Short recapitulation from last week

$(M, \mathcal{A})$ .. smooth manifold of dim. m with $\mathcal{A} = \{\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^m : \alpha \in A\}$
let $\varphi : \mathcal{U} \to \mathbb{R}^m$ be a chart around $x_0 \in M$ with $\varphi(x_0) = 0$
$x_1, \dots, x_m$ be coordinate function for φ
$\mathcal{C}^\infty_{\mathcal{A}}(M)$.. the set of all smooth functions $M \to \mathbb{R}$

Defined (last week): 1) $T_{x_0}M = \{[\gamma] \mid \gamma : (-\varepsilon_\gamma, \varepsilon_\gamma) \to M$ smooth path with $\gamma(0) = x_0$ and $\varepsilon_\gamma > 0\}$

is called the tangent space of M at x_0 and
$\gamma_1 \sim \gamma_2$ if $\forall f \in \mathcal{C}^\infty_{\mathcal{A}}(M)$: $\frac{d}{dt} f(\gamma_1(t))\Big|_{t=0} = \frac{d}{dt} f(\gamma_2(t))\Big|_{t=0}$.

We call $[\gamma] \in T_{x_0}M$ a tangent vector at x_0.

2) A derivation at $x_0 \in M$ is an $\mathbb{R}$-linear map
$$X : \mathcal{C}^\infty_{\mathcal{A}}(M) \to \mathbb{R}$$
which satisfies Leibniz rule:
$$X(fg) = X(f) \cdot g(x_0) + f(x_0) X(g) \text{ for every } f, g \in \mathcal{C}^\infty_{\mathcal{A}}(M),$$
We denote the set of all derivations at x_0 by $D_{x_0}M$.

Then we showed that:

1) $\phi_{x_0} : \mathbb{R}^m \to T_{x_0}M$, $\phi_{x_0}(v) = [\gamma_v]$
$$\gamma_v(t) = \varphi^{-1}(vt), \quad t \in (-\varepsilon, \varepsilon)$$
where $\varepsilon > 0$ is so small so that
$$vt \in \varphi(\mathcal{U}).$$
The map ϕ_{x_0} is 1-1 and so there is a unique linear structure on $T_{x_0}M$ for which ϕ_{x_0} is a linear isomorphism.

2) $\Psi_{x_0} : T_{x_0}M \to D_{x_0}M$, $[\gamma] \mapsto D_{[\gamma]}$
$$D_{[\gamma]}(f) = \frac{d}{dt} f(\gamma(t))\Big|_{t=0}$$
is well defined by definition of $T_{x_0}M$.

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3) •) $D_{x_0}M$ is in a canonical way a vector space with operations
$$(X_1 + X_2)(f) := X_1(f) + X_2(f)$$
$$(\lambda X_1)(f) := \lambda(X(f))$$
when $X_1, X_2 \in D_{x_0}M$ and $\lambda \in \mathbb{R}$.

•) If $f, g \in \mathcal{C}^\infty_{\mathcal{A}}(M)$ with $f = g$ on some open neighbourhood of x_0, then $X(f) = X(g)$ for any $X \in D_{x_0}M$.

•) If $f : \mathcal{U} \to \mathbb{R}$ is smooth (that is $f \circ \varphi^{-1} : \varphi(\mathcal{U}) \to \mathbb{R}$ is smooth), then there $\exists f^{ex} \in \mathcal{C}^\infty_{\mathcal{A}}(M)$ such that $f^{ex} = f$ on some open neigh. of x_0. And then by the previous property of derivations, we can define $X(f) := X(f^{ex})$ for any $X \in D_{x_0}M$.

4) For $f \in \mathcal{C}^\infty_{\mathcal{U}}(M)$ we put $\quad (f \circ \varphi^{-1} : \underset{\overset{\psi}{0}}{\varphi(\mathcal{U})} \to \mathbb{R})$

$$\frac{\partial}{\partial x_i} f := \frac{\partial f \circ \varphi^{-1}}{\partial x_i}(0), \quad i = 1, \dots, m.$$

Then $\frac{\partial}{\partial x_i} \in D_{x_0}M$ and $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$ is a basis of $D_{x_0}M$.

If $X \in D_{x_0}M$, then $X = \sum_{i=1}^m a_i \frac{\partial}{\partial x_i} \iff$

$$a_i = \frac{\partial}{\partial x_i} x_i, \quad i = 1, \dots, m.$$

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5) $\mathbb{D} : \mathbb{R}^m \to D_{x_0}M, \quad (\mathbb{D}(v))(f) := \frac{d}{dt} f(\varphi^{-1}(vt))\,|_{t=0}$

is a linear isomorphism and sends the canonical basis $\{e_1, \dots, e_m)$ of $\mathbb{R}^m$ to $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$.

$$\begin{tikzcd}
\mathbb{R}^m \arrow[rr, "\phi_{x_0}"] \arrow[dr, "\mathbb{D}"'] & \arrow[d, phantom, "\circ"] & T_{x_0}M \arrow[dl, "\psi_{x_0}"] \\
& D_{x_0}M &
\end{tikzcd}$$

diagram commutes and maps ϕ_{x_0} and $\mathbb{D}$ are linear isomorphisms $\Rightarrow$ ψ_{x_0} is a linear isomorphism $\Rightarrow$

<u>Corollary</u> The linear structure on $T_{x_0}M$ does not depend on the choice of φ.

<u>Lecture two</u>

<u>Tangent bundle of a manifold</u>

Assume that $\psi : V \to \mathbb{R}^m$ is another chart on M from $\mathcal{U}$ around x_0 with coordinate functions $y_1, \dots, y_m$ and $\psi(x_0) = 0$.

Then we obtain two bases of $T_{x_0}M$

$$B^{\varphi}_{x_0} = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\} \quad \text{and} \quad B^{\psi}_{x_0} = \{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m}\}.$$

Question: what is the change of basis matrix.

Let $f \in \mathcal{C}^\infty_{\mathcal{U}}(M)$. Then

$$\frac{\partial}{\partial x_j} f = \frac{\partial f \circ \varphi^{-1}}{\partial x_j}(0) = \frac{\partial (f \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})}{\partial x_j}(0) =$$

(C1)

$$= \sum_{i=1}^m \frac{\partial f \circ \psi^{-1}}{\partial y_i}(\underbrace{\psi \circ \varphi^{-1}(0)}_{= 0}) \cdot \frac{\partial (\psi \circ \varphi^{-1})_i}{\partial x_j}(0) = \sum_{i=1}^m \frac{\partial}{\partial y_i} f \, \frac{\partial (\psi \circ \varphi^{-1})_i}{\partial x_j}(0)$$

We see that $[\mathrm{Id}]^{B^{\varphi}_{x_0}}_{B^{\psi}_{x_0}} = (a_{ij})_{i=1,\dots,m}^{j=1,\dots,m}$ where $a_{ij} = \dfrac{\partial (\psi \circ \varphi^{-1})_i}{\partial x_j}(0)$

is the change of basis matrix. This is the Jacobi matrix of the transition function $\psi \circ \varphi^{-1}$ at 0.

Note that
$$B^{\varphi}_{x} = \{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \}$$
is a basis for $T_x M$ for every $x \in \mathcal{U}$. Also note that the coefficients of the change of basis matrix are smooth functions of $(x_1, \dots, x_m)$ resp. $(y_1, \dots, y_m)$ on $\mathcal{U} \cap \mathcal{V}$.

Let $TM = \bigcup_{x \in M} T_x M$ ... disjoint union and
$$p_{TM} : TM \longrightarrow M, \quad v \in T_x M \longmapsto x.$$
(We will write also (x, v) instead $v \in T_x M$.)

Definition We call TM the tangent bundle of M and p_{TM} is called the canonical projection.

$$TM = \bigcup_{\alpha \in A} p^{-1}(\mathcal{U}_\alpha), \quad M = \bigcup_{\alpha \in A} \mathcal{U}_\alpha \quad \text{where } p = p_{TM}.$$

For every $\alpha \in A$:
$$\Phi_\alpha : p^{-1}(\mathcal{U}_\alpha) \longrightarrow \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$$
$$\Phi_\alpha(x, v) = (\varphi_\alpha(x), [v]_{B^{\varphi}_{x}})$$

Note that
$$\Phi_\alpha^{-1} : \varphi(\mathcal{U}) \times \mathbb{R}^m \longrightarrow p^{-1}(\mathcal{U}_\alpha)$$
$$(\underbrace{(x_1, \dots, x_m)}_{\vec{x}}, (a_1, \dots, a_m)) \longmapsto \left(\varphi_\alpha^{-1}(x), \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i}\right)$$

$$\mathcal{A}_{TM} = \{ \Phi_\alpha : p^{-1}(\mathcal{U}_\alpha) \longrightarrow \mathbb{R}^{2m} \mid \alpha \in A \}$$

<u>Lemma</u> There is a unique topology τ_{TM} on TM for which all maps Φ_α, $\alpha \in A$, are homeomorphisms onto their images. Moreover, τ_{TM} is Hausdorff and second countable.

<u>Corollary</u> $(TM, \mathcal{A}_{TM})$ is a smooth manifold of dim $2m$ and $p_{TM} : TM \longrightarrow M$ is smooth.

Proof: The first claim follows from previous Lemma and computation (C1). Let us go over the proof of the second claim. By Proposition on page 2 from lecture notes for the second week it is enough to show:

$\forall (x,v)\in TM$ $\exists\, \alpha\in A$ and $\exists\, \beta\in A$ such that $p^{-1}(U_\alpha)\ni (x,v)$ and $U_\beta\ni p(x,v)=x$ and the composition

$$\Phi_\alpha(p^{-1}(U_\alpha\cap U_\beta)) \xrightarrow{\Phi_\alpha^{-1}} \underbrace{p^{-1}(U_\alpha)\cap p^{-1}(U_\beta)}_{p^{-1}(U_\alpha\cap U_\beta)} \xrightarrow{p} \underbrace{U_\beta\cap p(p^{-1}(U_\alpha))}_{U_\beta\cap U_\alpha} \xrightarrow{\varphi_\beta} \varphi_\beta(U_\beta\cap U_\alpha)$$

is smooth. But here we can for α we can put $\beta=\alpha$ and then it is enough to show that

$$\Phi_\alpha(p^{-1}(U_\alpha)) \xrightarrow{\Phi_\alpha^{-1}} p^{-1}(U_\alpha) \xrightarrow{p} U_\alpha \xrightarrow{\varphi_\alpha} \varphi_\alpha(U_\alpha)$$

$\Phi_\alpha(p^{-1}(U_\alpha)) = \varphi_\alpha(U_\alpha)\times\mathbb{R}^m$ is smooth.

$$(\underbrace{(x_1,\dots,x_m)}_{\vec{x}},(a_1,\dots,a_m)) \longmapsto \left(\varphi_\alpha^{-1}(\vec{x}), \sum_{i=1}^m a_i \frac{\partial}{\partial x_i}\right) \longmapsto (\varphi_\alpha^{-1}(\vec{x})) \longmapsto \vec{x} = \varphi_\alpha\circ\varphi_\alpha^{-1}(\vec{x})$$

We see that the composition is the canonical projection

$$\varphi_\alpha(U_\alpha)\times\mathbb{R}^m \longrightarrow \varphi_\alpha(U_\alpha)$$

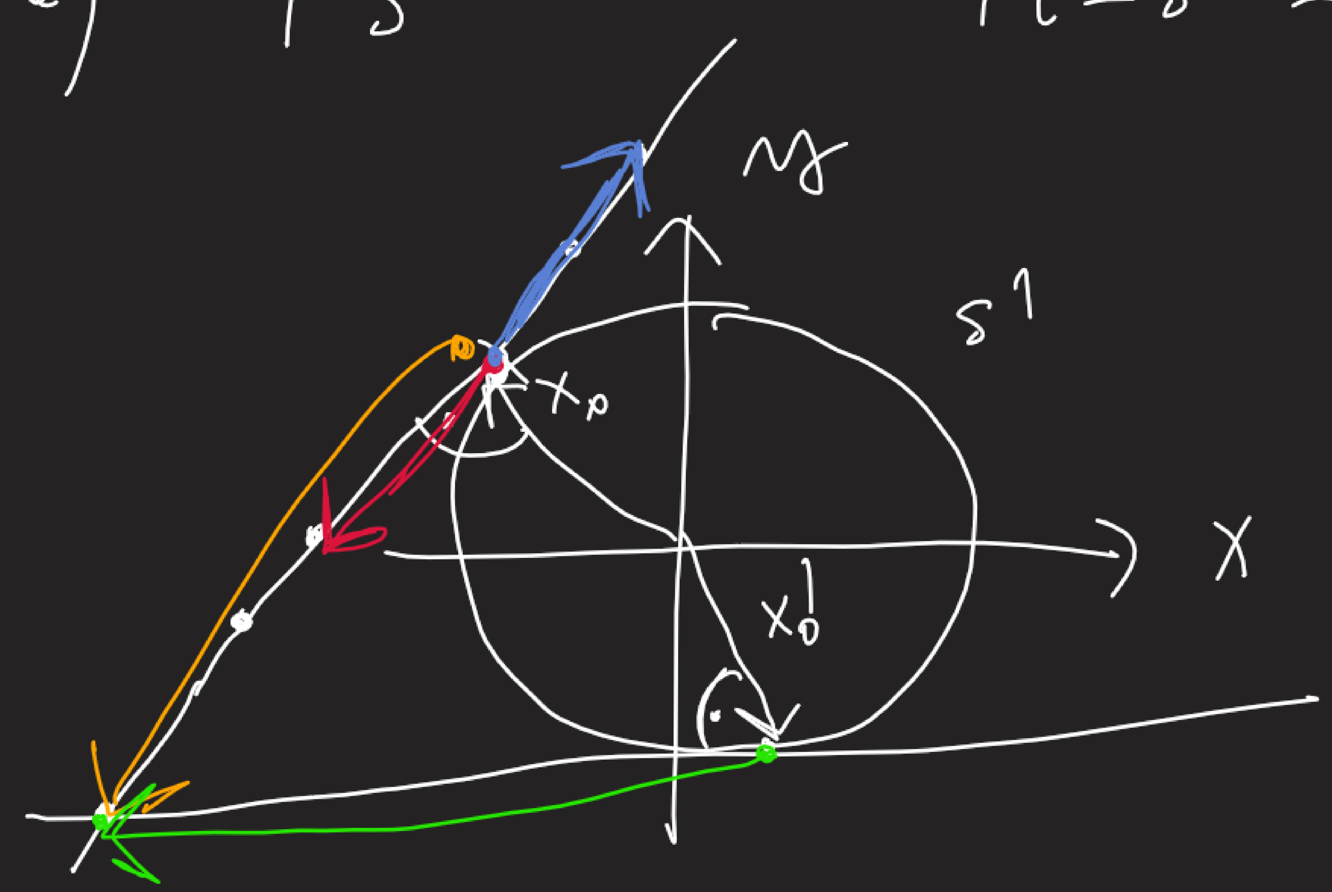
onto the first factor. This is clearly smooth. □

Examples

•) $M=\mathbb{R}^m$, $TM\cong\mathbb{R}^m\times\mathbb{R}^m=\mathbb{R}^{2m}$

$$\left((x_1,\dots,x_m), \sum a_i\frac{\partial}{\partial x_i}\right) \longleftarrow\!\shortmid ((x_1,\dots,x_m),(a_1,\dots,a_m))$$

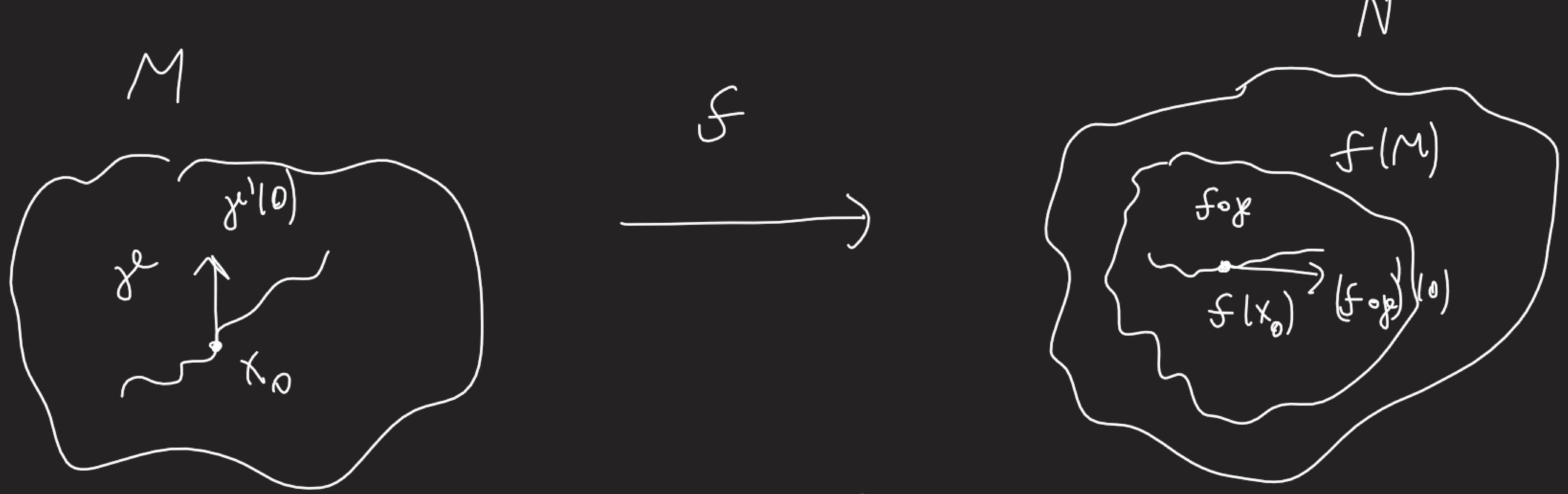
•) TS^1 $\qquad M=S^1\subseteq\mathbb{R}^2$



Tangent map of a smooth map between manifolds

Let us assume that $(N,\mathcal{A}_N)$ is a smooth manifold of dim n and $(M,\mathcal{A}_M)$ be the smooth manifold as above. Let $f:M\to N$ be smooth.

If $v = [\gamma] \in T_{x_0}M$, $x_0 \in M$, then it is easy to see that the equivalence class $f \circ \gamma$ in $T_{f(x_0)}N$ does not depend on γ.



Hence we have the following

<u>Definition</u> We call the map

$$T_{x_0}f : T_{x_0}M \longrightarrow T_{f(x_0)}N$$

$$T_{x_0}f([\gamma]) = [f\circ\gamma]$$

is called the tangent map of f at x_0.

Ex. $\phi_{x_0} : \mathbb{R}^m \longrightarrow T_{x_0}M$, $\phi_{x_0}(v) = [\varphi^{-1}(vt)]$
$\quad\overset{\shortparallel}{T_0\mathbb{R}^m}$ $\qquad \varphi : U \longrightarrow \mathbb{R}^m$ a chart on M

$\phi_{x_0} = T_0\varphi^{-1}$, 0.. origin of $\mathbb{R}^m$

Hence we have the following commutative digram

$$\begin{tikzcd} T_{x_0}M \arrow[r, "T_{x_0}f"] & T_{y_0}N \arrow[d, "T_{y_0}\rho"] \\ \mathbb{R}^m = T_0\mathbb{R}^m \arrow[u, "T_0\varphi^{-1}"] \arrow[r, "L"] & \mathbb{R}^n = T_0\mathbb{R}^n \arrow[ul, phantom, "\circ"] \end{tikzcd}$$

where $\rho : W \longrightarrow \mathbb{R}^n$ is a chart from $\mathcal{A}_N$ around $f(x_0) = y_0$ with coordinates $z_1, .., z_n$ and $\rho(y_0) = 0$ and where L is total derivative of the smooth map

$\rho\circ f\circ\varphi^{-1}$ at the point 0.

Note that $\rho\circ f\circ\varphi^{-1}$ is a smooth map of m real variables which is defined on some open neighb. of $0 \in \mathbb{R}^m$. So it has a total derivative or total differential $L = d(\rho\circ f\circ\varphi^{-1})(0)$ at 0.

We know that L is a linear map which is represented by its Jacobi matrix

$$\left(\frac{\partial \tilde{f}_i}{\partial x_j}\right)_{i=1,\dots,n}^{j=1,\dots,m} \quad \text{where } \tilde{f} = \rho \circ f \circ \varphi^{-1}$$

and we also know $T_0\varphi^{-1}$, $T_y \circ \rho$ are linear isomorphism.

Hence, by the commutativity of the previous diagram we get

<u>Corollary</u> •) The map $T_{x_0} f$ is linear.

•) If $g: N \to L$ is smooth where $(L, \mathcal{A}_L)$ is a smooth manifold, then

$$T_{x_1}(g \circ f) = T_{f(x_0)} g \circ T_{x_0} f.$$

•) $Tf: TM \to TN$, $Tf(x,v) = T_x f(v)$ is a smooth map of manifolds.

Vector fields on M

<u>Definition</u> A smooth vector field on a manifold $(M, \mathcal{A})$ is a smooth map

$$X: M \to TM$$

such that $p_{TM} \circ X = Id_M$. We denote by $\mathfrak{X}(M)$ the set of all vector fields on M.

<u>Local description of vector fields on M</u>

Let $\varphi: U \to \mathbb{R}^m$ be a chart on M with coordinate functions $x_1, \dots, x_m$. Then by definition

$$\varphi(U) \xrightarrow{\varphi^{-1}} U \xrightarrow{X} p^{-1}(U) \xrightarrow{\Phi} \Phi(p^{-1}(U)) = \varphi(U) \times \mathbb{R}^m$$

is smooth if X is smooth vector field. This composition

$$(x_1, \dots, x_m) = \vec{x} \longmapsto (\vec{x}, (a_1(x_1, \dots, x_m), \dots, a_m(x_1, \dots, x_m)))$$

Since $p \circ X = Id_M$ and $p|_{\Phi(p^{-1}(U))} = \varphi(U) \times \mathbb{R}^m$ is the canonical projection $\varphi(U) \times \mathbb{R}^m \to \mathbb{R}^n$ onto the first factor.

Moreover, we see that

$$X|_U = \sum_{i=1}^m a_i(x_1, \dots, x_m) \frac{\partial}{\partial x_i}.$$

We see that a smooth

vector field on M is on $\mathcal{U}$ given just by m-tuple of smooth functions $a_1, \dots, a_m$ which are functions of $x_1, \dots, x_m$.

$$\mathbb{X}(x) = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i}, \quad x \in \mathcal{U}.$$

Since $T_x M$ is a vector space and if $\mathbb{Y} \in \mathfrak{X}(M)$ with

$$\mathbb{Y}|_{\mathcal{U}} = \sum_{i=1}^{m} b_i \frac{\partial}{\partial x_i}$$ where again b_i are smooth function of $x_1, \dots, x_m$, then

we see that the operations:

$$(OP) \begin{cases} (\mathbb{X} + \mathbb{Y})(x) = \mathbb{X}(x) + \mathbb{Y}(x) \\ (\lambda \mathbb{X})(x) = \lambda \cdot \mathbb{X}(x) \end{cases}, \quad \lambda \in \mathbb{R}, \ x \in M$$

are well defined and since

$$(\mathbb{X} + \mathbb{Y})|_{\mathcal{U}} = \sum_{i=1}^{m} (a_i + b_i) \frac{\partial}{\partial x_i}$$ and

$$(\lambda \mathbb{X})|_{\mathcal{U}} = \sum_{i=1}^{m} \lambda a_i \frac{\partial}{\partial x_i}$$

where clearly $a_i + b_i$, λa_i are smooth functions of $x_1, \dots, x_m$, then we see that

Corollary $\mathfrak{X}(M)$ is a real vector space with operations $+, \cdot$ given in (OP).

There is another bilinear operation of $\mathfrak{X}(M)$, called the Lie bracket, which turns $\mathfrak{X}(M)$ into a Lie algebra. In order to define the Lie bracket we will need the following alternative description of $\mathfrak{X}(M)$.

Definition An $\mathbb{R}$-linear map

$$D : \mathcal{C}^\infty_{\mathcal{A}}(M) \to \mathcal{C}^\infty_{\mathcal{A}}(M)$$

is called a derivation on M if it satisfies Leibniz rule:

$$D(fg) = (Df) \cdot g + f \cdot (Dg)$$

where $f, g \in \mathcal{C}^\infty_{\mathcal{A}}(M)$. We denote by $D(M)$ the space of all derivations on M.

Lemma 1) $D(M)$ is a vector space with operations
$$(D_1+D_2)(f) = D_1 f + D_2 f \quad \text{and}$$
$$(\lambda D_1)(f) = \lambda \cdot D_1 f \quad \text{where}$$
$$f \in \mathcal{C}^\infty_{\mathcal{U}}(M),\ \lambda \in \mathbb{R},\ D_1, D_2 \in D(M).$$

2) If $x \in M$ is fixed and $D \in D(M)$, then the composition
$$\mathcal{C}^\infty_{\mathcal{U}}(M) \ni f \longrightarrow \underset{\mathcal{C}^\infty_{\mathcal{U}}(M)}{\underset{\cap}{Df}} \longrightarrow Df(x) \qquad \text{is a derivation at } x.$$

3) If $f \in \mathcal{C}^\infty_{\mathcal{U}}(M)$, $D \in D(M)$ and $\mathcal{U} \subseteq M$ is open, then $Df|_{\mathcal{U}}$ depends only on $f|_{\mathcal{U}}$.

4) If $D_1, D_2 \in D(M)$, then also $[D_1, D_2] \in D(M)$ where
$$[D_1, D_2] f = D_1(D_2 f) - D_2(D_1 f),\ f \in \mathcal{C}^\infty_{\mathcal{U}}(M).$$

Proof: 4) It is clear $[D_1, D_2]$ is $\mathbb{R}$-linear map $\mathcal{C}^\infty_{\mathcal{U}}(M) \longrightarrow \mathcal{C}^\infty_{\mathcal{U}}(M)$. Let us check that $[D_1, D_2]$ satisfies Leibniz rule. Let $f, g \in \mathcal{C}^\infty_{\mathcal{U}}(M)$, then
$$
\begin{aligned}
[D_1, D_2](fg) &= D_1(D_2(fg)) - D_2(D_1(fg)) = \\
&= D_1\big((D_2 f) g + f \cdot D_2 g\big) - D_2\big((D_1 f)\cdot g + f \cdot D_1 g\big) \\
&= D_1(D_2 f)\, g + \cancel{(D_2 f)(D_1 g)} + \cancel{(D_1 f)(D_2 g)} \\
&\quad + f \cdot (D_1(D_2 g)) - (D_2(D_1 f))\cdot g - \cancel{(D_1 f)(D_2 g)} \\
&\quad - \cancel{(D_2 f)\cdot(D_1 g)} - f\,(D_2(D_1 g)) \\
&= ([D_1, D_2] f)\cdot g + f \cdot ([D_1, D_2] g) \qquad \square
\end{aligned}
$$

Proposition There is a canonical linear isomorphism
$$\mathfrak{X}(M) \longrightarrow D(M).$$

Sketch of proof: Let $X \in \mathfrak{X}(M)$. Then $X(x) \in T_x M = D_x M, x \in M$.
$$f \in \mathcal{C}^\infty_{\mathcal{U}}(M) \longmapsto \underbrace{\big(M \ni x \mapsto (D_X f)(x) := X(x)\, f\big)}_{\text{a function on } M}$$

We have defined an operator $D_X : \mathcal{C}^\infty_{\mathcal{U}}(M) \longrightarrow \left\{ \begin{array}{l} f: M \to \mathbb{R}, \\ f \text{ is a function} \end{array} \right\}$

The question one: Is the function $D_{\mathbb{X}} f$ smooth?

Answer: YES, since if $\mathcal{U} \subseteq M$ is open, then

$$D_{\mathbb{X}} f|_{\mathcal{U}} = \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i} f|_{\mathcal{U}} \quad \text{if} \quad \mathbb{X}|_{\mathcal{U}} = \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i}.$$

By assumption, $f|_{\mathcal{U}}$ is smooth, a_i are smooth functions. Indeed $D_{\mathbb{X}} f|_{\mathcal{U}}$ is smooth. And since $\mathcal{U}$ was arbitrary, it follows that $D_{\mathbb{X}} f$ is smooth on M.

The question two: Is $D_{\mathbb{X}}$ $\mathbb{R}$-linear and satisfies Leibniz rule.

Answer: YES, linearity is clear and Leibniz rule follows from the fact that $\mathbb{X}(x) \in D_x M$ is a derivation at x.

$$D \in \mathcal{D}(M) \longrightarrow \mathbb{X}_D \in \mathfrak{X}(M)$$

Since $f \ni \mathcal{C}^\infty_{\mathbb{R}}(M) \longmapsto Df \longmapsto Df(x)$ is a derivation at x, denote it by $D(x) \in D_x M = T_x M$,

we define $\mathbb{X}_D(x) = D(x)$. Then indeed

$\mathbb{X}_D$ is a smooth vector field on M.

Also note that if $\varphi : \mathcal{U} \to \mathbb{R}^m$ is a chart on M, then $\mathbb{X}_D = \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i}$ when $a_i = D x_i$, $i = 1, \dots, m$

where again $x_1, \dots, x_m$ are coordinate functions on $\mathcal{U}$.

Clearly these two assignements

$$\mathfrak{X}(M) \ni \mathbb{X} \mapsto D_{\mathbb{X}} \in \mathcal{D}(M) \quad \text{and} \quad \mathcal{D}(M) \ni D \mapsto \mathbb{X}_D \in \mathfrak{X}(M)$$

are inverse to each other and linear. $\square$

<u>Definition</u> If $\mathbb{X}, \mathbb{Y} \in \mathfrak{X}(M)$, then $\mathbb{Z} = [\mathbb{X}, \mathbb{Y}] \in \mathfrak{X}(M)$ is the unique vector field on M such that

$$D_{\mathbb{Z}} = \underbrace{[D_{\mathbb{X}}, D_{\mathbb{Y}}]}_{\text{bracket in } \mathcal{D}(M)}.$$

We call $\mathbb{Z}$ the Lie bracket of $\mathbb{X}$ and $\mathbb{Y}$.

Proposition (Local formula for the Lie bracket)

Let $\mathbb{X}, \mathbb{Y} \in \mathfrak{X}(M)$ and assume that

$$\mathbb{X}|_{\mathcal{U}} = \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad \mathbb{Y}|_{\mathcal{U}} = \sum_{i=1}^{m} b_i \frac{\partial}{\partial y_i}$$

where $a_i, b_i, i=1,\dots,m$ are smooth functions on $\mathcal{U} \subseteq M$ (open).

Then $$[\mathbb{X}, \mathbb{Y}]|_{\mathcal{U}} = \sum_{i=1}^{m} \Big(\underbrace{\sum_{j=1}^{m} \Big(a_j \Big(\frac{\partial}{\partial x_j} b_i\Big) - b_j \Big(\frac{\partial}{\partial x_j} a_i\Big)}_{\text{smooth functions on } \mathcal{U}}\Big)\Big) \frac{\partial}{\partial x_i}.$$

Proof: We have $D_{\mathbb{X}} f|_{\mathcal{U}} = \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i} f$ and

$$D_{\mathbb{Y}} f|_{\mathcal{U}} = \sum_{i=1}^{m} b_i \frac{\partial}{\partial x_i} f, \quad f \in \mathcal{C}^{\infty}_{\mathbb{A}}(M).$$

By definition, we have to compute

$$[D_{\mathbb{X}}, D_{\mathbb{Y}}] f|_{\mathcal{U}} = \sum_{i,j=1}^{m} \Big(a_i \frac{\partial}{\partial x_i} \Big(b_j \frac{\partial}{\partial x_j} f\Big) - b_i \frac{\partial}{\partial x_i} \Big(a_j \frac{\partial}{\partial x_j} f\Big)\Big)$$

$$= \sum_{i,j=1}^{m} \Big(a_i \Big(\frac{\partial}{\partial x_i} b_j\Big) \frac{\partial}{\partial x_j} f + a_i b_j \Big(\frac{\partial}{\partial x_i}\Big(\frac{\partial}{\partial x_j} f\Big)\Big) -$$

$$- b_i \Big(\frac{\partial}{\partial x_i} a_j\Big) \frac{\partial}{\partial x_j} f - a_j b_i \Big(\frac{\partial}{\partial x_i}\Big(\frac{\partial}{\partial x_j} f\Big)\Big)\Big)$$

$$= \sum_{i=1}^{m} \Big(\sum_{j=1}^{m} \Big(a_j \Big(\frac{\partial}{\partial x_j} b_i\Big) \frac{\partial}{\partial x_i} f - b_j \Big(\frac{\partial}{\partial x_j} a_i\Big) \frac{\partial}{\partial x_i} f\Big)\Big)$$

$$+ \sum_{i,j=1}^{m} (a_i b_j) \Big(\underbrace{\frac{\partial^2 f \circ \varphi^{-1}}{\partial x_i \partial x_j} - \frac{\partial f \circ \varphi^{-1}}{\partial x_j \partial x_i}}_{A} \Big).$$

Here φ is a chart $\varphi: \mathcal{U} \to M$ on M. We see that $A = 0$ since the partial derivatives in $\mathbb{R}^m$ commutes if $f \circ \varphi^{-1}$ is $\mathcal{C}^2(\mathcal{U})$. $\square$

Theorem Properties of $[\,,\,]: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$.

1) $[\,,\,]$ is bilinear in both arguments.

2) $[\mathbb{X}, [\mathbb{Y}, \mathbb{Z}]] = [[\mathbb{X}, \mathbb{Y}], \mathbb{Z}] + [\mathbb{Y}, [\mathbb{X}, \mathbb{Z}]]$ Jacobi identity

where $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \in \mathfrak{X}(M)$.

3) $[f\mathbb{X}, g\mathbb{Y}] = fg[\mathbb{X}, \mathbb{Y}] + f(\mathbb{X} g)\mathbb{Y} - g(\mathbb{Y} f)\mathbb{X}$

where $\mathbb{X}, \mathbb{Y} \in \mathfrak{X}(M)$, $f, g \in \mathcal{C}^{\infty}_{\mathbb{A}}(M)$ and we write $\mathbb{X} g = D_{\mathbb{X}} g$.

Proof is left as an exercise.

Corollary $(\mathcal{X}(M), [-,-])$ is a Lie algebra (that is $[-,-]$ satisfies the properties listed in 1) and 2)).

Homework 2 Lie group G ... group & smooth manifold

multiplication $G \times G \longrightarrow G$
inversion $G \longrightarrow G$ } are smooth

e ... identity of G

$\mathfrak{g} := T_e G$ ... Lie algebra, i.e. $\mathfrak{g}$ is a vector space with Lie bracket $[,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which is bilinear and satisfies Jacobi identity.

This Lie algebra structure on $\mathfrak{g}$ it has a geometric origin, this Lie bracket $\mathfrak{g}$ is the Lie bracket of so called left invariant vector fields on G

In Homework 2 you have to compute this Lie bracket of left invariant fields on $G = GL(n, \mathbb{R})$.

$GL(n, \mathbb{R}) \leq \mathbb{R}^{n^2} = M_{n \times n}(\mathbb{R})$

we can coordinates x_{ij}, $i, j = 1, \dots, n$ on $M_{n \times n}(\mathbb{R})$ as coordinates on $GL(n, \mathbb{R})$. So a vector field on $GL(n, \mathbb{R})$ is given by

$$\mathbb{X}_A = \sum_{i,j} f_{ij}(x) \frac{\partial}{\partial x_{ij}}$$

where $f_{ij}(x)$ are smooth functions of x_{kl}

Compute $f_{ij}(x)$ for $\mathbb{X}_A$ and $A \in M_{n \times n}(\mathbb{R})$.