatible atlases on M, then also $\mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas on M that is compatible with $\mathcal{A}_1$ and $\mathcal{A}_2$. Hence each atlas $\mathcal{A}$ on M determines a unique maximal atlas $\mathcal{A}_{MAX}$ on M which is compatible with the atlas.

Definition Let $(M, \mathcal{A})$ be a smooth manifold. By a smooth structure on M determined by $\mathcal{A}$ we understand the smooth manifold $(M, \mathcal{A}_{MAX})$.

Examples: 1) The atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on $\mathbb{R}$ are compatible and they determine the same smooth structure on $\mathbb{R}$. (Standard smooth structure.)
2) On the other hand, the smooth structure on $\mathbb{R}$ determined by $\mathcal{A}_3$ is not the standard smooth structure.

Remark: Although $\mathcal{A}_1$ and $\mathcal{A}_3$ are not compatible atlases, we have seen in Example A (3) above that φ^{-1} is a diffeomorphism of $(\mathbb{R}, \mathcal{A}_1)$ and $(\mathbb{R}, \mathcal{A}_3)$. Thus $(\varphi^{-1})^* \mathcal{C}^\infty_{\mathcal{A}_3}(\mathbb{R}) = \mathcal{C}^\infty_{\mathcal{A}_1}(\mathbb{R})$.

Definition We say that two smooth structures $\mathcal{A}_1$ and $\mathcal{A}_2$ on M are equivalent if there exists a homeomorphism $\Phi: M \to M$ such that $\Phi^* \mathcal{C}^\infty_{\mathcal{A}_2}(M) = \mathcal{C}^\infty_{\mathcal{A}_1}(M)$.

Facts: 1) If $n \neq 4$, then any two smooth structures on $\mathbb{R}^n$ are equivalent.
2) The are infinitely many non-equivalent smooth structures on $\mathbb{R}^4$.

3) There are topological manifolds (in dim $n \geq 4$) which admit no smooth structure.

4) Open problem: how many non-equivalent smooth structures are there on S^4?

5) There are 28 non-equivalent smooth structures on S^7.

6) Assume that $(M, \mathcal{A})$ is a closed manifold of dim. n and that $f \in \mathcal{C}^\infty_{\mathcal{A}}(M)$ has only two critical points. Then M is homeomorphic to S^n.

(Due to Kervaire, Milnor, Freedman, Donaldson, ...)

Review of calculus

Let $f : \mathcal{U} \to \mathbb{R}^n$ be a map, where $\mathcal{U}$ is an open subset of $\mathbb{R}^m$. We say that f is differentiable at $x_0 \in \mathcal{U}$ if there exists a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ such that:

$$\lim_{h \to 0} \frac{\| f(x_0 + h) - f(x_0) - L(h) \|_{\mathbb{R}^n}}{\| h \|_{\mathbb{R}^m}} = 0 .$$

Here $\| \cdot \|_{\mathbb{R}^n}$ denotes the Euclidean (or equivalently any norm) on $\mathbb{R}^n$.

Then L is called the total derivative of f at x_0 and we denote it also by $Df(x_0)$.

If $v \in \mathbb{R}^n$, then $D_v f(x_0) = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$ (if the limit exists) is called the directional derivative of f at x_0 along v. If f is differentiable at x_0, then

$$D_v f(x_0) = Df(x_0)(v) = L(v).$$

If $e_j = (0, \dots, 1, \dots, 0)$ (j-th position) is the j-th element from the standard basis of $\mathbb{R}^m$, then $D_{e_j} f(x_0)$ is called the first partial derivative of f at x_0 and usually it is denoted by $\frac{\partial f}{\partial x_j}(x_0)$.

If $f = (f_1, \dots, f_n) : \mathcal{U} \to \mathbb{R}^n$ (so that $f_i : \mathcal{U} \to \mathbb{R}$ is the i-th component of f), then $Df(x_0)$ is represented by (or corresponds to) the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_m}(x_0) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x_0) & \cdots & \frac{\partial f_n}{\partial x_m}(x_0) \end{pmatrix} \in M_{n \times m}(\mathbb{R}).$$

This matrix is called the Jacobi matrix.

Facts 1) If $g : \mathcal{U} \to \mathbb{R}$ is a function and $\partial_i(\partial_j g)$ and $\partial_j(\partial_i g)$ are continuous in $\mathcal{U}$, then $\partial_i(\partial_j g) = \partial_j(\partial_i g)$.

2) If $\frac{\partial f_i}{\partial x_j}$ are continuous in $\mathcal{U}$ for every $i = 1, \dots, n$ and $j = 1, \dots, m$, then $Df(x)$ exists at every point $x \in \mathcal{U}$.

Notation: Assuming that g is smooth on $\mathcal{U}$, then we write $\partial^\alpha g = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_m^{\alpha_m} g$, where $\alpha = (\alpha_1, \dots, \alpha_m)$, $\alpha_i \in \mathbb{N} \cup \{0\}$, $i = 1, \dots, m$.

We say that g is smooth at $x_0 \in U$ if g has all partial derivatives of all orders on some open neighbourhood of x_0.

We say that g is of class $\mathcal{C}^k$ if all partial derivatives of g up to order k exist on an open neighbourhood of x_0 and are continuous there.

Theorem A Let $f: U \to \mathbb{R}^n$ be differentiable at x_0 and $h: V \to \mathbb{R}^\ell$ be differentiable at $f(x_0)$ where U and V are open in $\mathbb{R}^m$ and $\mathbb{R}^n$, resp. Then $g \circ f$ is differentiable at x_0 and

$$D(g \circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0).$$

Definition Let U be open in $\mathbb{R}^m$ and V be open in $\mathbb{R}^n$. A map $f: U \to V$ is a diffeomorphism if f is smooth, bijective and its inverse is also smooth.

Theorem Let U and V be as above. If $f: U \to V$ is a diffeomorphism, then $Df(x)$ is a linear isomorphism at every $x \in U$. (And so $m = n$.)

Proof: As $f^{-1} \circ f = Id_U$, it follows by Theorem A that

$$Df^{-1}(f(x)) \circ Df(x) = D(f^{-1} \circ f)(x) = D(Id_U)(x) \text{ for every } x \in U.$$

$D(Id_U)(x): \mathbb{R}^m \to \mathbb{R}^m$ is the identity map, we see that $Df^{-1}(f(x))$ is the inverse to the linear map $Df(x)$. □

Tangent space

Recall that a curve on a smooth manifold $(M, \mathcal{A})$ is a smooth map $\gamma: (a,b) \to M$ where $a < b$.

Definition (Geometric definition of a tangent vector)

Let $(M, \mathcal{A})$ be a smooth manifold. Then the tangent space $T_x M$ of M at $x \in M$ is defined as the set of all equivalence classes of smooth curves $\gamma: (-\varepsilon_\gamma, \varepsilon_\gamma) \to M$, $\gamma(0) = x$ where $\varepsilon_\gamma > 0$ and two curves γ_1, γ_2 are equivalent if

$$\frac{d}{dt} f(\gamma_1(t))\Big|_{t=0} = \frac{d}{dt} f(\gamma_2(t))\Big|_{t=0} \qquad \left(\frac{d}{dt} f(t) \Big|_{t=0} = \lim_{t \to 0} \frac{f(t) - f(0)}{t} \right)$$

for every $f \in \mathcal{C}^\infty(M)$. Equivalence class $[\gamma] \in T_x M$ is called a tangent vector at x (determined by γ).

Definition (Algebraic definition of a tangent vector)

A derivation at point $x \in M$ is an $\mathbb{R}$-linear map $X: \mathcal{C}^\infty_{\mathcal{A}}(M) \to \mathbb{R}$ which satisfies Leibniz rule

$X(fg) = (Xf)\, g(x) + f(x)\, Xg$. We denote by $D_x M$ the set of all derivations at x.

Notation We will in the rest of this section use the following notation. We will fix a point $x_0 \in M$ and a chart $\varphi: U \to \mathbb{R}^m$ from $\mathcal{A}$ around x_0. We denote by $x_1, \dots, x_m$ the associated coordinate functions $x_i = \varphi_i$ and $\varphi = (\varphi_1, \dots, \varphi_m)$. We assume that $\varphi(x_0) = 0$.

Theorem H Let x_0, M and U be as above. Then there is a smooth function $\tilde{\varkappa}: \varphi(U) \to \mathbb{R}$ such that
(i) $\varkappa = 1$ on some open neighbourhood of 0 and
(ii) $\varkappa = 0$ outside some compact subset of $\varphi(U)$.

Proof: See B. Andrews: Lectures on differential geometry, lecture 4, page 26. available at https://maths-people.anu.edu.au/~andrews/DG/ . $\square$

Remark (Ex) With the notation of Theorem H, put $\varkappa = \tilde{\varkappa} \circ \varphi : U \to \mathbb{R}$. Then $\varkappa \in \mathcal{C}^\infty_{\mathcal{A}|_U}(U)$, $\varkappa = 1$ on some open neighbourhood of x_0, $\varkappa = 0$ outside a compact subset of K and $\varkappa \geq 0$.
Let $f \in \mathcal{C}^\infty_{\mathcal{A}|_U}(U)$. Then

$$f^{ex}(x) = \begin{cases} f(x)\,\varkappa(x), & x \in U \\ 0, & x \notin U \end{cases}$$

belongs to $\mathcal{C}^\infty_{\mathcal{A}}(M)$ and $f = \tilde{f}$ on some open neighbourhood of x_0.
We will later need the function 1^{ex} where 1 is the constant function in U.

Proposition A 1) The space of derivations $D_{x_0}M$ at x_0 is a real vector space.
2) If $X \in D_{x_0}M$ and $c \in \mathbb{R}$ is a constant function, then $X(c) = 0$.
3) If $f, g \in \mathcal{C}^\infty_{\mathcal{A}}(M)$ agree on some open neighbourhood of x_0, then $X(f) = X(g)$.

Proof: ad(1) Let $\lambda \in \mathbb{R}$, $X, Y \in D_x M$ and $f, g \in \mathcal{C}^\infty_{\mathcal{A}}(M)$. Then $\lambda X, X+Y$ are clearly $\mathbb{R}$-linear maps (where $(\lambda X)(f) = \lambda(X(f))$ and $(X+Y)(f) = X(f) + Y(f)$). It remains verify Leibniz rule:

$$(\lambda X)(fg) = \lambda(X(fg)) = \lambda(X(f)g(x_0) + f(x_0)X(g)) = (\lambda X)(f)g(x_0) + f(x_0)(\lambda X)(g)$$ and

$$(X+Y)(fg) = X(fg) + Y(fg) = X(f)g(x_0) + f(x_0)X(g) + Y(f)g(x_0) + f(x_0)Y(g)$$
$$= (X+Y)(f)g(x_0) + f(x_0)(X+Y)(g).$$

ad(2) By $\mathbb{R}$-linearity of X: $X(c) = X(c \cdot 1) = cX(1)$, where 1 is the constant function $M \ni m \mapsto 1$. As $X(1) = X(1^2) = X(1) \cdot 1(m) + 1(m)X(1) = 2X(1) \Rightarrow X(1) = 0$. Hence $X(c) = cX(1) = 0$.

ad(3) By shrinking U if necessary, we may assume that $f = g$ in U.
Then $1^{ex}(f-g) = 0$ on M and thus

$$0 = X(1^{ex}(f-g)) = X(1^{ex})(f-g)(x_0) + 1^{ex}(x_0)\,X(f-g) = X(f) - X(g). \qquad \square$$

Remark (Der) Let $f \in \mathcal{C}^\infty_{\mathcal{A}|_U}(U)$ and $f^{ex} \in \mathcal{C}^\infty_{\mathcal{A}}(M)$ be its extension as in Remark (Ex).
Then we can define $X(f) := X(f^{ex})$ for every $X \in D_{x_0}M$ and so the space of derivations has a canonical action on the space of functions that are smooth on some open neighbourhood of x_0.
This definition makes sense by Proposition A (3).

Example B 1) It is straightforward to verify that

$$\frac{\partial}{\partial x_i}(f) := \frac{\partial f \circ \varphi^{-1}}{\partial x_i}(0) \in D_{x_0}M, \quad f \in \mathcal{C}^\infty_{\mathcal{A}}(M), \text{ for every } i = 1, \dots, m.$$

2) Let $[\gamma] \in T_{x_0}M$ and γ be its representative. Then the map $\mathcal{C}^\infty_{\mathcal{A}}(M) \ni f \mapsto D_\gamma(f) := \frac{d}{dt}(f(\gamma(t)))\big|_{t=0}$ is clearly $\mathbb{R}$-linear.
by Leibniz rule for function of one real variable:

$$D_\gamma(f \cdot g) = (D_\gamma f)\,g(x_0) + f(x_0)\,D_\gamma(g)$$ which shows that $D_\gamma \in D_{x_0}M$.

Moreover if $\gamma_1 \sim \gamma_2$, then $D_{\gamma_1} = D_{\gamma_2}$ by definition of $\sim$.
Thus the map $\gamma \mapsto D_\gamma$ descends to equivalence classes.
We denote by $D_{[\gamma]} \in D_{x_0}M$ the derivation determined by $[\gamma] \in T_{x_0}M$. Hence, there is a well defined map

$$\psi_{x_0}: T_{x_0}M \to D_{x_0}M, \ [\gamma] \mapsto D_{[\gamma]}.$$

3) Finally, consider the map $\phi_{x_0}: \mathbb{R}^m \to T_{x_0}M$, $\phi_{x_0}(v) = [\gamma_v]$ where $\gamma_v(t) = \varphi^{-1}(tv)$, $t \in (-\varepsilon, \varepsilon)$ and $\varepsilon > 0$ is so small so that $tv \in \varphi(U)$ for every $t \in (-\varepsilon, \varepsilon)$. We claim that the map ϕ_{x_0} is bijective.

Proof: (i) injectivity: let $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_m)$ be such that $u \neq v$. Then $u_i \neq v_i$ for some $i = 1, \dots, m$.

We have $D_{\gamma_u}(x_i) = \frac{d}{dt}\left(x_i(\varphi^{-1}(tv))\right)\Big|_{t=0} = \frac{d}{dt}(tv_i)\Big|_{t=0} = v_i$.

Similarly $D_{\gamma_v}(x_i) = u_i$. This shows that $[\gamma_u] \neq [\gamma_v]$.

(ii) surjectivity: let $[\gamma] \in T_{x_0}M$ and γ be its representative and $v = \frac{d}{dt}(\varphi \circ \gamma(t))\Big|_{t=0} \in \mathbb{R}^m$. Now it is enough to verify that $D_{\gamma_v} f = D_\gamma f$ for every $f \in \mathcal{C}^\infty_{x_0}(M)$. We have

$$D_\gamma(f) = \frac{d}{dt} f(\gamma(t))\Big|_{t=0} = \frac{d}{dt} f \circ \varphi^{-1}(\varphi \circ \gamma(t))\Big|_{t=0} =$$

$$= \sum_{i=1}^m \frac{\partial f \circ \varphi^{-1}}{\partial x_i}(\varphi \circ \gamma(t)) \frac{d}{dt}(\varphi_i \circ \gamma)(t)\Big|_{t=0} = \sum_{i=1}^m \frac{\partial f \circ \varphi^{-1}}{\partial x_i}(0)\, v_i \quad | \, v = (v_1, \dots, v_m)$$

$$= \frac{d}{dt}\Big|_{t=0}\left(f \circ \varphi^{-1}(vt)\right) = D_{\gamma_v}(f). \quad \square$$

Now there is a unique structure of a real vector space of dim. m on $T_{x_0}M$ for which the map ϕ_{x_0} is a linear isomorphism. However it is not at this moment clear whether this linear structure depends on φ. This will be clarified in the proof of next Theorem.

<u>Theorem B</u> Let $(M, \mathcal{A})$ be a smooth manifold of dimension m and $x_0 \in M$. Then

1) $D_{x_0}M$ is a real vector space of dimension m and
2) also $T_{x_0}M$ has a canonical structure of a real vector space of dimension m such that the maps ϕ_{x_0} and ψ_{x_0} (defined in Example B above) are isomorphisms of vector spaces.

In the proof of Theorem B we will need the following result.

<u>Theorem</u> (Taylor Theorem)

Let V be an open subset of $\mathbb{R}^n$ and $f: V \to \mathbb{R}$ be smooth. Assume that $0 \in V$. Then $f(x) = a + \sum_{i=1}^m a_i x_i + \sum_{i,j=1}^m a_{ij}(x) x_i x_j$ on some open rectangle $R = [A_1, B_1] \times \dots \times [A_m, B_m]$ where $A_i < 0$, $B_i > 0$, $a_i \in \mathbb{R}$ and $a_{ij}(x) \in \mathcal{C}^\infty(R)$ for every $i, j = 1, \dots, m$.

Proof: •) Let us first consider the case $n = 1$. By Fundamental Theorem of calculus: $g(1) - g(0) = \int_0^1 g'(t)\,dt$ where $g \in \mathcal{C}^\infty([0,1])$.

Fix $x \in U$ and write $g(t) = f(tx)$. Then $g'(t) = f'(tx)\,x$ and

$$f(x) - f(0) = g(1) - g(0) = \int_0^1 g'(t)\,dt = \int_0^1 f'(tx)\,x\,dt$$

$$= \left[f'(tx)\,x\,(t-1)\right]_0^{t=1} - \int_0^1 (t-1)\, x^2 f''(tx)\,dt$$

$$= -f'(0)\,x\,(-1) - x^2 \int_0^1 (t-1) f''(tx)\,dt$$

$= f'(0)\,x + x^2 h(x)$ where $h(x) = -\int_0^1 (t-1)\, f''(tx)\,dt$.

We obtain $f(x) = f(0) + x f'(0) + x^2 h(x)$ on some open neighbourhood of 0.

•) $n \geq 2$ is left as an exercise. □

Proof of Theorem B: 1) Let $\frac{\partial}{\partial x_i}$ be defined as in Example 1) and $X \in D_{x_0}M$.
Put $b_i := X(x_i) \in \mathbb{R}$ for $i = 1, \dots, m$ and $Y := X - \sum_{i=1}^m b_i \frac{\partial}{\partial x_i}$.
By Taylor's Theorem $f \in \mathcal{C}^\infty_{A}(M)$ has the representation

$$f \circ \varphi^{-1}(x) = a + \sum_{i=1}^m a_i x_i + \sum_{i,j=1}^m a_{ij}(x)\, x_i x_j .$$

Then

$$Y(f) = Y\Big(a + \sum_{i=1}^m a_i x_i + \sum_{i,j=1}^m a_{ij}(x)\, x_i x_j\Big) =$$
$$= a_i \sum_{i=1} Y(x_i) + \sum_{i,j=1} a_{ij}(x)\big(Y(x_i)\, x_j + x_i Y(x_j)\big)$$
$$= 0 + 0 = 0 .$$

This shows that $X = \sum_{i=1}^m b_i \frac{\partial}{\partial x_i}$. As $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$, it follows that $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$ is a basis of $D_{x_0}M$. In particular $\dim D_{x_0}M = m$.

2) We will keep notation set above in this section. By the proof of 1), any $X \in D_{x_0}M$ is of the form

$$(D)\quad \begin{tikzcd} \mathbb{R}^m \arrow[r, "\Phi_{x_0}"] \arrow[dr, "\mathbb{D}"'] & T_{x_0}M \arrow[d, "\Psi_{x_0}"] \\ & D_{x_0}M \end{tikzcd}$$

$$X(f) = \sum_{i=1}^m b_i \frac{\partial}{\partial x_i} f = \sum_{i=1}^m b_i \frac{\partial f \circ \varphi^{-1}}{\partial x_i}(0) = \frac{d}{dt}\big(f \circ \varphi^{-1}(b\,t)\big)\Big|_{t=0}$$
$$D_b(f \circ \varphi^{-1})(0) = \frac{d}{dt} f(\gamma_b(t))\Big|_{t=0}$$ for a unique $b = (b_1, \dots, b_m) \in \mathbb{R}^m$.

Let us also define $\mathbb{D} : \mathbb{R}^m \to D_{x_0}M$, $(\mathbb{D}(v))(f) = D_v(f \circ \varphi^{-1})(0)$.

It is clear that $\mathbb{D}(v) = D_{[\gamma_v]}$ and so (D) commutes.

Since the maps $\mathbb{D}$ and Φ_{x_0} are linear isomorphisms, also the map Ψ_{x_0} is a linear isomorphism which completes the proof. (Note that the linear structure on $T_{x_0}M$ really does not depend on φ.) □