Short recapitulation from lecture 3

Let $(M, \mathcal{A})$ be a smooth manifold of dimension m and $(TM, \mathcal{A}_{TM})$ be its tangent bundle with projection $\pi: TM \to M$, $\pi(x,v) = x$ where (x,v) stands for $v \in T_x M$. We denote by $\mathfrak{X}(M)$ the vector space of all smooth vector fields on M. Recall that a vector field on M is a smooth map

$$\mathbb{X}: M \to TM, \quad \pi \circ \mathbb{X} = \mathrm{Id}_M.$$

Now if $\varphi: \mathcal{U} \to \mathbb{R}^m$ is a chart on M from $\mathcal{A}$ with coordinate functions $x_1, \dots, x_m$, then $\mathbb{X}$ on $\mathcal{U}$ can be written as

$$\mathbb{X} = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i}$$

where $a_i(x)$ are smooth functions of $x_1, \dots, x_m$. Alternatively, we can view $\mathbb{X}$ as a derivation on M, that is as an $\mathbb{R}$-linear map

$$\mathcal{C}^\infty_{\mathcal{A}}(M) \to \mathcal{C}^\infty_{\mathcal{A}}(M)$$

which satisfies Leibniz rule. If $f \in \mathcal{C}^\infty_{\mathcal{A}}(M)$, then the restriction $\mathbb{X}f|_{\mathcal{U}}$ of $\mathbb{X}f$ to $\mathcal{U}$ is given by

$$\mathbb{X}f|_{\mathcal{U}} = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i} f|_{\mathcal{U}} = \sum_{i=1}^{m} a_i(x) \frac{\partial f\circ\varphi^{-1}}{\partial x_i}(\varphi(x)).$$

We have also proved that $\mathfrak{X}(M)$ is a Lie algebra with Lie bracket $[-,-]$. If $\mathbb{Y} \in \mathfrak{X}(M)$ and

$$\mathbb{Y} = \sum_{i=1}^{m} b_i(x) \frac{\partial}{\partial x_i}$$

on $\mathcal{U}$, then the vector field $[\mathbb{X}, \mathbb{Y}]$ is on $\mathcal{U}$ given by

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{m} \left(a_j \left(\frac{\partial}{\partial x_j} b_i\right) - b_j \left(\frac{\partial}{\partial x_j} a_i\right)\right)\right) \frac{\partial}{\partial x_i}.$$

Also recall that if $\Phi: M \to N$ is a smooth map of manifolds, then $T_x\Phi: T_x M \to T_{\Phi(x)} N$, $T_x\Phi([\gamma]) = [\Phi \circ \gamma]$ is called the tangent map of Φ at x. This map is linear and the map

$$T\Phi: TM \to TN, \quad T\Phi(x,v) = T_x\Phi(v)$$

is a smooth map of manifolds which fits into the following commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{T\Phi} & TN \\ \pi_{TM} \downarrow & & \downarrow \pi_{TN} \\ M & \xrightarrow{\Phi} & N \end{array}$$

where all maps are smooth.

Examples of Lie bracket

•) $M = \mathbb{R}^2$, $\mathbb{X}, \mathbb{Y} \in \mathfrak{X}(\mathbb{R}^2)$ $\quad \mathbb{X} = x_1 \frac{\partial}{\partial x_1} + x_2^2 x_1 \frac{\partial}{\partial x_2}$, $\mathbb{Y} = x_1 x_2 \frac{\partial}{\partial x_1} + x_2^3 \frac{\partial}{\partial x_2}$

$$[\mathbb{X}, \mathbb{Y}] = \left[x_1 \frac{\partial}{\partial x_1} + x_2^2 x_1 \frac{\partial}{\partial x_2},\ x_1 x_2 \frac{\partial}{\partial x_1} + x_2^3 \frac{\partial}{\partial x_2}\right]$$

$$= x_1\left(\frac{\partial}{\partial x_1} x_1 x_2\right)\frac{\partial}{\partial x_1} + x_1\left(\frac{\partial}{\partial x_1} x_2^3\right)\frac{\partial}{\partial x_2} + x_2^2 x_1\left(\frac{\partial}{\partial x_2} x_1 x_2\right)\frac{\partial}{\partial x_1}$$

$$+ x_2^2 x_1\left(\frac{\partial}{\partial x_2} x_2^3\right)\frac{\partial}{\partial x_2} - \left(x_1 x_2\left(\frac{\partial}{\partial x_1} x_1\right)\frac{\partial}{\partial x_1} + x_1 x_2\left(\frac{\partial}{\partial x_1} x_2^2 x_1 \frac{\partial}{\partial x_2}\right)\right.$$

$$\left. + x_2^3\left(\frac{\partial}{\partial x_2} x_1\right)\frac{\partial}{\partial x_1} + x_2^3\left(\frac{\partial}{\partial x_2} x_2^2 x_1\right)\frac{\partial}{\partial x_2}\right)$$

$$= \cancel{x_1 x_2 \frac{\partial}{\partial x_1}} + 0 + x_2^2 x_1^2 \frac{\partial}{\partial x_1} + 3 x_2^4 x_1 \frac{\partial}{\partial x_2} - \cancel{x_1 x_2} \frac{\partial}{\partial x_1} - x_1 x_2^3 \frac{\partial}{\partial x_2} + 0 + 2 x_2^4 x_1 \frac{\partial}{\partial x_2}$$

$$= x_1^2 x_2^2 \frac{\partial}{\partial x_1} + \left(2 x_1 x_2^4 - x_1 x_2^3 + 3 x_2^4 x_1\right)\frac{\partial}{\partial x_2}$$

•) $M = \mathbb{R}^3$, $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3 \in \mathfrak{X}(\mathbb{R}^3)$

$$\mathbb{X}_1 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \qquad [\mathbb{X}_1, \mathbb{X}_2] = -x_2 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_2} = \mathbb{X}_3$$

$$\mathbb{X}_2 = x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} \qquad [\mathbb{X}_1, \mathbb{X}_3] = +x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} = -\mathbb{X}_2$$

$$\mathbb{X}_3 = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} \qquad [\mathbb{X}_2, \mathbb{X}_3] = -x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_1} = \mathbb{X}_1$$

We see that X_1, X_2, X_3 generate 3-dimensional Lie subalgebra inside the infinite dimensional Lie algebra $\mathfrak{X}(\mathbb{R}^3)$

Integral curves and flows of vector fields

Definition Let $X \in \mathfrak{X}(M)$ where $(M, \mathcal{A})$ is as above. An integral curve of X is a smooth curve $c: I \longrightarrow M$ where I is an open interval with $0 \in I$ such that

$$c'(t) = \frac{d}{dt} c(t) = X(c(t))$$

for every $t \in I$. If $c_x(0) = c(0) = x$ we call c_x an integral curve of X at x.

Local description of integral curves

Let $\varphi: U \to \mathbb{R}^m$ be a chart on M and c be a curve on M whose image is contained in U. Then c is an integral curve for $X \in \mathfrak{X}(M)$ iff the curve $\tilde{c} = \varphi \circ c$ is a solution of the system of ordinary differential equations

$$\tilde{c}_i'(t) = \frac{d\tilde{c}}{dt}(t) = a_i(c(t)) \quad , \; i = 1, \dots, m,$$

where $X = \sum_{i=1}^m a_i \frac{\partial}{\partial x_i}$ on U and $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_m)$.

Theorem Let $X \in \mathfrak{X}(M)$. Then for any $x \in M$ there exists an open interval I_x containing 0 and an integral curve $c_x : I_x \to M$ of X at x. Moreover there exists a unique maximal integral curve of X at x (that is an integral curve of X at x which is defined on a maximal interval).

Proof: See Lemma 3.6 in [11]. $\square$

Definition Now let c_x be a maximal integral curve of X at $x \in M$ defined on I_x. Put $I(X) = \bigcup_{x \in M} I_x \times \{x\} \subseteq \mathbb{R} \times M$. We call

$$Fl^X : I(X) \longrightarrow M, \quad Fl^X_t(x) = Fl^X(t, x) := c_x(t)$$

the flow of X or the flow generated by X.

Theorem A For each $X \in \mathfrak{X}(M)$, the set $I(X)$ is an open neighbourhood of $\{0\} \times M$ in $\mathbb{R} \times M$ and the flow Fl^X is smooth (as a map of manifolds). We have

$$\text{(PF)} \qquad Fl^X(t+s, x) = Fl^X(t, Fl^X(s, x))$$

whenever the right hand side exists (and so the left hand side exists and we have equality) or $t, s > 0$ or $s, t < 0$ and the left hand side exists (and then also the right hand side exists and we have equality).

Moreover, for every $x_0 \in M$ there is an open interval J containing 0 and an open set U with $x_0 \in U$ such that for every $t \in J$ the map

$$U \longrightarrow M, \; x \longmapsto Fl^X_t(x) = Fl^X(t, x)$$

is a diffeomorphism onto its image.

Proof: See Theorem in Section 3.7 in [11]. $\square$

Definition By a one-parametric family of local diffeomorphisms on M we understand a smooth map $\Phi: I(\Phi) \longrightarrow M$ where $I(\Phi)$ is an open neighbourhood of $\{0\} \times M$ in $\mathbb{R} \times M$ such that

- $\Phi(0, x) = x$ for every $x \in M$,
- for every t the map $M \ni x \mapsto \Phi_t(x) = \Phi(t, x)$ is a local diffeomorphism on it's domain and
- $\Phi(t+s, x) = \Phi(t, \Phi(s, x))$ in the same sense as in theorem A.

Remark A: By Section 2.1 in [12], given a one-parametric family Φ of local diffeomorphisms on M, there is a unique vector field on M whose flow is this one-parametric family.

Definition A vector field X is called complete if $I(X) = \mathbb{R} \times M$.

Theorem Let X be a vector field on M with compact support, that is $X = 0$ outside a compact subset of K. Then X is complete.

Proof: See Lemma in Section 3.8 in [11]. □

Examples: •) Let $M = \mathbb{R}^2$ and $X = \frac{\partial}{\partial x_1}$. Then $c_x(t) = (x_1 + t, x_2)$, $t \in \mathbb{R}$, $x = (x_1, x_2)$, is a maximal integral curve of X at x. The flow of X is

$$Fl^X : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad Fl^X(t, x) = x + (t, 0).$$

We see that X is complete.

•) Let $X_1 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$, $X_2 = x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}$, $X_3 = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}$ be as in Examples of Lie bracket. Then the integral curve of X_1 at x is a solution of

$$c_1'(t) = c_2(t), \quad c_2'(t) = -c_1(t), \quad c_3'(t) = 0$$

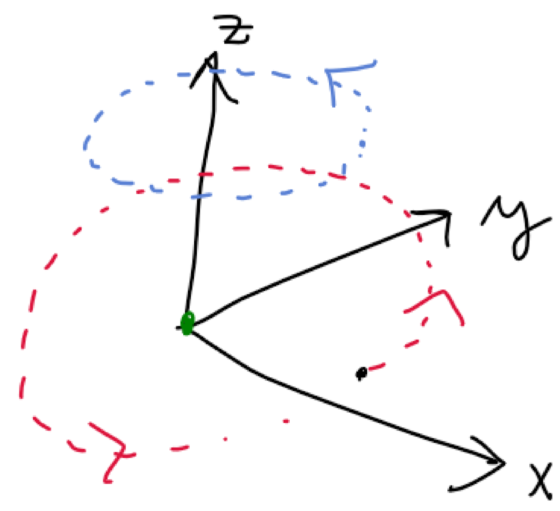
with the initial condition $c(0) = x$. A solution is

$$c_x(t) = (x_1 \cos t - x_2 \sin t,\ x_1 \sin t + x_2 \cos t,\ x_3), \quad t \in \mathbb{R}$$

and the flow is

$$Fl^{X_1}(t, x) = (x_1 \cos t - x_2 \sin t,\ x_1 \sin t + x_2 \cos t,\ x_3), \quad t \in \mathbb{R}$$

Note that the flow of X_1 is rotation along z-axis with constant angular velocity.



Hence X_1 is complete and it is sometimes called the infinitesimal generator of rotation along z-axis.

Exercise: compute the flow of X_2, X_3.

•) $M = \mathbb{R}^2$ and $X = x_1^2 \frac{\partial}{\partial x_1}$. If $x = (x_1, x_2)$, then $c_x = c = (c_1, c_2)$ is a solution of

$$c_1'(t) = c_1^2(t), \quad c_2'(t) = 0$$

with initial condition $c_1(0) = x_1$, $c_2(0) = x_2$. We clearly have that $c_2(t) = x_2$ for all t. On the other hand, writing $c_1(t) = x(t)$, we have to solve

$$\text{(ODE)} \quad \frac{dx}{dt} = x^2(t)$$

which is a first order ODE with separation of variables. Hence,

$$-x^{-1} = \int \frac{dx}{x^2}, \qquad \int dt = t + d, \quad d \in \mathbb{R}$$

and so $x(t) = \frac{-1}{t + d}$ is a solution of (ODE).

Now $\frac{-1}{0 + d} = x(0) = c_1(0) = x_1$ and so $d = -\frac{1}{x_1}$, $x_1 \neq 0$.

If $x_1 \neq 0$, then

$$c_x(t) = \left(\frac{x_1}{1 - t x_1}, x_2 \right) \quad \begin{cases} t \in (-\infty, \frac{1}{x_1}), & x_1 > 0. \\ t \in (\frac{1}{x_1}, +\infty), & x_1 < 0. \end{cases}$$

is the maximal integral curve of X at x.

If $x_1 = 0$, then $c_x(t) = (0, 0)$, $t \in \mathbb{R}$, is the maximal integral curve. We see that X is not complete.

Lie derivative of vector fields revisited

Let us assume that $\Phi : M \longrightarrow N$ is diffeomorphism of manifolds. If $X \in \mathfrak{X}(M)$, then we put $\Phi_* X : N \longrightarrow TN$, $\Phi_* X(x) = T_{\Phi^{-1}(x)} \Phi \circ X \circ \Phi^{-1}(x)$. Note that by definition, the following diagram

$$\begin{array}{ccc} TM & \xrightarrow{T\Phi} & TN \\ X\uparrow & \Phi^{-1}\circ & \uparrow \Phi_* X \\ M & \longleftarrow & N \end{array}$$

commutes. We see that $\Phi_* X$ is a smooth map and $\Phi_* X(x) \in T_x N$. Hence $\Phi_* X$ is a smooth vector field on N.

Definition We call $\Phi_* X \in \mathfrak{X}(N)$ the pushforward of $X \in \mathfrak{X}(M)$.

Let $X \in \mathfrak{X}(M)$ and Fl^X be its flow. We know that for any $x_0 \in M$ there is an open neighbourhood $\mathcal{U}$ of x_0 with compact closure and an open interval I containing 0 such that $Fl^X(-t,x)$ is defined for every $t \in I$, $x \in \mathcal{U}$ and that for every $t \in I$ the map

$$\mathcal{U} \ni x \longmapsto Fl^X_{-t}(x) = Fl^X(-t,x)$$

is a diffeomorphism onto its image. If $Y \in \mathfrak{X}(M)$, then there is some (possibly smaller) open neighbourhood $\mathcal{U}'$ of x and an open interval I' containing 0 such that the vector field

$$(Fl^X_{-t})_* Y$$

is defined on $\mathcal{U}'$ for every $t \in I'$! Hence the following definition makes sense

Definition (Geometric definition of Lie derivative)
Let $X, Y \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty_{\mathbb{R}}(M)$. Then we define

$$\mathcal{L}_X f(x) = \frac{d}{dt} f(Fl^X(t,x))\big|_{t=0} \quad \text{and}$$

$$\mathcal{L}_X Y = \frac{d}{dt} (Fl^X_{-t})_* Y \big|_{t=0}.$$

Lemma Let $X \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty_{\mathbb{R}}(M)$. Then

$$\mathcal{L}_X f = Xf = D_X f \in \mathcal{C}^\infty_{\mathbb{R}}(M).$$

Proof is left as an exercise.

Theorem B Let $X, Y \in \mathfrak{X}(M)$. Then

$$\mathcal{L}_X Y = [X, Y] \in \mathfrak{X}(M).$$

In order to proof Theorem B we will need some preliminary results which will be useful also later on. Let $X \in \mathfrak{X}(M)$ and Fl^X be its flow

$$Fl^X : I(X) \longrightarrow M, \ (t,x) \mapsto Fl^X(t,x).$$

By definition, $I(X)$ is an open subset of $\mathbb{R}\times M$ of all points (t,x) for which the flow $Fl^X(t,x)$ is defined. Now for every $t \in \mathbb{R}$, put $M_t = I(X) \cap \{t\} \times M$. Now M_t is the domain of the flow Fl^X_t of X at time t and we know that M_t is open subset of M and that

$$Fl^X_t : M_t \longrightarrow M, \ x \mapsto Fl^X_t(x)$$

is smooth. Hence we can consider its tangent map

$$TFl^X_t : TM_t \longrightarrow TM, \quad TFl^X_t(x,v) = T_x Fl^X_t(v).$$

Hence, there is a smooth map of manifolds

$$\widetilde{Fl}^X : \tilde{I}(X) \longrightarrow TM, \quad \widetilde{Fl}^X(t,(x,v)) = T_x Fl^X_t(v)$$

where $\tilde{I}(X) = \bigcup_{(x,v)\in TM} I_x \times \{(x,v)\}$. Now note that we have

$$(FP)' \quad \widetilde{Fl}^X(t+s,(x,v)) = \widetilde{Fl}^X(t, \widetilde{Fl}^X(s,(x,v)))$$

in the same sense as in Theorem A and so $\widetilde{Fl}^X$ is a one-parametric family of local diffeomorphism on TM. We have

Theorem Let $X \in \mathfrak{X}(M)$. Then there is a unique vector field $\tilde{X}$ on TM whose flow is

$$Fl^{\tilde{X}}(t,(x,v)) = \widetilde{Fl}^X(t,(x,v))$$

for every $(t,(x,v)) \in \tilde{I}(X)$.

Moreover, if $\varphi: \mathcal{U} \to \mathbb{R}^m$ is a chart on M and

$$\mathfrak{X} = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i} \quad \text{on } \mathcal{U}$$

and $(x_1, \dots, x_m, p_1, \dots, p_m)$ are the associated coordinate functions on

$$\Phi: \pi^{-1}(\mathcal{U}) \longrightarrow \mathbb{R}^m \times \mathbb{R}^m, \text{ then}$$

$$\tilde{X} = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{m} p_j \left(\frac{\partial}{\partial x_j} a_i(x) \right) \frac{\partial}{\partial p_i} \quad \text{on } \pi^{-1}(\mathcal{U}).$$

Proof: The first claim was already discussed in Remark A. Hence, it is enough to show the second claim. In the coordinates on $\mathcal{U}$, assume that on some open subset $\mathcal{U}'$ of $\mathcal{U}$ the flow $Fl^{\mathfrak{X}}$ of $\mathfrak{X}$ is given by

$$\Phi = \Phi(t, (x_1, \dots, x_m)): I' \times \mathcal{U}' \to \mathcal{U}.$$

We write $\Phi_t(x_1, \dots, x_m) = \Phi(t, x_1, \dots, x_m)$ and denote by Φ_i and $(\Phi_t)_i$ be the i-th component of Φ and Φ_t, respectively. Then the flow $\widetilde{Fl}^{\mathfrak{X}}$ is on $\pi^{-1}(\mathcal{U}')$ given by

$$\tilde{\Phi} = \tilde{\Phi}(t, x_1, \dots, x_m, p_1, \dots, p_m): I' \times \pi^{-1}(\mathcal{U}') \longrightarrow \pi^{-1}(\mathcal{U})$$

$$\tilde{\Phi}(t, x_1, \dots, x_m, p_1, \dots, p_m) = \left((\Phi_t)_1, \dots, (\Phi_t)_m, \sum_{j=1}^{m} \frac{\partial (\Phi_t)_1}{\partial x_j} p_j, \dots, \sum_{j=1}^{m} \frac{\partial (\Phi_t)_m}{\partial x_j} p_j \right).$$

Let us now fix a point $(x_1, \dots, x_m, p_1, \dots, p_m) \in \pi^{-1}(\mathcal{U})$. If we differentiate all components w.r. to t, we get

$$\frac{\partial (\Phi_t)_i (x_1, \dots, x_m)}{\partial t} \Big|_{t=0} = a_i(x_1, \dots, x_m), \quad i = 1, \dots, m,$$

$$\sum_{j=1}^{m} \frac{\partial^2 (\Phi_t)_i}{\partial t \, \partial x_j} p_j = \sum_{j=1}^{m} \frac{\partial^2 (\Phi_t)_i}{\partial x_j \, \partial t} p_j$$

$$= \sum_{j=1}^{m} \frac{\partial a_i}{\partial x_j}(x_1, \dots, x_m) p_j, \quad i = 1, \dots, m.$$

These are obviously the components of $\tilde{\mathfrak{X}}$ w.r. to $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_m}$ □

Proof of Theorem B: Again it is enough to verify the claim in the chart $\varphi: \mathcal{U} \to \mathbb{R}^m$ as in the proof of the previous Theorem. Assume that

$$Y = \sum_{i=1}^{m} b_i(x_1, \dots, x_m) \frac{\partial}{\partial x_i} \quad \text{on } \mathcal{U}.$$

Fix $x \in \mathcal{U}'$ and I be as above, then

$$T_x M \ni (\widetilde{Fl}^{\mathfrak{X}}_{-t})(Y)(x) = (T Fl^{\mathfrak{X}}_{-t}(Y(Fl^{\mathfrak{X}}_t(x)))) = \sum c_i(t) \frac{\partial}{\partial x_i}, \quad t \in I.$$

Let us differentiate w.r. to t and evaluate at $t = 0$ we obtain

$$\frac{d}{dt} c_i(t) \Big|_{t=0} = \frac{d}{dt} \left(\sum_{j=1}^{m} \frac{\partial (\Phi_{-t})_i}{\partial x_j} (\Phi_t(x_1, \dots, x_m)) \, b_j (\Phi_t(x_1, \dots, x_m)) \right) \Big|_{t=0}$$

$$= \left(\sum_{j=1}^{m} \frac{\partial (-a_i)}{\partial x_j} (\Phi_t(x_1, \dots, x_m)) \, b_j (\Phi_t(x_1, \dots, x_m)) \right.$$

$$\left. + \sum_{j,k=1}^{m} \frac{\partial (\Phi_{-t})_i}{\partial x_j} (\Phi_t(x_1, \dots, x_m)) \frac{\partial b_j}{\partial x_k} a_k (\Phi_t(x_1, \dots, x_m)) \right) \Big|_{t=0}$$

$$= \sum_{j=1}^{m} \left(-b_j \frac{\partial a_i}{\partial x_j} + \frac{\partial b_j}{\partial x_i} a_i \right)(x_1, \dots, x_m).$$

Here we use that $\frac{\partial (\Phi_0)_i}{\partial x_j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$, since $\Phi_0: M \to M$ is the identity.

This proves that $\mathcal{L}_{\mathfrak{X}} Y(x) = [\mathfrak{X}, Y](x)$. As $x \in M$ is arbitrary, the proof is complete. □

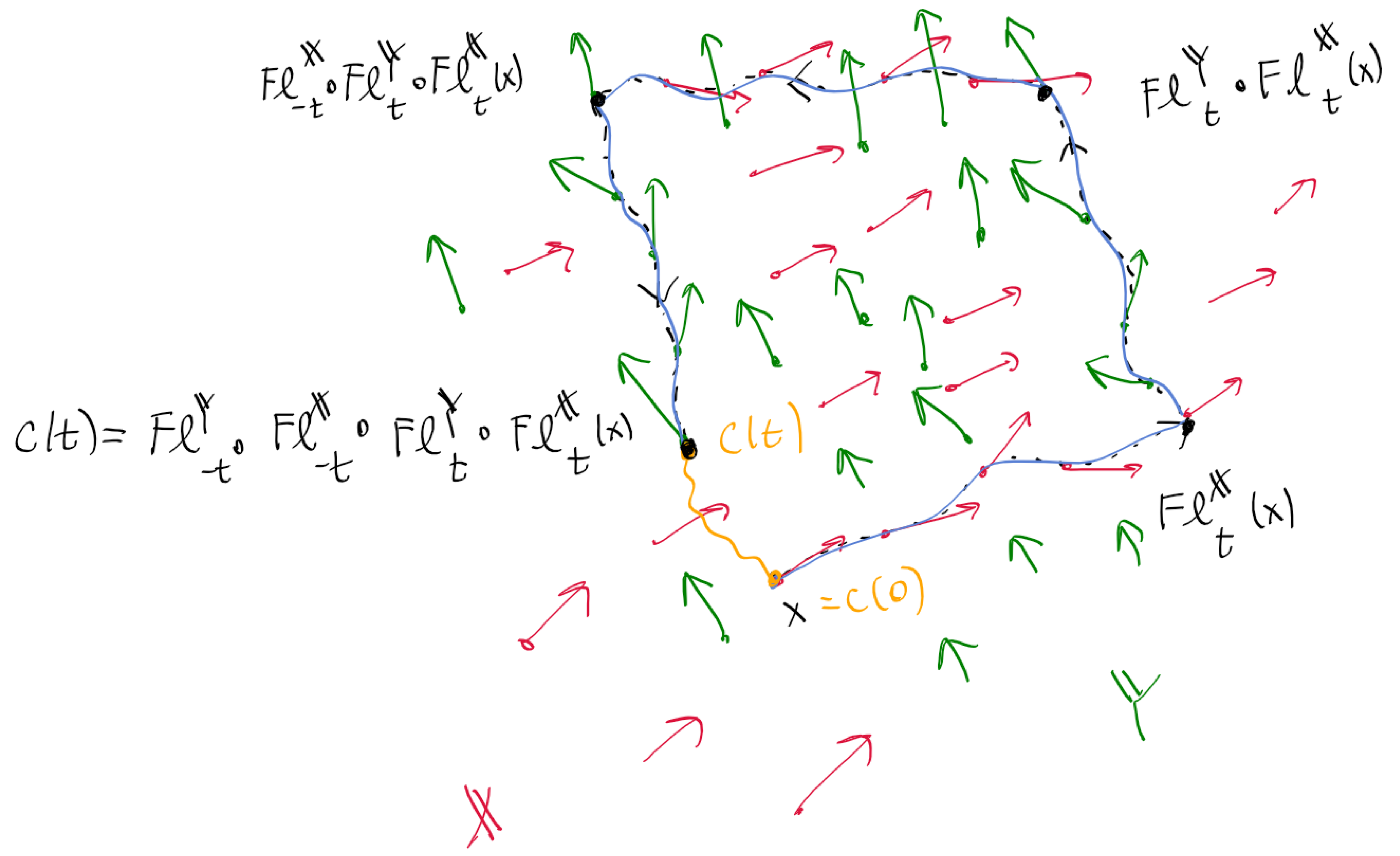
Geometric interpretation of Lie bracket

Let $X, Y \in \mathfrak{X}(M)$ and $x \in M$ be fixed. Let $\varepsilon > 0$ be small so that Fl^X_t, Fl^Y_t are defined on some open neighbourhood of x for all $|t| < \varepsilon$. Then

$$[X,Y](x) = \frac{1}{2}\frac{d^2}{dt^2}\Big(\underbrace{Fl^Y_{-t}\circ Fl^X_{-t}\circ Fl^Y_t\circ Fl^X_t(x)}_{c(t)}\Big)\Big|_{t=0}$$

$$= \frac{d}{dt}\Big(\underbrace{Fl^Y_{-\sqrt{t}}\circ Fl^X_{-\sqrt{t}}\circ Fl^Y_{\sqrt{t}}\circ Fl^X_{\sqrt{t}}(x)}_{\tilde{c}(t)}\Big)\Big|_{t=0}$$

This can be visualized in the following picture.



Now $c(t)$ is a smooth curve defined on $(-\varepsilon,\varepsilon)$ and $c(0) = x$, $c'(0) = 0$. If we put $\tilde{c}(s^2) = c(s)$, then also $\tilde{c}$ is smooth and

$$\frac{d\tilde{c}}{ds}(s)\Big|_{s=0^+} = [X,Y](x)$$