Riemannian geometry

Definition Topological manifold (of dim n) is a top-space (X,τ) with atlas $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^n : \alpha \in A\}$ s.t.

(TM1) $\forall \alpha \in A$: $U_\alpha \in \tau$ and $\bigcup_{\alpha \in A} U_\alpha = X$,

(TM2) $\forall \alpha \in A$: φ_α is a homeomorphism onto its image and $\varphi_\alpha(U_\alpha)$ is open in $\mathbb{R}^n$,

(TM3) (X,τ) is Hausdorff and

(TM4) (X,τ) is second countable.

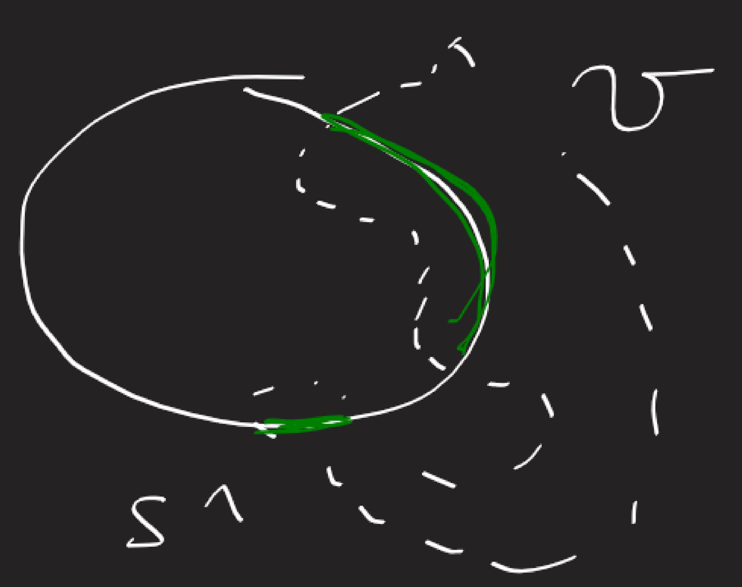
(TM1) & (TM2) $\Leftrightarrow$ $\forall x \in X$ has an open neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$

max. atlas $\mathcal{A}_{MAX}$ = collection of all maps $\varphi : U \to \mathbb{R}^n$ where $U \in \tau$ and φ is a homeomorphism onto its image, $\varphi(U)$ is in $\mathbb{R}^n$

The definition of continuous function $f : X \to \mathbb{R}$ does not depend on $\mathcal{A}$

Ex. •) $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

$S^1 \subseteq \mathbb{R}^2$ on S^1 we consider the subspace top. τ_{S^1}



$U \subseteq S^1$ is open $\Leftrightarrow$ $\exists V$ open in $\mathbb{R}^2$

$U = V \cap S^1$

$\mathcal{A}_{S^1} = \{\varphi_r : U_r \to \mathbb{R} : r \in \mathbb{R}\}$

$V_r = (r, r+2\pi)$, $r \in \mathbb{R}$

$\varphi_r^{-1}(t) = (\cos t, \sin t)$

φ_r^{-1} is a homeomorphism onto its image

its inverse φ_r is a map from

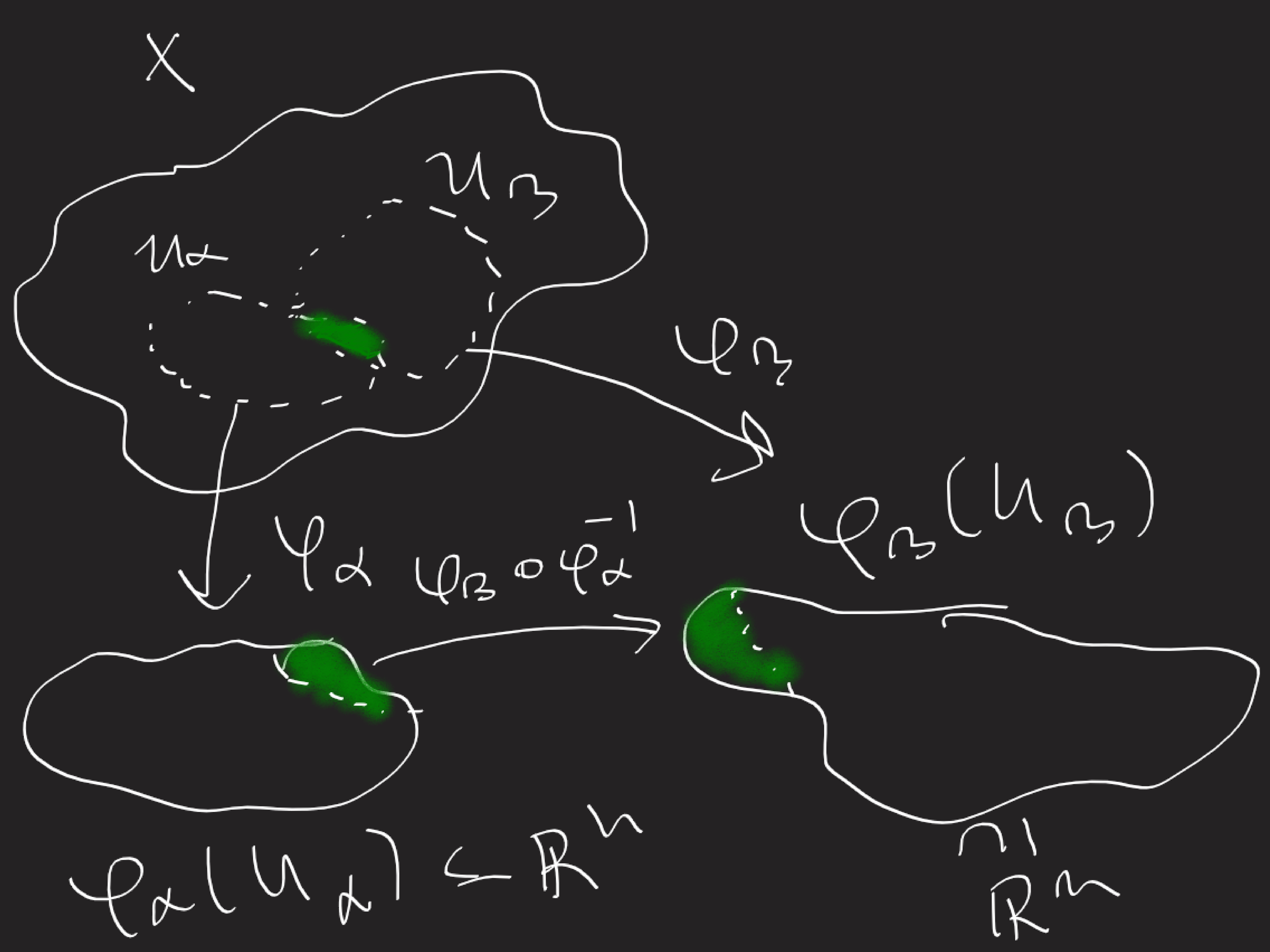
$U_r = \varphi_r^{-1}(V_r)$ to $\mathbb{R}$

(S^1, τ_{S^1}) with $\mathcal{A}_{S^1}$ is a top. manifold

Def Smooth manifold of dimension n is a top. manifold (X,τ) with atlas $\mathcal{A}$ as above such that $\forall \alpha, \beta \in A$:

(TF)
$$\underbrace{\varphi_\alpha(\underbrace{U_\alpha \cap U_\beta}_{\in \tau})}_{\subseteq \mathbb{R}^n} \xrightarrow{\varphi_\beta \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap U_\beta)}} \varphi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$$

is smooth



We call a map φ_α from atlas a chart,

$\varphi_\beta \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap U_\beta)}$

is called a transition function

Examples .) (S^1, τ_{S^1}) with $\mathcal{A}_{S^1}$ is a smooth man. of dim 1.

.) $(\mathbb{R}, \tau^1)$ $\quad \mathcal{A}_1 = \{ Id : \mathbb{R} \to \mathbb{R} \}$
is a smooth man. of dim 1

.) $(\mathbb{R}, \tau^1)$ $\quad \mathcal{A}_2 = \{ \exp : \mathbb{R} \to \mathbb{R} \}$

.) $(\mathbb{R}, \tau^1)$ $\quad \mathcal{A}_3 = \{ \varphi : \mathbb{R} \to \mathbb{R}, \ \varphi(x) = \sqrt[3]{x} \}$

φ is continuous, increasing and its inverse is continuous, φ is a homeomorphism

This data gives a topological man.
There is only one map in atlas $\Rightarrow$
This is even a smooth manifold

Notation We will denote a smooth manifold simply as $(M, \mathcal{A})$ where M is the underlying top-space and $\mathcal{A}$ is the given atlas on M

Def A function $f : M \to \mathbb{R}$ on a smooth man. $(M, \mathcal{A})$ is called smooth if :

$$\forall \alpha \in \mathcal{A} \qquad \underset{\mathbb{R}^n}{\varphi_\alpha(U_\alpha)} \xrightarrow{\varphi_\alpha^{-1}} \underset{M}{U_\alpha} \xrightarrow{f} \mathbb{R}$$

is smooth. We will denote $\mathcal{C}^\infty_{\mathcal{A}}(M)$ the space of all smooth functions on M.

<u>Example</u> $\cdot)$ $\mathcal{C}^\infty_{\mathcal{A}_1}(\mathbb{R}) \ni f \qquad f: \mathbb{R} \longrightarrow \mathbb{R}$

$\Longleftrightarrow \mathbb{R} \xrightarrow{(\mathrm{Id})^{-1}} \mathbb{R} \xrightarrow{f} \mathbb{R}$ is smooth

$\mathcal{A}_1 = \{ \mathrm{Id}: \mathbb{R} \to \mathbb{R} \}$

$\Longleftrightarrow \quad \mathbb{R} \xrightarrow{f} \mathbb{R}$ is smooth

$\mathcal{C}^\infty_{\mathcal{A}_1}(\mathbb{R}) = \mathcal{C}^\infty(\mathbb{R})$.

$\cdot)$ $\mathcal{C}^\infty_{\mathcal{A}_2}(\mathbb{R}) = \mathcal{C}^\infty(\mathbb{R}) \qquad \mathcal{A}_2 = \{ \exp: \mathbb{R} \to \mathbb{R} \}$

exp has a smooth inverse given log

$\cdot)$ $\mathcal{C}^\infty_{\mathcal{A}_3}(\mathbb{R}) \qquad \mathcal{A}_3 = \{ \varphi: \mathbb{R} \to \mathbb{R},\ \varphi(x) = \sqrt[3]{x} \}$

claim $\varphi \in \mathcal{C}^\infty_{\mathcal{A}_3}(\mathbb{R})$. indeed since

$\varphi \in \mathcal{C}^\infty_{\mathcal{A}_3}(\mathbb{R}) \Longleftrightarrow \mathbb{R} \xrightarrow{\varphi^{-1}} \mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ is smooth

$\Longleftrightarrow \quad \mathbb{R} \xrightarrow{\mathrm{Id}} \mathbb{R}$ is smooth

$\varphi \in \mathcal{C}^\infty_{\mathcal{A}_3}(\mathbb{R}) \neq \mathcal{C}^\infty(\mathbb{R}) \not\ni \varphi$

<u>Remark</u>: We see that the space $\mathcal{C}^\infty_{\mathcal{A}}(M)$ depends on the choice of atlas.

<u>Def</u> Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be smooth manifolds of dimensions m and n, resp., where $\mathcal{A}_M = \{ \varphi_\alpha : U_\alpha \to \mathbb{R}^m,\ \alpha \in A \}$ and

$$\mathcal{A}_N = \{ \psi_\beta : V_\beta \longrightarrow \mathbb{R}^n,\ \beta \in B \}.$$

A map $\Phi: M \longrightarrow N$ is smooth if $\forall \alpha \in A$ and $\forall \beta \in B$:

$$\begin{array}{ccccccc} \varphi_\alpha(U_\alpha \cap \Phi^{-1}(V_\beta)) & \xrightarrow{\varphi_\alpha^{-1}} & U_\alpha \cap \Phi^{-1}(V_\beta) & \xrightarrow{\Phi} & V_\beta & \xrightarrow{\psi_\beta} & \psi_\beta(V_\beta) \\ \cap\!\shortmid & & \cap\!\shortmid & & \cap\!\shortmid & & \cap\!\shortmid \\ \mathbb{R}^m & & M & & N & & \mathbb{R}^n \end{array}$$

is smooth. If Φ has an inverse and Φ^{-1} is also smooth, then Φ is called a diffeom.

Example
- $\mathrm{Id}: \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of $(\mathbb{R}, \mathcal{A}_1)$ and $(\mathbb{R}, \mathcal{A}_2)$
- $\mathrm{Id}: \mathbb{R} \to \mathbb{R}$ is not a diffeomorphism of $(\mathbb{R}, \mathcal{A}_1)$ and $(\mathbb{R}, \mathcal{A}_3)$

  $\mathbb{R} \xrightarrow{(\mathrm{Id})^{-1}} \mathbb{R} \xrightarrow{\mathrm{Id}} \mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ $\quad (\varphi(x) = \sqrt[3]{x})$

  is not smooth as $\varphi \notin \mathcal{C}^\infty(\mathbb{R})$.
- $\varphi^{-1}: \mathbb{R} \to \mathbb{R}$, $\varphi^{-1}(x) = x^3$ then φ^{-1} is a smooth map of $(\mathbb{R}, \mathcal{A}_1)$ and $(\mathbb{R}, \mathcal{A}_3)$, since

  $\mathbb{R} \xrightarrow{\mathrm{Id}^{-1}} \mathbb{R} \xrightarrow{\varphi^{-1}} \mathbb{R} \xrightarrow{\varphi} \mathbb{R}$

  is smooth $\mathbb{R} \to \mathbb{R}$ (in the usual sense).

Remark We see that $(\mathbb{R}, \mathcal{A}_1)$ and $(\mathbb{R}, \mathcal{A}_2)$ give same structures of smooth manifolds on $\mathbb{R}$ as $\mathcal{C}^\infty_{\mathcal{A}_1}(\mathbb{R}) = \mathcal{C}^\infty_{\mathcal{A}_2}(\mathbb{R})$ and Id is a diffeom.

Definition Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two smooth atlases on M. (Smooth manifolds $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$ with the same underlying top. space M.) We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible if $\mathrm{Id}: M \to M$ is smooth or equivalently $\forall \alpha \in A_1$ and $\forall \beta \in A_2$

$$\varphi_\alpha(U_\alpha \cap V_\beta) \xrightarrow{\psi_\beta \circ \varphi_\alpha^{-1}} \psi_\beta(U_\alpha \cap V_\beta)$$

is smooth where

$\mathcal{A}_1 = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^n \mid \alpha \in A_1\}$ and

$\mathcal{A}_2 = \{\psi_\beta : V_\beta \to \mathbb{R}^n \mid \beta \in A_2\}$.

Remark If $\mathcal{A}_1, \mathcal{A}_2$ are two compatible atlases, then $\mathcal{A}_1 \cup \mathcal{A}_2$ is also an atlas on M. So there is a unique smooth maximal atlas on M which contains every atlas on M that is compatible with the given smooth atlas $\mathcal{A}$ on M.

This maximal atlas $\mathcal{A}_{MAX} = \bigcup \mathcal{A}'$

union is taken over all atlases on M that are compatible with $\mathcal{A}$.

<u>Definition</u> We call $(M, \mathcal{A}_{MAX})$ a smooth structure (determined by the atlas $\mathcal{A}$).

<u>Definition</u> Two smooth structures $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$ are called equivalent if there is a homeomorphism $\Phi: M \to M$ such that $\underbrace{\Phi^* \mathcal{C}^\infty_{\mathcal{A}_1}(M)}_{(D)} = \mathcal{C}^\infty_{\mathcal{A}_2}(M)$.

<u>Definition</u> Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be two smooth manifolds. Let $\Phi: M \to N$ be smooth. Then the map $\Phi^*: \mathcal{C}^\infty_{\mathcal{A}_N}(N) \to \mathcal{C}^\infty_{\mathcal{A}_M}(M)$, $\Phi^*(f) = f \circ \Phi$ is called the pullback of Φ.

Exercise: Verify Φ^* is correct.

We denote by $\Phi^* \mathcal{C}^\infty_{\mathcal{A}_N}(N) = \{ f \circ \Phi : f \in \mathcal{C}^\infty_{\mathcal{A}_N}(N) \}$
$\subseteq \mathcal{C}^\infty_{\mathcal{A}_M}(M)$.

(D) $\Phi^* \mathcal{C}^\infty_{\mathcal{A}_1}(M) = \{ f \circ \Phi : f \in \mathcal{C}^\infty_{\mathcal{A}_1}(M) \}$

<u>Example</u> The map $\varphi^{-1}: \mathbb{R} \to \mathbb{R}$ shows that the smooth structure det. $\mathcal{A}_1$ is equivalent to the smooth str. det. $\mathcal{A}_3$.

<u>Facts</u>: 1) All smooth structures on $\mathbb{R}^n$ are equivalent if $n \neq 4$.

2) There are $+\infty$ non-equivalent smooth structures on $\mathbb{R}^4$.

3) Open problem: how many non-equivalent smooth structures are there on S^4?

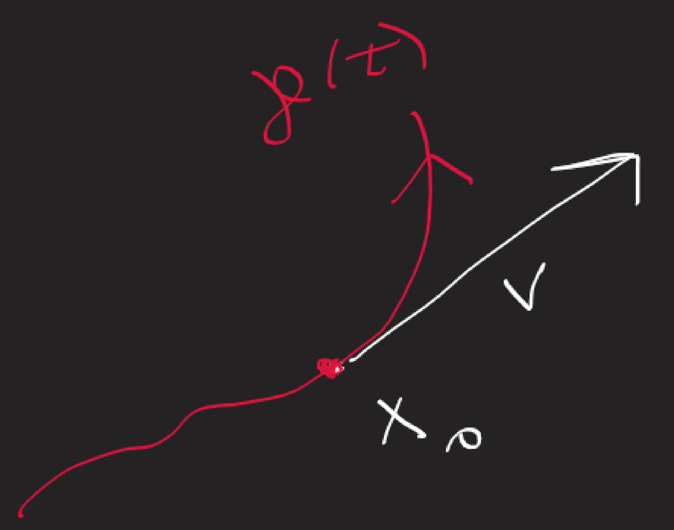
4) There are top. man. that admit no structure of a smooth atlas.

5) Let M^n be a closed (compact without boundary) manifold of dim n which admits a smooth function with only two critical points (one point is a strict global max. and the other is a strict global min). Then M^n is homeomorphic to the sphere S^n.

Tangent space and tangent vector

We want to give a definition of a tangent vector on a manifold.

$M=\mathbb{R}^n$, $x_0\in\mathbb{R}$, a tangent vector at x_0 is just a usual vector $v\in\mathbb{R}^n$



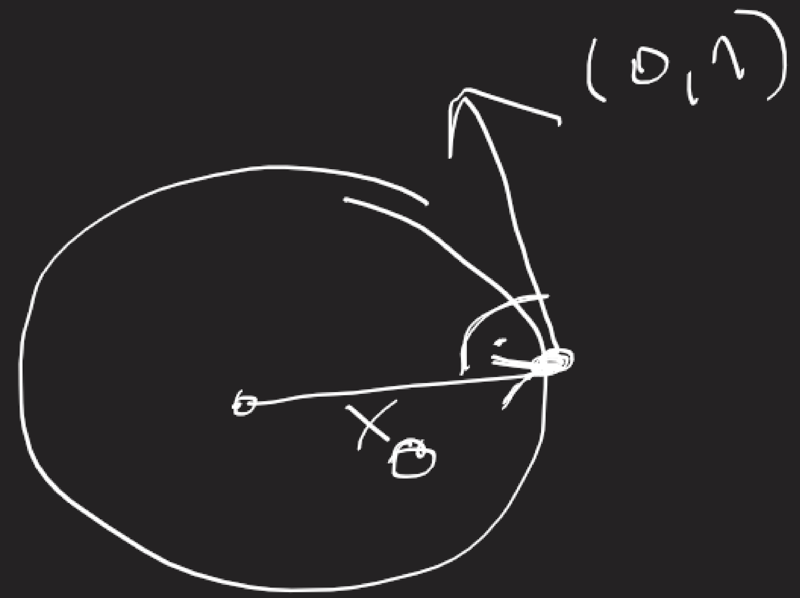
and a geometric point of view is that v is tangent to a curve $\gamma:(-\varepsilon,\varepsilon)\longrightarrow\mathbb{R}^n$ which passes trough x_0 at time $t=0$

$$v=\gamma'(0)=\frac{d}{dt}\Big)_{t=0}\gamma(t)$$

M is a subset (a smooth submanifold) of $\mathbb{R}^m$ and $x_0\in M$, then a tangent vector v at x_0 to M is a $v\in\mathbb{R}^m$ such that $\exists$ curve

$$\gamma:(-\varepsilon,\varepsilon)\longrightarrow M \text{ with } \gamma(0)=x_0,\ \gamma'(0)=v$$

•) $M=S^1$



$x_0=(1,0)$

$\gamma(t)=(\cos t,\sin t)$

$\gamma'(t)\Big|_{t=0}=(-\sin t,\cos t)\big|_{t=0}=(0,1)$

the angle between $x_0=(1,0)$ and $\gamma'(t)=(0,1)$ is $90°$

<u>Observation</u> If $M=\mathbb{R}^n$, $x_0\in\mathbb{R}^n$, two curves in $\mathbb{R}^n$, say γ_1 and γ_2, such that

$\gamma_1(0)=x_0=\gamma_2(0)$ and $\gamma_1:(-\varepsilon_1,\varepsilon_1)\longrightarrow\mathbb{R}^n$
$\gamma_2:(-\varepsilon_2,\varepsilon_2)\longrightarrow\mathbb{R}^n$,

then $\gamma_1'(0)=\gamma_2'(0)\iff$

$$\forall f\in\mathcal{C}^\infty(\mathbb{R}):\ \frac{d}{dt}f(\gamma_1(t))\Big|_{t=0}=\frac{d}{dt}f(\gamma_2(t))\Big|_{t=0}$$

$\gamma_1'(0)=(v_1,\dots,v_n)$, $f=x_i$ .. i-th coordinate function on $\mathbb{R}^n$

then $\frac{d}{dt}x_i(\gamma_1(t))\Big|_{t=0}=v_i$.

__Definition__ Let $(M, \mathcal{A})$ be a smooth manifold of dimension m. Then the tangent space $T_{x_0}M$ of M at point $x_0 \in M$ is defined as the set of all equivalence classes of (smooth) curves $\gamma : (-\varepsilon_\gamma, \varepsilon_\gamma) \longrightarrow M$ with $\gamma(0) = x_0$ where

$$\gamma_1 \sim \gamma_2 \text{ if}$$

$$\forall f \in C^\infty_{\mathcal{A}}(M): \quad \frac{d}{dt} f(\gamma_1(t))\Big|_{t=0} = \frac{d}{dt} f(\gamma_2(t))\Big|_{t=0}.$$

We denote by $[\gamma]$ the equivalence class of γ and we call $[\gamma]$ a tangent vector at x_0.

Now we want to show that $T_{x_0}M$ has a canonical structure of a real vector space of dim m.

In order to do that, let us fix a chart $\varphi : U \to \mathbb{R}^m$ from the atlas $\mathcal{A}$ such that $x_0 \in U$ and $\varphi(x_0) = 0$. Then it is clear that any curve in M which passes through x_0 at time $t=0$ can be written as $\gamma(t) = \varphi^{-1}(\tilde{\gamma}(t))$ where

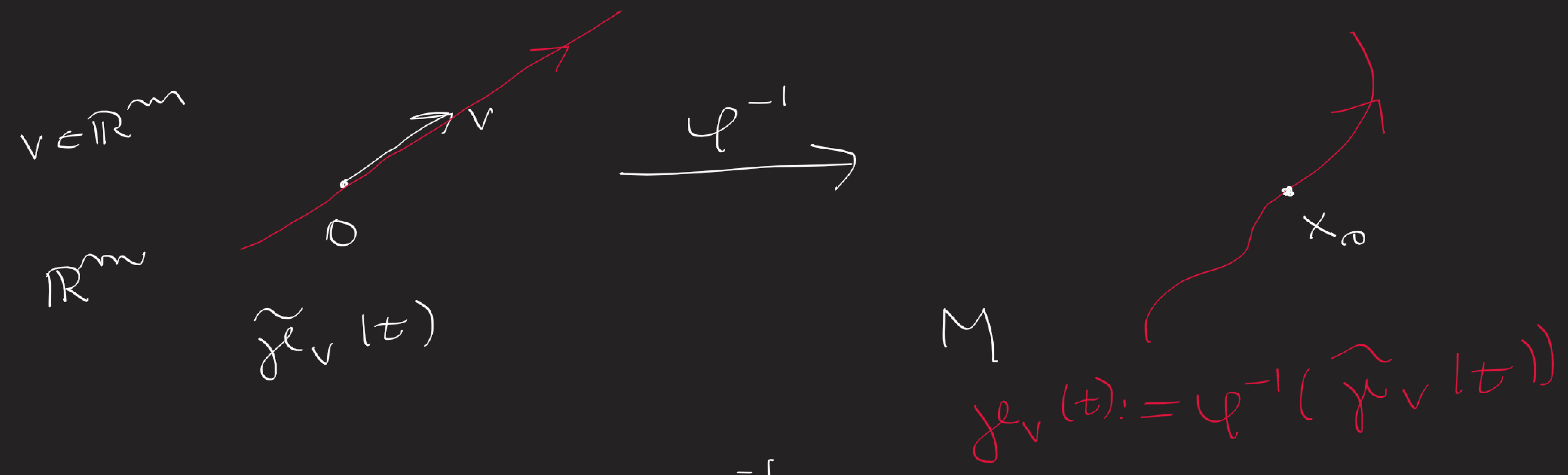
$$\tilde{\gamma} : (-\varepsilon, \varepsilon) \longrightarrow \varphi(U) \subset \mathbb{R}^m$$

for t small enough. So we can now define a map:

$$\Phi_{x_0} : \mathbb{R}^m \longrightarrow T_{x_0}M$$

$$\Phi_{x_0}(v) = [\varphi^{-1}(\tilde{\gamma}_v(t))]$$

$\tilde{\gamma}_v(t) = tv$, $t \in (-\varepsilon, \varepsilon)$ so that $tv \in \varphi(U)$ for all $t \in (-\varepsilon, \varepsilon)$.



we want to verify that this map

$$\mathbb{R}^m \ni v \longmapsto [\gamma_v] \in T_{x_0}M$$

is well defined and that this is a bijection of sets.

Sketch proof: 1) is well defined since we are just taking the equivalence classes

2) this is injective ?

3) this is surjective ?

ad 2) $v, w \in \mathbb{R}^m$: $v \neq w \Rightarrow [\gamma_v] \neq [\gamma_w]$

in order to show this we argue like this so $[\gamma_v] \neq [\gamma_w]$ if we show that there is a function $f \in C^\infty_A(M)$ such

$$\frac{d}{dt} f(\gamma_v(t))\Big|_{t=0} \neq \frac{d}{dt} f(\gamma_w(t))\Big|_{t=0}.$$

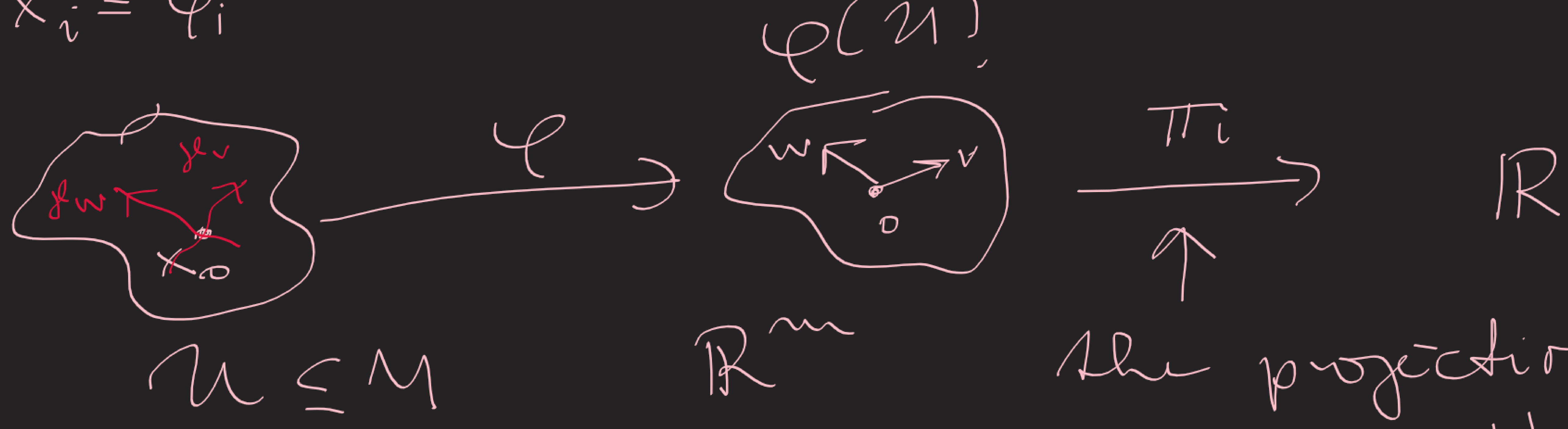
we assume that $v \neq w$ and so $v = (v_1, \dots, v_m)$, $w = (w_1, \dots, w_m) \Rightarrow \exists\, i = 1, \dots, m$: $v_i \neq w_i$.

$$\frac{d}{dt} x_i(vt)\Big|_{t=0} = v_i \neq \frac{d}{dt} x_i(wt) = w_i$$

i-th coordinate function on $\mathbb{R}^m$

If we use the map φ, then $x_i = \varphi_i$ - the i-th component of φ, this is a function on U which is smooth.

$x_i = \varphi_i$



the projection on the i-th component

$x_i = \pi_i \circ \varphi$ is a smooth function on $\mathcal{U}$ (coordinate functions on $\mathcal{U}$)

γ_v, γ_w are smooth curves in M which are contained in $\mathcal{U}$

$$\frac{d}{dt} x_i(\gamma_v(t))\Big|_{t=0} = \frac{d}{dt} x_i(\varphi^{-1}(vt))\Big|_{t=0} =$$

$\gamma_v = \varphi^{-1}(vt)$

$$= \frac{d}{dt} \pi_i(\varphi \circ \varphi^{-1}(vt))\Big|_{t=0}$$

$$= \frac{d}{dt} \pi_i(vt)\Big|_{t=0}$$

$$= \frac{d}{dt} v_i t\Big|_{t=0}$$

$$= v_i \neq w_i = \frac{d}{dt} x_i(\gamma_w(t))\Big|_{t=0}$$

The only remaining point is to show that there is a global smooth function x_i^{ext} on M which agrees with x_i on some open neighbourhood of x_0.

$x_i = x_i^{ext}$ on some open neigh. of x_0 $\Rightarrow$

$$v_i = \frac{d}{dt} x_i(\gamma_v(t))\Big|_{t=0} = \frac{d}{dt} x_i^{ext}(\gamma_v(t))\Big|_{t=0}$$

$\neq$

$$w_i = \frac{d}{dt} x_i(\gamma_w(t))\Big|_{t=0} = \frac{d}{dt} x_i^{ext}(\gamma_w(t))\Big|_{t=0}$$

We are done with the proof since $x_i^{ext} \in \mathcal{C}^\infty_{\mathbb{R}}(M)$ and the computation above shows that $[\gamma_v]$ and $[\gamma_w]$ are different.

Let f be a smooth function on $\mathcal{U} \Longleftrightarrow$

$f = \tilde{f} \circ \varphi$ where $\tilde{f} \in \mathcal{C}^\infty(\varphi(\mathcal{U}))$.

$\varphi(\mathcal{U}) \subseteq \mathbb{R}^m$

$\exists\ \chi \in \mathcal{C}^\infty(\varphi(\mathcal{U}))$:

1) $\chi = 1$ on some open neigh. of 0

2) $\chi = 0$ outside some compact subset K of $\mathcal{U}$.

$\chi \circ \varphi$ is a smooth function in $\mathcal{U}$ which 1) $= 1$ on some open neigh. of x_0

2) $= 0$ outside some compact subset of $\mathcal{U}$.

f is smooth in $\mathcal{U}$, then

$$f^{ex}(x) = \begin{cases} f \cdot \chi \circ \varphi(x) & , x \in \mathcal{U} \\ 0 & , x \notin \mathcal{U}. \end{cases}$$

$\cdot$ pointwise product

$f^{ex}(x) \in \mathcal{C}^\infty_A(M)$ and $f^{ex} = f$ on some open neigh. of x_0.

end of the proof of injectivity.

3) surjectivity?

$\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$, $\gamma(0) = x_0$, $\forall t \in (-\varepsilon, \varepsilon)$ $\varphi(t) \in \mathcal{U}$

$[\gamma] = [\gamma_v] = \Phi_{x_0}(v)$

$\bar{\gamma} = \varphi \circ \gamma : (-\varepsilon, \varepsilon) \longrightarrow \varphi(\mathcal{U}) \subseteq \mathbb{R}^m$

$\bar{\gamma}(0) = 0$, $v = \frac{d}{dt}\bar{\gamma}(t)\big|_{t=0} \in \mathbb{R}^m$

$\gamma_v(t) = \varphi^{-1}(vt)$, $t \in (-\varepsilon, \varepsilon)$

$\Phi_{x_0} : \mathbb{R}^m \longrightarrow T_{x_0}M$ is $1-1$

so there is unique structure on $T_{x_0}M$ so that Φ_{x_0} is a linear isomorphism

Question: does this linear structure depend on the choice of $\mathcal{C}$?

<u>Definiton</u> A derivation at $x_0 \in M$ is an $\mathbb{R}$-linear map

$$D : \mathcal{C}^\infty_{\mathcal{A}}(M) \longrightarrow \mathbb{R}$$

which satisfies

(Leibniz rule) $D(fg) = Df \cdot g(x_0) + f(x_0) \cdot Dg$

where $f, g \in \mathcal{C}^\infty_{\mathcal{A}}(M)$. We denote by $D_{x_0}M$ the space of all derivations at $x_0 \in M$.

<u>Theorem</u> 1) $D_{x_0}M$ is a real vector space.

2) $D \in D_{x_0}M$ and c is a constant function on M, then $Dc = 0$.

3) $Df = Dg$ whenever $f, g \in \mathcal{C}^\infty_{\mathcal{A}}(M)$ and $f = g$ on some open neigh. of x_0.

$$
\begin{tikzcd}
\mathbb{R}^m \arrow[rr, "\Phi_{x_0}"] \arrow[dr, "\mathbb{D}"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & T_{x_0}M \arrow[dl, "\Psi_{x_0}"] \\
& \mathbb{D}_{x_0}M &
\end{tikzcd}
$$

•) $\Phi_{x_0}(v) = [\gamma_v]$

$\gamma_v(t) = \varphi^{-1}(vt)$
$t \in (-\varepsilon, \varepsilon)$

•) $(\mathbb{D}(v))(f) = \frac{d}{dt}(f(\varphi^{-1}(vt))\Big|_{t=0}$

$\mathbb{D}(v)$ is indeed a derivation

•) $\Big(\underbrace{\Psi_{x_0}[\gamma]}_{\in\, \mathbb{D}_{x_0}M}\Big)(f) = \frac{d}{dt} f(\gamma(t))\Big|_{t=0}$.

•) The diagram is indeed commutative.

•) The map $\mathbb{D}$ is an isomorphism of vector spaces.

•) Ψ_{x_0} is also an isomorphism of vector spaces of dimension m and in particular the ~~vs~~ linear structure on $T_{x_0}M$ does not depend on the choice of chart φ.