Vector fields and tangent bundle

Short recapitulation from last lecture.

Let $(M,\mathcal{A})$ be a smooth manifold of dimension m and $\varphi: \mathcal{U} \to \mathbb{R}^m$ be a chart around x_0 from $\mathcal{A}$ with $\varphi(x_0)=0$ and coordinate functions $x_1,\dots,x_m$.

We defined the tangent space $T_{x_0}M$ at x_0 as the space of all equivalence classes of curves $\gamma: (-\varepsilon_\gamma, \varepsilon_\gamma) \to M$, $\gamma(0)=x_0$, $\varepsilon_\gamma > 0$ where $\gamma_1 \sim \gamma_2$ if $\forall f \in \mathcal{C}^\infty_{\mathcal{A}}(M)$: $\frac{d}{dt} f(\gamma_1(t))|_{t=0} = \frac{d}{dt} f(\gamma_2(t))|_{t=0}$.

We have also proved that the map

$$\phi_{x_0}: \mathbb{R}^m \to T_{x_0}M, \ \phi_x(v) = [\gamma_v], \quad \gamma_v(t) = \varphi^{-1}(vt),$$

where $t\in(-\varepsilon,\varepsilon)$ and $\varepsilon>0$ is so small so that $vt \in \varphi(\mathcal{U})$, is bijective and so there is a unique linear structure on $T_{x_0}M$ so that ϕ_{x_0} is a linear isomorphism.

Moreover we have defined a derivation at $x_0 \in M$, recall that a derivation at x_0 is a linear map $X: \mathcal{C}^\infty_{\mathcal{A}}(M) \to \mathbb{R}$ which satisfies Leibniz rule: $X(fg) = Xf\cdot g(x_0) + f(x_0) Xf$ for every $f,g \in \mathcal{C}^\infty_{\mathcal{A}}(M)$.

Then $D_{x_0}M$ denotes the space of all derivations at x_0.

We have proved that:

1) $D_{x_0}M$ is a vector space.
2) if $f,g \in \mathcal{C}^\infty_{\mathcal{A}}(M)$ agree on some open neighbourhood of x_0, then $Xf = Xg$ for every $X \in D_{x_0}M$.
3) if f is smooth on some open neighbourhood of x_0, then there is $f^{ex} \in \mathcal{C}^\infty_{\mathcal{A}}(M)$ so that $f^{ex} = f$ on some (possibly smaller) open neighbourhood, then we define $Xf := Xf^{ex}$. (This definition makes sense by the previous result 2).)

Moreover, we have verified that there is a well defined map

$$\Psi_{x_0}: T_{x_0}M \to D_{x_0}M, \ (\Psi_{x_0}([\gamma]))(f) = \frac{d}{dt} f(\gamma(t))|_{t=0}.$$

Then we have defined a map

$$\mathbb{D}: \mathbb{R}^m \to D_{x_0}M, \ (\mathbb{D}(v))(f) = D_v(f\circ\varphi^{-1})(0)$$

where D_v is the directional derivative in the direction (or along) $v\in\mathbb{R}^m$, that $D_v g(0) = \frac{d}{dt} g(tv)|_{t=0}$ and g is a differentiable function at $0 \in \mathbb{R}^m$.

From definitions, it is clear that

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "\phi_{x_0}"] \arrow[dr, "\mathbb{D}"'] & T_{x_0}M \arrow[d, "\Psi_{x_0}"] \\ & D_{x_0}M \end{tikzcd}$$

commutes and we have verified that $\mathbb{D}$ is a linear isomorphism. Thus also Ψ_{x_0} is a linear isomorphism which proves that the linear structure on $T_{x_0}M$ does not depend on φ.

We have also shown that $B = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$ is a basis of $D_{x_0}M$ (and hence of $T_{x_0}M$) where

$$\frac{\partial}{\partial x_i} f = \frac{\partial f\circ\varphi^{-1}}{\partial x_i}(0), \quad i=1,\dots,m \text{ and } f \in \mathcal{C}^\infty_{\mathcal{A}}(M).$$

Also recall that $X = \sum_{i=1}^m a_i \frac{\partial}{\partial x_i} \Leftrightarrow a_i := \frac{\partial}{\partial x_i} x_i$, $i=1,\dots,m$.

Tangent bundle of a manifold

Let $(M,\mathcal{A})$ be a smooth manifold of dimension m. We will denote a tangent vector at $x\in M$ simply as $u, v, \dots$ rather than $[\gamma]$. We write (x,v) in order to point out that $v \in T_x M$.

Definition The tangent bundle of $(M,\mathcal{A})$ is defined as $\bigcup_{x\in M} T_x M$ (the disjoint union) and let $\pi: TM \to M$ be the canonical projection $T_{x_0}M \ni v \longmapsto x_0$.

Let $\varphi : \mathcal{U} \to M$ be a chart from $\mathcal{A}$, $x_1, \dots, x_m$ be the coordinate functions on $\mathcal{U}$. Now let $x \in \mathcal{U}$ be arbitrary and then

$$B^{\varphi}_x := \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \right\} \quad \text{where} \quad \frac{\partial}{\partial x_i} f = \frac{\partial f \circ \varphi}{\partial x_i}(\varphi(x))$$

is a basis of $D_x M$ (and thus of $T_x M$). Thus $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \right\}$ is a basis of $T_x M$ for every $x \in \mathcal{U}$ (this might look a bit suspicious but it is a standard notation). If $v \in T_x M$, let $[v]_\varphi$ be the coordinates of v with respect to B^{φ}_x.

Let us consider the map

$$\Phi : \pi^{-1}(\mathcal{U}) \longrightarrow \mathbb{R}^m \times \mathbb{R}^m, \quad (x, v) \longmapsto (\varphi(x), [v]_\varphi).$$

This map is clearly bijective and its inverse is

$$\Phi^{-1}\left((x_1, \dots, x_m), (p_1, \dots, p_m)\right) = \left(\varphi^{-1}(x_1, \dots, x_m), \sum_{i=1}^m p_i \frac{\partial}{\partial x_i}\right).$$

Assume now that $\psi : \mathcal{V} \to M$ is a different chart from $\mathcal{A}$, $y_1, \dots, y_m$ be the associated coordinate functions on $\mathcal{V}$ and $B^{\psi}_{x'} = \left\{ \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m} \right\}$ be a basis of $T_{x'} M$ with $x' \in \mathcal{V}$.

Assume that $x \in \mathcal{U} \cap \mathcal{V}$, then B^{φ}_x and B^{ψ}_x are two bases of $T_x M$. We would like to compute the change of basis matrix. By definition, for $f \in \mathcal{C}^\infty_{\mathcal{A}}(M)$

$$\frac{\partial}{\partial x_j} f = \frac{\partial f \circ \varphi^{-1}}{\partial x_j}(\varphi(x)) = \frac{\partial f \circ \psi^{-1} \circ \psi \circ \varphi^{-1}}{\partial x_j}(\varphi(x)) = \sum_{i=1}^m \frac{\partial f \circ \psi^{-1}}{\partial y_i}(\psi \circ \varphi^{-1} \circ \varphi(x)) \, \frac{\partial (\psi \circ \varphi^{-1})_i}{\partial x_j}(\varphi(x))$$

$$(\mathrm{CoB}) \qquad = \sum_{i=1}^m \frac{\partial f \circ \psi^{-1}}{\partial y_i}(\psi(x)) \, \frac{\partial (\psi \circ \varphi^{-1})_i}{\partial x_j}(\varphi(x)) = \frac{\partial}{\partial y_i} f \sum_{i=1}^m a_{ij}$$

where $(\psi \circ \varphi^{-1})_i$ is the i-th component of $\psi \circ \varphi^{-1}$ and

$$a_{ij} := \frac{\partial (\psi \circ \varphi^{-1})_i}{\partial x_j}; \quad i, j = 1, \dots, m.$$

We see that $A = (a_{ij})_{i=1,\dots,m}^{j=1,\dots,m}$ is the change of basis matrix from B^{ψ}_x to B^{φ}_x.
Note that A is the Jacobi matrix of $\psi \circ \varphi^{-1}$ at the point $\varphi(x)$.
As $\psi \circ \varphi^{-1}$ is smooth, the coefficients a_{ij} of A are smooth functions of $x_1, \dots, x_m$.

We have thus proved:

Lemma A Let $\Psi : \pi^{-1}(\mathcal{V}) \to \mathbb{R}^m \times \mathbb{R}^m$, $\Psi(x', v) = (\psi(x'), [v]_\psi)$ and Ψ^{-1} be its inverse. Then with the notation established above, we have that the composition

$$\Psi \circ \Phi^{-1} : \Phi(\pi^{-1}(\mathcal{U}) \cap \pi^{-1}(\mathcal{V})) \longrightarrow \Psi(\pi^{-1}(\mathcal{U}) \cap \pi^{-1}(\mathcal{V}))$$

$$\Psi \circ \Phi^{-1}\left((x_1, \dots, x_m), (p_1, \dots, p_m)\right) = \left(\psi \circ \varphi^{-1}(x_1, \dots, x_m), \left(\sum_{i=1}^m a_{i1} p_i, \dots, \sum_{i=1}^m a_{im} p_i\right)\right)$$

is smooth.

Note that $\pi^{-1}(\mathcal{U}) \cap \pi^{-1}(\mathcal{V}) = \pi^{-1}(\mathcal{U} \cap \mathcal{V})$ and that $\Phi(\pi^{-1}(\mathcal{U} \cap \mathcal{V})) = \varphi(\mathcal{U} \cap \mathcal{V}) \times \mathbb{R}^m$ which is clearly an open subset of $\Phi(\pi^{-1}(\mathcal{U})) = \varphi(\mathcal{U}) \times \mathbb{R}^m$ (since $\varphi(\mathcal{U} \cap \mathcal{V})$ is open in $\varphi(\mathcal{U})$). Now we have proved in Lemma A that the map $\Psi \circ \Phi^{-1}$ is smooth. Since it is 1-1 and since the role of Φ and Ψ is symmetric, thus also $\Phi \circ \Psi^{-1}$ is smooth. Hence $\Psi \circ \Phi^{-1}$ is a diffeomorphism and also a homeomorphism.

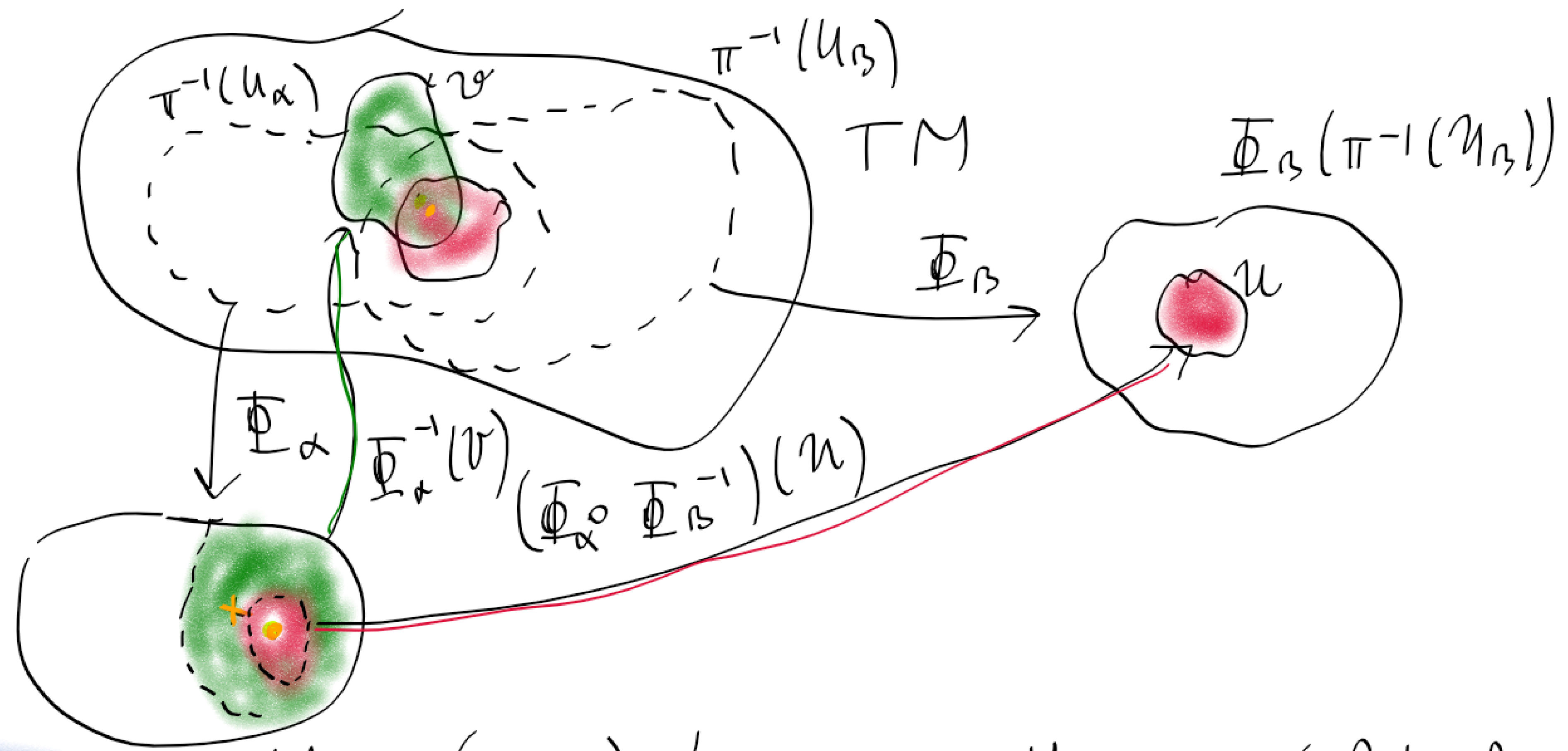
Now it is easy to see that the collection

$$\mathcal{B} = \{ \Phi_\alpha^{-1}(V) : \alpha \in A, \ V \text{ open in } \Phi_\alpha(\pi^{-1}(\mathcal{U}_\alpha)) \}$$

satisfies (B1) and (B2). Let τ_{TM} be the topology on TM generated by $\mathcal{B}$. We now claim that

Lemma B For any $\alpha \in A$ the map
$$\Phi_\alpha^{-1} : \Phi_\alpha(\pi^{-1}(U_\alpha)) \longrightarrow TM$$
is a homeomorphism onto its image.

Proof: By definition, Φ_α^{-1} is open, i.e. it maps open sets to open sets. Thus it remains to show that Φ_α^{-1} is also continuous. Let V be open in TM and consider $\Phi_\alpha^{-1}(V) \subseteq \Phi_\alpha(\pi^{-1}(U_\alpha))$. If $x \in \Phi_\alpha^{-1}(V)$, then $\Phi_\alpha^{-1}(x) \in V$ and by definition, there is $B \in B$ and an open subset U of $\Phi_B(\pi^{-1}(U_B))$ such that $\Phi_\alpha^{-1}(x) \in \Phi_B^{-1}(U) \subseteq V$. But then $(\Phi_\alpha \circ \Phi_B^{-1})^{-1}(U)$ is open in $\Phi_\alpha(\pi^{-1}(U_\alpha))$ and it contains x. Hence $\Phi_\alpha^{-1}(V)$ is open in $\Phi_\alpha(\pi^{-1}(U_\alpha))$. $\square$



Theorem A Let (M, A) be a smooth manifold of dimension m with atlas $A = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^m : \alpha \in A\}$. Then (TM, A_{TM}) where
$$A_{TM} = \{\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m} : \alpha \in A\}$$
is a smooth manifold of dimension $2m$ and $\pi : TM \to M$ is smooth.

Proof: By Lemma A, transition functions are smooth. Obviously $\bigcup_{\alpha\in A} \pi^{-1}(U_\alpha) = TM$. By Lemma B, each map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to \Phi_\alpha(\pi^{-1}(U_\alpha))$ is a homeomorphism onto its image which is clearly open in $\mathbb{R}^{2m}$ (Hence we have (TM1) & (TM2)). Now it also clear that the topology is Hausdorff and that it is second countable (left as an exercise).
The composition $\varphi_\alpha(U_\alpha) \times \mathbb{R}^m = \Phi_\alpha(\pi^{-1}(U_\alpha)) \xrightarrow{\Phi_\alpha^{-1}} \pi^{-1}(U_\alpha) \xrightarrow{\pi} U_\alpha \xrightarrow{\varphi_\alpha} \varphi_\alpha(U)$ is the canonical projection onto the first factor and it is obviously smooth. This proves the second claim. $\square$

Tangent map of smooth map

Assume that (M, A_M) and (N, A_N) are smooth manifolds and that $\Phi : M \to N$ is smooth. Let us fix $x_0 \in M$ and consider $[\gamma] \in T_{x_0}M$. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be its representative. Then $\Phi \circ \gamma(t)$ is a smooth curve with $\Phi \circ \gamma(0) = \Phi(x_0)$. Now if $f \in C^\infty_{A_N}(N)$, then $f \circ \Phi \in C^\infty_{A_M}(M)$ and so
$$\frac{d}{dt} f(\Phi \circ \gamma_1(t))\Big|_{t=0} = \frac{d}{dt}(f \circ \Phi)(\gamma_1(t))\Big|_{t=0} = \frac{d}{dt}(f \circ \Phi)(\gamma_2(t))\Big|_{t=0} = \frac{d}{dt} f(\Phi \circ \gamma_2(t))\Big|_{t=0}$$
if $\gamma_1 \sim \gamma_2$. This shows $[\gamma_1] = [\gamma_2] \in T_{x_0}M$. Hence
$$[\Phi \circ \gamma_1] = [\Phi \circ \gamma_2] \in T_{\Phi(x_0)} M.$$

Definition With the notation set above, we call the map
$$T_{x_0}\Phi : T_{x_0}M \longrightarrow T_{\Phi(x_0)}N, \quad T_{x_0}\Phi([\gamma]) = [\Phi \circ \gamma]$$
the tangent map of Φ at x_0.

Example If $\varphi : U \to \mathbb{R}^m$ is a chart around x_0 as in Lecture II, then
$$T_0\varphi^{-1} : T_0\mathbb{R}^m = \mathbb{R}^m \longrightarrow T_{x_0}M$$ coincides with the map denoted by ϕ_{x_0}.

Assume now that $\varphi : U \to \mathbb{R}^m$ is a chart around x_0 with $\varphi(x_0) = 0$ with coordinate functions $x_1, \dots, x_m$ and that $\rho : W \to \mathbb{R}^n$ is a chart around $\Phi(x_0)$ with $\rho(\Phi(x_0)) = 0$ and coordinate functions $y_1, \dots, y_m$. Then $\psi \circ \Phi \circ \varphi^{-1}$ is a smooth map defined on some open neighbourhood of $0 \in \mathbb{R}^m$ and let us write $y_i(x_1, \dots, x_m) = \psi_i \circ \Phi \circ \varphi^{-1}(x_1, \dots, x_m)$, here ψ_i is the i-th component of ψ, $i = 1, \dots, n$.

Then the Jacobi matrix of $D(\psi \circ \Phi \circ \varphi^{-1}(0))$ is

$$M = \begin{pmatrix} \frac{\partial y_1}{\partial x_1}(0) & \cdots & \frac{\partial y_1}{\partial x_m}(0) \\ \vdots & & \vdots \\ \frac{\partial y_1}{\partial x_m}(0) & \cdots & \frac{\partial y_m}{\partial x_m}(0) \end{pmatrix}.$$

Thus the following commutative diagram

$$\text{(D1)} \quad \begin{array}{ccc} T_{x_0}M & \xrightarrow{T_{x_0}\Phi} & T_{\Phi(x_0)}N \\ T_{x_0}\varphi \downarrow & \circ & \downarrow T_{\Phi(x_0)}\psi \\ \mathbb{R}^m = T_0\mathbb{R}^m & \xrightarrow{L} & \mathbb{R}^n = T_0\mathbb{R}^n \end{array}$$

where the linear map L is given by the matrix M.

Corollary 1) The tangent map $T_{x_0}\Phi$ is linear.

2) Assume that $\Psi: N \to L$ is a smooth map between $(N, \mathcal{A}_N)$ and a smooth manifold $(L, \mathcal{A}_L)$. If Φ is as above, then

$$T_{x_0}(\Psi \circ \Phi) = T_{\Phi(x_0)}\Psi \circ T_{x_0}\Phi.$$

3) The map $Tf: TM \to TN$, $Tf(x,v) = T_x f(v)$ is a smooth map of manifolds

Proof: ad 1) By the commutativity of (D1), $T_{x_0}\Phi = T_{\Phi(x_0)}\psi^{-1} \circ L \circ T_{x_0}\varphi$ and the right hand side is a composition of linear maps. ad 2) Is clear.

ad 3) Follows from the fact that the coefficients of L are smooth functions of $x_1, \dots, x_m$.

Vector fields

Definition Let $(M, \mathcal{A})$ be a smooth manifold of dimension m and let $(TM, \mathcal{A}_{TM})$ be the associated tangent bundle with projection $\pi: TM \to M$.

A smooth vector field on M is a smooth map

$$X: M \to TM \quad \text{such that} \quad \pi \circ X = Id_M.$$

We denote by $\mathcal{X}(M)$ the set of all vector fields on M.

Local description of vector fields

Let $\varphi: U \to \mathbb{R}^m$ be a chart on M from atlas $\mathcal{A}$. As we have seen, then the set $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\} = B^\varphi_x$ is a basis of $T_x M$ (for every $x \in U$).

By definition, for $X \in \mathcal{X}(M)$ and $x \in M$: $X(x) \in T_x M$ and so there exist unique functions $a_1, \dots, a_m$ on U such that

$$X(x) = \sum_{i=1}^m a_i(x) \frac{\partial}{\partial x_i}.$$

Now since X is a smooth map $M \to TM$, then we have that the composition

$$(A): \varphi(U) \xrightarrow{\varphi_\alpha^{-1}} U \xrightarrow{X} \pi^{-1}(U) \xrightarrow{\Phi_\alpha} \Phi(\pi^{-1}(U)) = \varphi(U) \times \mathbb{R}^m$$

is smooth. If $x_1, \dots, x_m$ are the coordinate functions on U, then (A) is

$$\varphi(U) \ni (x_1, \dots, x_m) \longmapsto ((x_1, \dots, x_m), (a_1(x_1, \dots, x_m), \dots, a_m(x_1, \dots, x_m))) \in \varphi(U) \times \mathbb{R}^m.$$

We see that the coefficients $a_1, \dots, a_m$ are smooth functions of the coordinate functions $x_1, \dots, x_m$.

If $\psi: V \to \mathbb{R}^m$ is the other chart on M from (CoB) that gives rise to the basis $B^\psi_x = \{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m}\}$ of $T_x M$ for every $x \in V$, then for every $x \in U \cap V$ we have that

$$X(x) = \sum_{j=1}^m b_j(x) \frac{\partial}{\partial y_j}$$

where $b_1, \dots, b_m$ are by (CoB) related by

$$X(x) = \sum_{j=1}^m a_j \frac{\partial}{\partial x_j} = \sum_{j=1}^m a_j \sum_{i=1}^m a_{ij} \frac{\partial}{\partial y_i} = \sum_{i=1}^m b_i \frac{\partial}{\partial y_i} = X(x)$$

and so

$$\sum_{j=1}^m a_j a_{ij} = b_i$$

where $a_{ij} = \dfrac{\partial(\psi \circ \varphi)^{-1}_i}{\partial x_j}$; $i, j = 1, \dots, m$.

Since $\mathbb{X}(x) \in T_x M$ and $T_x M$ is a vector space, we can define the following operations for $\mathbb{X}, \mathbb{Y} \in \mathfrak{X}(M)$ and $\lambda \in \mathbb{R}$:

(O1) $(\mathbb{X}+\mathbb{Y})(x) = \mathbb{X}(x) + \mathbb{Y}(x)$,

(O2) $(\lambda \mathbb{X})(x) = \lambda \mathbb{X}(x)$.

If $c_1, \dots, c_m$ are coefficients of $\mathbb{Y}$ on $\mathcal{U}$ w.r. to $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$, then $a_1+c_1, \dots, a_m+c_m$ are coefficients of $\mathbb{X}+\mathbb{Y}$ and $\lambda a_1, \dots, \lambda a_m$ are coefficients of $\lambda\mathbb{X}$ on $\mathcal{U}$ w.r. to $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$. Since $a_1+c_1, \dots, a_m+c_m$ and $\lambda a_1, \dots, \lambda a_m$ are smooth functions of $x_1, \dots, x_m$, then we have proved that

Corollary Operations (O1) and (O2) turn $\mathfrak{X}(M)$ into a real vector space.

The next goal is to show that there is also another binary operation on $\mathfrak{X}(M)$, called the Lie bracket. In order to define this operation, we will introduce an alternative description of $\mathfrak{X}(M)$.

Definition A derivation on $(M, \mathcal{A})$ is an $\mathbb{R}$-linear map
$$D: \mathcal{C}^\infty_{\mathcal{A}}(M) \longrightarrow \mathcal{C}^\infty_{\mathcal{A}}(M)$$
that satisfies Leibniz rule
$$\forall x \in M: \quad D(fg)(x) = (Df)(x)\, g(x) + f(x)(Dg)(x).$$
We denote by $D(M)$ the set of all derivations on M.

Remark: We used in the definition of a derivation that $\mathcal{C}^\infty_{\mathcal{A}}(M)$ is a vector space (this fact should be clear).

Theorem 1) The space $D(M)$ is a real vector space.

2) Fix $x \in M$ and $D \in D(M)$. Then $\mathcal{C}^\infty_{\mathcal{A}}(M) \ni f \longmapsto Df \in \mathcal{C}^\infty_{\mathcal{A}}(M) \longmapsto (Df)(x)$ is a derivation at x.

3) If $D_1, D_2 \in D(M)$, then also the assignment
$$\mathcal{C}^\infty_{\mathcal{A}}(M) \ni f \longmapsto [D_1, D_2] f := D_1 D_2 f - D_2 D_1 f$$
is a derivation on M.

Proof: 1) If $D_1, D_2 \in D(M)$ and $\lambda \in \mathbb{R}$, then also
$$(D_1+D_2)(f) = D_1 f + D_2 f, \quad (\lambda D_1) f = \lambda (D_1 f)$$
are derivations on M. This proves the first claim.

2) Is obvious.

3) Let $f, g \in \mathcal{C}^\infty_{\mathcal{A}}(M)$ and $D_1, D_2 \in D(M)$. It is clear that $[D_1, D_2]$ is $\mathbb{R}$-linear since composition and difference of linear operators is linear. It remains to verify that $[D_1, D_2]$ satisfies Leibniz rule. We have
$$D_1(D_2(fg)) = D_1(D_2 f \cdot g + f \cdot D_2 g) = (D_1(D_2 f)) \cdot g + D_2 f \cdot D_1 g + D_1 f \cdot D_2 g + f\,(D_1(D_2 g)).$$
Thus
$$D_1(D_2(fg)) - D_2(D_1(fg)) = (D_1(D_2 f))g + f(D_1(D_2 g)) - (D_2(D_1 f))g - f(D_2(D_1 g))$$
$$= ([D_1, D_2] f) g + f([D_1, D_2] g).$$
Here we use that the multiplication $\cdot$ on $\mathcal{C}^\infty_{\mathcal{A}}(M)$ is commutative.

Definition We call $[D_1, D_2]$ the commutator of D_1 and D_2.

Let $\mathbb{X} \in \mathfrak{X}(M)$ and $\mathbb{X}(x) = [\gamma_x] \in T_x M$. Recall from lecture two that
$$D_{[\gamma_x]} f = \frac{d}{dt} f(\gamma_x(t))\big|_{t=0} \quad \text{is a derivation at } x \in M, \text{ i.e.}$$
$D_{[\gamma_x]} \in D_x M$, where $f \in \mathcal{C}^\infty_{\mathcal{A}}(M)$. Let $D_{\mathbb{X}} f$ be the function on M defined by $(M \ni x \longmapsto D_{\mathbb{X}} f(x) := D_{[\gamma_x]} f)$.

Theorem 1) Let $\mathfrak{X} \in \mathfrak{X}(M)$. Then $D_{\mathfrak{X}} f \in \mathcal{C}^{\infty}_{\mathbb{R}}(M)$ if $f \in \mathcal{C}^{\infty}_{\mathbb{R}}(M)$ and

$$D_{\mathfrak{X}} : \mathcal{C}^{\infty}_{\mathbb{R}}(M) \longrightarrow \mathcal{C}^{\infty}_{\mathbb{R}}(M), \; f \longmapsto D_{\mathfrak{X}} f$$

is a derivation on M.

2) The map $\mathfrak{X}(M) \longrightarrow D(M)$, $\mathfrak{X} \longmapsto D_{\mathfrak{X}}$ is an isomorphism of real vector spaces.

Proof: ad 1) We know from lecture two that $D_{[\gamma_x]} \in D_x M$ and so $D_{\mathfrak{X}}$ satisfies Leibniz rule. It is also clear that

$$D_{\mathfrak{X}}(f+g) = D_{\mathfrak{X}} f + D_{\mathfrak{X}} g, \quad D_{\mathfrak{X}}(\lambda f) = \lambda D_{\mathfrak{X}} f.$$

Hence it remains to show that $D_{\mathfrak{X}} f \in \mathcal{C}^{\infty}_{\mathbb{R}}(M)$ for $f \in \mathcal{C}^{\infty}_{\mathbb{R}}(M)$.

It is enough to verify $D_{\mathfrak{X}} f$ is smooth in any chart on M.

Let $\varphi : U \to \mathbb{R}^m$ be a chart on M with coordinate functions $x_1, \ldots, x_m$ and the corresponding bases $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\}$ of every $T_x M$ with $x \in U$.

Assume that

$$\mathfrak{X} = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i} \quad \text{and} \quad f \in \mathcal{C}^{\infty}_{\mathbb{R}}(M).$$

Then we know that a_i are smooth functions of coordinate functions $x_1, \ldots, x_m$ and the same is true for $f \circ \varphi^{-1}$. Then by definition, $D_{\mathfrak{X}} f$ as a function of $x_1, \ldots, x_m$ is given by $\sum_{i=1}^{m} a_i(x) \frac{\partial f \circ \varphi^{-1}}{\partial x_i}$. We see that $D_{\mathfrak{X}} f$ is a smooth function on U and since U was arbitrary, $D_{\mathfrak{X}} f$ is smooth on the whole of M.

ad 2) By 1), we know that for fixed $x \in M$ and $D \in D(M)$ the composition

$$f \longmapsto Df(x), \quad f \in \mathcal{C}^{\infty}(M),$$

is a derivation at x.

Let $X_x \in D_x M$ be this derivation at x. We also know that

$$X_x = \sum_i a_i(x) \frac{\partial}{\partial x_i} \Longleftrightarrow X_x x_i = a_i(x), \quad x = 1, \ldots, m.$$

But $X_x x_i = (D x_i)(x)$ (by Remark (Der) from lecture 2.) is a smooth function on U for every $i = 1, \ldots, m$ and so $U \ni x \longmapsto a_i(x)$ is smooth.

This shows that the map

$$\mathfrak{X}_D : M \longrightarrow TM, \; \mathfrak{X}_D(x) = X_x \in D_x M \cong T_x M$$

is a smooth map of manifolds. As clearly $\pi \circ \mathfrak{X}_D = \mathrm{Id}_M$, $\mathfrak{X}_D$ is a smooth vector field on M. Since the maps

$$\mathfrak{X}(M) \longrightarrow D(M), \; \mathfrak{X} \longmapsto \mathfrak{X}_D \quad \text{and} \quad D(M) \longrightarrow \mathfrak{X}(M), \; D \longrightarrow D_{\mathfrak{X}}$$

are inverse to each other and linear, the proof is complete. $\square$

Notation We will write $\mathfrak{X} f := D_{\mathfrak{X}} f$ for $\mathfrak{X} \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^{\infty}_{\mathbb{R}}(M)$.

Definition Let $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{X}(M)$. We denote by Z the unique vector field on M such that $D_Z = [D_{\mathfrak{X}}, D_{\mathfrak{Y}}]$. We call Z the Lie bracket of vector fields $\mathfrak{X}$ and $\mathfrak{Y}$ and write $Z = [\mathfrak{X}, \mathfrak{Y}]$.

Theorem (Local form of Lie bracket)

Let $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{X}(M)$ and $\varphi : U \to \mathbb{R}^m$ be a chart on M. Assume that on U

$$\mathfrak{X} = \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i}, \quad \mathfrak{Y} = \sum_{j=1}^{m} b_j \frac{\partial}{\partial x_j} \quad \text{for some (unique) smooth functions on } U.$$

Then $Z = [\mathfrak{X}, \mathfrak{Y}]$ on U has the form

$$[D_{\mathfrak{X}}, D_{\mathfrak{Y}}] = \sum_{j=1}^{m} \left(\sum_{i=1}^{m} \left(a_i \left(\frac{\partial}{\partial x_i} b_j \right) - b_i \left(\frac{\partial}{\partial x_i} a_j \right) \right) \right) \frac{\partial}{\partial x_j}.$$

Proof: Let $D_{\mathfrak{X}}$ and $D_{\mathfrak{Y}}$ be the corresponding derivations on M. By definition, we need to show that for every $f \in \mathcal{C}^{\infty}_{\mathbb{R}}(M)$.

$$[D_{\mathfrak{X}}, D_{\mathfrak{Y}}] f = \sum_{j=1}^{m} \left(\sum_{i=1}^{m} \left(a_i \left(\frac{\partial}{\partial x_i} b_j \right) - b_i \left(\frac{\partial}{\partial x_i} a_j \right) \right) \right) \frac{\partial}{\partial x_j} f.$$

On U we have

$$(D_X D_Y f)(x) = \sum_{i,j=1}^{m} a_i \left(\frac{\partial(b_j \circ \varphi^{-1})}{\partial x_i}\right)(\varphi(x)) \left(\frac{\partial(f\circ\varphi^{-1})}{\partial x_j}\right)(\varphi(x)) + \sum_{i,j=1}^{m} a_i b_j \left(\frac{\partial^2(f\circ\varphi^{-1})}{\partial x_i \partial x_j}\right)(\varphi(x))$$

$$(D_Y D_X f)(x) = \sum_{i,j=1}^{m} b_j \left(\frac{\partial(a_i \circ \varphi^{-1})}{\partial x_i}\right)(\varphi(x)) \left(\frac{\partial(f\circ\varphi^{-1})}{\partial x_j}\right)(\varphi(x)) + \sum_{i,j=1}^{m} a_i b_j \left(\frac{\partial^2(f\circ\varphi^{-1})}{\partial x_i \partial x_j}\right)(\varphi(x)).$$

But since $\frac{\partial^2(f\circ\varphi^{-1})}{\partial x_i \partial x_j}(\varphi(x)) = \frac{\partial^2(f\circ\varphi^{-1})}{\partial x_j \partial x_i}(\varphi(x))$ we obtain the result. □

Theorem (Properties of Lie bracket)
Let $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty_{\mathbb{R}}(M)$ and $\alpha, \beta \in \mathbb{R}$. Then

1) $[\alpha X + \beta Y, Z] = [\alpha X, Z] + [\beta Y, Z]$
2) $[X, \alpha Y + \beta Z] = [X, \alpha Y] + [X, \beta Z]$
3) $[X, Y] = -[Y, X]$ skew-symmetry
4) $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$ Jacobi identity
5) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$

Proof: It is enough to prove these identities for derivations $D_1, D_2, D_3 \in D(M)$.
ad 1) & ad 2) & ad 3) are obvious identities for linear operators on $C^\infty_{\mathbb{R}}(M)$.
ad 4)

$$[D_1, [D_2, D_3]] = [D_1, (D_2D_3 - D_3D_2)] = \underline{D_1D_2D_3} - \underline{D_1D_3D_2} - \underline{D_2D_3D_1} + \underline{D_3D_2D_1}$$

$$[[D_1, D_2], D_3] = \underline{D_1D_2D_3} - \cancel{D_2D_1D_3} - \cancel{D_3D_1D_2} + \underline{D_3D_2D_1}$$

$$[D_2, [D_1, D_3]] = \cancel{D_2D_1D_3} - \underline{D_2D_3D_1} - \underline{D_1D_3D_2} + \cancel{D_3D_1D_2}$$

We have verified that both sides in 4) agree.
ad 5) Let $h \in C^\infty_{\mathbb{R}}(M)$. We have to show that

$$[fD_1, gD_2]h = fg[D_1, D_2]h + f(D_1g)D_2h - g(D_2f)D_1h.$$

The left hand side is

$$fD_1(gD_2h) - gD_2(fD_1h) = fgD_1(D_2h) - gfD_1(D_2h)$$
$$+ f(D_1g)D_2h - g(D_2f)D_1h =$$
$$fg[D_1, D_2]h - f(D_1g)D_2h - g(D_2f)D_1h.$$ □

Corollary $(\mathfrak{X}(M), [-,-])$ is an (infinite dimensional) Lie algebra.