Integral curves, flow of vector field and Lie bracket

Recapitulation from last week

$(M, \mathcal{A})$ ... smooth manifold with atlas $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^m \mid \alpha \in A\}$ of dimension m

$\varphi : U \to \mathbb{R}^m$ a chart from $\mathcal{A}$ with coordinate functions $x_1, \dots, x_m$

$B_x^\varphi = \{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \}$ ... a basis of $T_x M$ for any $x \in U$

•) tangent bundle $(TM, \mathcal{A}_{TM})$ of M, $TM = \bigcup_{x \in M} T_x M$, $v \in T_x M$ .. (x, v)
is a smooth manifold of dim $2m$, $\pi : TM \to M$, $\pi(x, v) = x$

$$\mathcal{A}_{TM} = \{ \Phi_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2m} \mid \alpha \in A \}$$

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$$

$$\Phi_\alpha(x, v) = (\varphi_\alpha(x), [v]_{B_x^{\varphi_\alpha}})$$

$[v]_{B_x^{\varphi_\alpha}}$ ... coordinates of v w.r.to $B_x^{\varphi_\alpha}$

$$\Phi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \times \mathbb{R}^m \longrightarrow TM$$

$$\Phi_\alpha^{-1}(x_1, \dots, x_m, p_1, \dots, p_m) = \left(\varphi_\alpha^{-1}(x_1, \dots, x_m), \sum_{i=1}^m p_i \frac{\partial}{\partial x_i} \right)$$

•) $(N, \mathcal{A}_N)$ ... smooth manifold of dim n

$f : M \to N$ be smooth, $x_0 \in M$

$$T_{x_0} M \xrightarrow{T_{x_0} f} T_{f(x_0)} N$$

$T_{x_0} f([\gamma]) = [f \circ \gamma]$ $\qquad [\gamma] \in T_{x_0} M$

$T_{x_0} f$ ... the tangent map of f at x_0

$$\begin{tikzcd}
T_{x_0} M \arrow[r, "T_{x_0} f"] \arrow[d, shift right, "T_{x_0}\varphi"'] & T_{y_0} N \arrow[d, shift left, "T_{y_0}\rho"] \\
T_0 \mathbb{R}^m = \mathbb{R}^m \arrow[r, "L"] \arrow[u, shift right, "T_0 \varphi^{-1}"'] & T_0 \mathbb{R}^n = \mathbb{R}^n \arrow[u, shift left, "T_0 \rho^{-1}"]
\end{tikzcd}$$

$y_0 = f(x_0)$, $x_0 \in U$
$\varphi(x_0) = 0$

$\rho : V \to \mathbb{R}^n$ is a chart from $\mathcal{A}_N$
$\rho(y_0) = 0$

$T_{x_0}\varphi$, $T_{y_0}\rho$ are linear isomorphism

$L = d\tilde{f}(0)$ .. total derivative or total differential of

$\tilde{f} = \rho \circ f \circ \varphi^{-1}$

$\tilde{f}$ .. smooth map $\mathbb{R}^m \to \mathbb{R}^n$

$\tilde{f}(0) = 0$

L is a linear map $\mathbb{R}^m \to \mathbb{R}^n$ represented by $\left(\frac{\partial \tilde{f}_i}{\partial x_j}(0) \right)_{i=1,\dots,n}^{j=1,\dots,m} \in M_{n \times m}(\mathbb{R})$

$\Rightarrow T_{x_0} f$ is linear map

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\pi_{TM}"'] \arrow[dr, phantom, "\circlearrowright"] & TN \arrow[d, "\pi_{TN}"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

$Tf(x,v) = T_x f(v)$

since the coefficients of the Jacobi matrix of L depend smoothly on $x_1, \dots, x_m \Rightarrow Tf$ is a smooth map of manifolds

$\pi_{TM} = \pi$

•) Vector field on M is $\mathbb{X} : M \to TM$ such that $\underline{\mathbb{X} \text{ is smooth and}}$ $\pi \circ \mathbb{X} = Id_M$.

$\forall x \in M$: $\mathbb{X}(x) \in T_x M$ and this assignment depends smoothly on the footpoint

$\varphi : U \to \mathbb{R}^m$ is as above $\qquad (x \in U)$

$$\mathbb{X}(x) = \sum_{i=1}^m a_i(x) \underset{\substack{\uparrow \\ \text{is a basis of } T_xM}}{\frac{\partial}{\partial x_i}}$$

the smoothness of $\mathbb{X}$ just means that $a_i(x)$ are smooth functions of $x_1, \dots, x_m$

•) $\mathfrak{X}(M)$.. the vector space of all smooth vector fields on M

$\mathfrak{X}(M)$ is an (infinite dimensional) Lie algebra with Lie bracket $[\,,\,] : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

1) $[-,-]$ bilinear
2) $[\mathbb{X}, \mathbb{Y}] = -[\mathbb{Y}, \mathbb{X}]$
3) $[\mathbb{X}, [\mathbb{Y}, \mathbb{Z}]] = [[\mathbb{X}, \mathbb{Y}], \mathbb{Z}] + [\mathbb{Y}, [\mathbb{X}, \mathbb{Z}]]$ Jacobi identity

$\mathbb{X}, \mathbb{Y}, \mathbb{Z} \in \mathfrak{X}(M)$

• $\mathbb{Y}(x) = \sum_{i=1}^m b_i(x) \frac{\partial}{\partial x_i}$ on U

$$[\mathbb{X}, \mathbb{Y}](x) = \sum_{i=1}^m c_i(x) \frac{\partial}{\partial x_i}$$

$$c_i(x) = \sum_{j=1}^m \Big(a_j(x) \underbrace{\Big(\frac{\partial}{\partial x_j} b_i\Big)(x)}_{\frac{\partial b_i \circ \varphi^{-1}}{\partial x_j}(\varphi(x))} - b_j \Big(\frac{\partial}{\partial x_j} a_i\Big)(x) \Big)$$

Examples 1) $M=\mathbb{R}^2$, $\quad X = x_1^3 x_2 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2}$, $\quad Y = x_1 \frac{\partial}{\partial x_2}$

$$[X,Y] = \left[x_1^3 x_2 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2}, \, x_1 \frac{\partial}{\partial x_2} \right]$$

$$= x_1^3 x_2 \Big(\underbrace{\frac{\partial}{\partial x_1} x_1}_{1}\Big) \frac{\partial}{\partial x_2} + x_2^2 \Big(\underbrace{\frac{\partial}{\partial x_2} x_1}_{=0}\Big) - x_1 \Big(\underbrace{\frac{\partial}{\partial x_1} x_1^3 x_2}_{3x_1^2 x_2}\Big) \frac{\partial}{\partial x_1} - x_1 \Big(\frac{\partial}{\partial x_2} x_2^2\Big) \frac{\partial}{\partial x_2}$$

$$= x_1^3 x_2 \frac{\partial}{\partial x_2} - 3 x_1^3 x_2 \frac{\partial}{\partial x_1} - x_1 2 x_2 \frac{\partial}{\partial x_2}$$

$$= -3 x_1^3 x_2 \frac{\partial}{\partial x_1} + (x_1^3 x_2 - 2 x_1 x_2) \frac{\partial}{\partial x_2}.$$

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2) $M = \mathbb{R}^3$ $\quad X_1 = -x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_1}$ $\quad X_2 = -x_1 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_1}$

$X_3 = -x_2 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_2}$

$$[X_1, X_2] = -x_2 \frac{\partial}{\partial x_3} + \Big(+x_3 \frac{\partial}{\partial x_2}\Big) = X_3$$

$$[X_1, X_3] = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} = -X_2$$

$$[X_2, X_3] = -x_1 \frac{\partial}{\partial x_2} + \Big(+x_2 \frac{\partial}{\partial x_1}\Big) = X_1$$

The linear span of X_1, X_2, X_3 is a 3-dimensional Lie subalgebra in $\mathfrak{X}(\mathbb{R}^3)$, (this is $\mathfrak{so}(3)$)

Integral curve

Definition Let $X \in \mathfrak{X}(M)$. A curve $c_X = c : I \to M$ where I is an open interval containing 0 is called integral for X at $x \in M$ if $c_X'(t) = X(c_X(t))$ for every $t \in I$ and $c_X(0) = x$.

Pic.

Let us assume for simplicity that $c_x(I) \subseteq U$, $\varphi(x) = 0$, x fixed.

Local description of integral curves

$\varphi : U \to \mathbb{R}^m$ with coordinate functions $x_1, \dots, x_m$, $\quad X(x) = \sum_{i=1}^m a_i(x) \frac{\partial}{\partial x_i}$ on U

$\tilde{c} = \varphi \circ c$ $\qquad \tilde{c} : I \to \mathbb{R}^m$

$c = c_x$ is integral for X at x iff $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_m)$ are solutions of

$$\tilde{c}_i'(t) = a_i(\tilde{c}(t)) \quad \forall t \in I, \text{ for every } i = 1, \dots, m$$
$$\tilde{c}_i(0) = 0$$

1-order ODE for m-functions $\tilde{c}_i$, $i = 1, \dots, m$

Theorem Let $\mathbb{X} \in \mathfrak{X}(M)$ and $x \in M$. Then there exists an open interval I_x containing 0 and an integral curve $c_x : I_x \to M$ of $\mathbb{X}$ at x. Moreover there exists a unique maximal integral curve of $\mathbb{X}$ at x (i.e. a solution $c : I \to M$, $c'(t) = X(c(t))$, $c_x(0) = x$ which is defined on a maximal interval I).

Proof: Follows from the Picard–Lindelöw theorem. $\square$

Flow of vector fields

Definition Let $\mathbb{X} \in \mathfrak{X}(M)$ and c_x be the maximal integral curve of $\mathbb{X}$ at a fixed $x \in M$ defined on I_x (with $0 \ni I_x$).

Put $I(\mathbb{X}) = \bigcup_{x \in M} I_x \times \{x\} \subseteq \mathbb{R} \times M$. We call

$$Fl^{\mathbb{X}} : I(\mathbb{X}) \longrightarrow M, \quad Fl^{\mathbb{X}}(t,x) = Fl^{\mathbb{X}}_t(x) = c_x(t)$$

the flow of $\mathbb{X}$ or the flow generated by $\mathbb{X}$.

Theorem A $I(\mathbb{X})$ is open $\mathbb{R} \times M$ and $Fl^{\mathbb{X}}$ is smooth (as map of manifolds). We have

$$\text{(PF)} \qquad Fl^{\mathbb{X}}(t+s, x) = Fl^{\mathbb{X}}(t, Fl^{\mathbb{X}}(s,x))$$

whenever
- the right hand side exists (and so the left side exists and $=$) or
- the left hand side exists and $t,s > 0$ or $t,s < 0$ (and so the right hand side exists and $=$).

Moreover, for every $x_0 \in M$ there is an open neigh. $\mathcal{U}$ of x_0 and an open interval J with $0 \in J$ such that for every $t \in J$ the map

$$\mathcal{U} \to M, \ x \mapsto Fl^{\mathbb{X}}_t(x)$$

exists and it is a diffeomorphism onto its image.

Definition By a 1-parametric family of local diffeomorphisms on M we understand a smooth map

$$\Phi : I(\Phi) \longrightarrow M$$

where $I(\Phi)$ is an open neighbourhood of $\{0\} \times M$ in $\mathbb{R} \times M$ such that:

1) for every t the map $M \ni x \mapsto \Phi_t(x) = \Phi(t,x)$ is a local diffeom. (on its domain)
2) Φ_0 is the identity on M and
3) $\Phi(t+s, x) = \Phi(t, \Phi(s,x))$ in the same sense as in Theorem A.

$\mathbb{X} \ni \mathfrak{X}(M) \longrightarrow$ flow $Fl^{\mathbb{X}}$ is a 1-par. family of local diffeom. on M

it can be shown that this assignement is 1–1.

<u>Definition</u> $\mathbb{X} \in \mathfrak{X}(M)$ is called complete $I(\mathbb{X}) = \mathbb{R} \times M$.

<u>Theorem</u> If $\mathbb{X} \in \mathfrak{X}(M)$ has compact support ($=0$ outside a compact subset of M), then $\mathbb{X}$ is complete.

<u>Examples</u> 1) $\mathbb{X}_1 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \in \mathfrak{X}(\mathbb{R}^3)$

$x_0 = (\tilde{a}, \tilde{b}, \tilde{c}) \in \mathbb{R}^3$

$c = c_{x_0} = (c_1, c_2, c_3)$

c is a solution of

$$\boxed{\begin{aligned} c_1'(t) &= x_2(c(t)) = c_2(t) & c_1(0) &= \tilde{a} \\ c_2'(t) &= -x_1(c(t)) = -c_1(t) & c_2(0) &= \tilde{b} \\ c_3'(t) &= 0(c(t)) = 0 & c_3(0) &= \tilde{c} \end{aligned}}$$

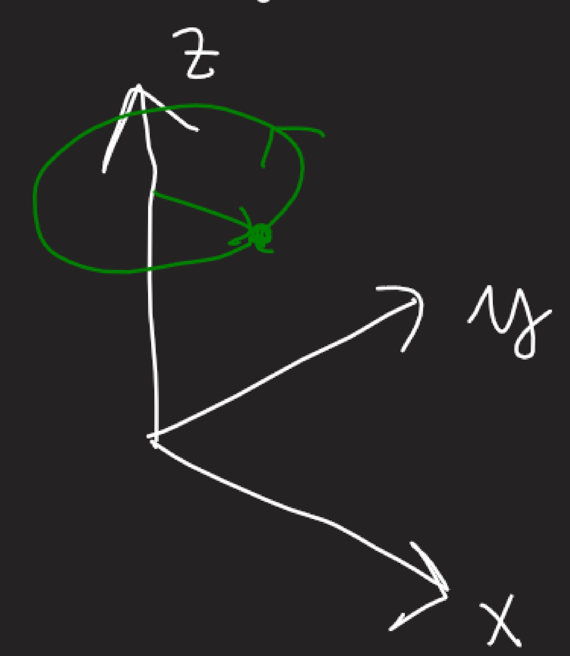
$c_3(t) = \tilde{c}$

$$\boxed{c(t) = (\tilde{a}\cos t - \tilde{b}\sin t, \ \tilde{a}\sin t + \tilde{b}\cos t, \ \tilde{c}), \ t \in \mathbb{R}}$$

is the integral curve of $\mathbb{X}_1$ at $(\tilde{a}, \tilde{b}, \tilde{c})$

$Fl^{\mathbb{X}_1}(t, y) = (y_1 \cos t - y_2 \sin t, \ y_1 \sin t + y_2 \cos t, \ y_3), \ t \in \mathbb{R}$

$y = (y_1, y_2, y_3)$



is rotation in $\mathbb{R}^3$ along z-axis with constant angular velocity

$\mathbb{X}_1$ .. infinitesimal generator of rotation in $\mathbb{R}^3$

$\mathbb{X}_1$ is complete

2) $M = \mathbb{R}^2$, $\mathbb{X} = x_1^2 \frac{\partial}{\partial x_1}$, $y = (y_1, y_2)$

c_y ... int. curve of $\mathbb{X}$ at y

$c = (c_1, c_2)$ $\qquad c_1'(t) = c_1^2(t) \qquad c_1(0) = y_1$

$\qquad c_2'(t) = 0 \qquad c_2(0) = y_2$

$\boxed{c_2(t) = y_2}$ $\qquad x(t) = c_1(t) \Rightarrow x'(t) = x^2(t)$

$x'(t) = x^2(t)$ 1. ODE in separation of variables

$$\boxed{\frac{dx}{dt} = x^2} \qquad \int \frac{dx}{x^2} = -x^{-1} \qquad -\frac{1}{x} = t + d$$

$$\int dt = t + d \qquad \boxed{x(t) = \frac{-1}{t+d}}$$

$$c_1(t) = \frac{-1}{t+d}$$

$$c_1(0) = y_1 = -\frac{1}{d} \qquad d = -\frac{1}{y_1}$$

makes sense when $y_1 \neq 0$

$$\boxed{c_1(t) = \frac{1}{1 - y_1 t}} \qquad \begin{cases} t \in (-\infty, \frac{1}{y_1}), & y_1 > 0 \\ t \in (\frac{1}{y_1}, +\infty), & y_1 < 0 \end{cases}$$

$\mathbb{X}$ is not complete.

Lie derivative of vector fields

Definition Assume that $f : M \to N$ is a differ. of smooth man. (M, ∇_M) and (N, ∇_N). If $\mathbb{X} \in \mathfrak{X}(M)$, then $\Phi_* \mathbb{X} \in \mathfrak{X}(N)$ is defined by

$$\Phi_* \mathbb{X}(x) = T_{\Phi^{-1}(x)} \Phi \circ \mathbb{X} \circ \Phi^{-1}(x), \quad x \in N.$$

It is called the pushforward of $\mathbb{X}$.

We have to verify: 1) $\Phi_* \mathbb{X}(x) \in T_x N$ & 2) $\Phi_* \mathbb{X}$ is smooth.

ad 1) $\mathbb{X} \circ \Phi^{-1}(x) \in T_{\Phi^{-1}(x)} M$ (Φ has a smooth inverse Φ^{-1})

$$T_{\Phi^{-1}(x)} \Phi : T_{\Phi^{-1}(x)} M \to T_{\Phi(\Phi^{-1}(x))} N = T_x N$$

ad 2) $\Phi_* \mathbb{X}$ is a composition of smooth maps $\Rightarrow$ it is also smooth

Note that

$$\begin{tikzcd} TM \arrow[r, "T\Phi"] & TN \\ M \arrow[u, "\mathbb{X}"] \arrow[r, phantom, "\circ"] & N \arrow[l, "\Phi^{-1}"] \arrow[u, "\Phi_* \mathbb{X}"'] \end{tikzcd}$$

Let $\mathbb{X} \in \mathfrak{X}(M)$, if $x_0 \in M$ then we know that there is an open neigh. $\mathcal{U}$ of x_0 and an open interval J with $0 \in J$ such that

$$x \longmapsto Fl^{\mathbb{X}}_{-t}(x) \quad \text{is defined for every } x \in \mathcal{U} \text{ and every } t \in J$$

and this map is a diffeomorphism onto its image for every fixed $t \in J$.
Now if $\mathbb{Y} \in \mathfrak{X}(M)$, then $\forall t \in J$ the vector field

$$(Fl^{\mathbb{X}}_{-t})_* \mathbb{Y} \quad \text{is well defined on } \mathcal{U}.$$

Since this is true for any $x_0 \in M$, we have

<u>Definition</u> Let $\mathbb{X}, \mathbb{Y} \in \mathfrak{X}(M)$. Then

$$\mathcal{L}_{\mathbb{X}} \mathbb{Y} = \frac{d}{dt}\left((Fl^{\mathbb{X}}_{-t})_* \mathbb{Y} \right)\Big|_{t=0} \in \mathfrak{X}(M)$$

is called the Lie derivative of $\mathbb{Y}$ along $\mathbb{X}$.

<u>Theorem</u> If $\mathbb{X}, \mathbb{Y} \in \mathfrak{X}(M)$, then

$$\mathcal{L}_{\mathbb{X}} \mathbb{Y} = [\mathbb{X}, \mathbb{Y}].$$

Sketch of proof: We fix a chart around x_0, say $\varphi : \mathcal{U} \to \mathbb{R}^m$, $\varphi(x_0) = 0$, assume that $Fl^{\mathbb{X}}_{-t}$ is defined for every $x \in \mathcal{U}$ and for every $-t \in J$.

$$Fl^{\mathbb{X}} : I(\mathbb{X}) \longrightarrow M \text{ is smooth}, \quad I(\mathbb{X}) \subseteq \mathbb{R} \times M \text{ open}$$

for every $t \in \mathbb{R}$, let $M_t = I(\mathbb{X}) \cap (\{t\} \times M)$

M_t is the domain of $Fl^{\mathbb{X}}_t$, we know that

$Fl^{\mathbb{X}}_t : M_t \longrightarrow M$ is smooth, its tangent map is

$$TFl^{\mathbb{X}}_t : TM_t \to TM, \quad TFl^{\mathbb{X}}_t(x, v) = T_x Fl^{\mathbb{X}}_t(v)$$

$$\widetilde{Fl}^{X}_t(x,v) := T_x Fl^X_t(v) \quad \text{so that}$$

$$\widetilde{Fl}^{X}_t : T_x M_t \longrightarrow T_{Fl^X_t(x)} M \qquad I(X) = \bigcup_{x\in M} I_x \times \{x\}$$

Consider
$$\widetilde{Fl}^{X} : \tilde{I}(X) \longrightarrow TM \qquad \tilde{I}(X) = \bigcup_{(x,v)\in TM} I_x \times \{(x,v)\}$$
$$\widetilde{Fl}^{X}(t,(x,v)) = \widetilde{Fl}^{X}_t(x,v)$$

Now $\widetilde{Fl}^{X}$ is a 1-par. family of local diffeom. on TM, we know that there is a unique vector field on TM, whose flow is this 1-parame. family

$$\Big(X \in \mathcal{X}(M) \longrightarrow Fl^X \text{ on } M \longrightarrow \widetilde{Fl}^{X} \text{ on } TM \longrightarrow \tilde{X} \in \mathcal{X}(TM)$$

$\tilde{X}$ is determined by
$$Fl^{\tilde{X}} = \widetilde{Fl}^{X} \Big)$$

Def. $\tilde{X}$ is called the lift X to TM.

How does $\tilde{X}$ look in coordinates on TM? Assume that $\varphi : U \to \mathbb{R}^m$ is a chart on M with coordinate functions $x_1, \dots, x_m$ and that $X = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i}$ on U. In these coordinates the flow of X is given by $(t, x_1, \dots, x_m) \longmapsto \big(\Phi_1(t, x_1, \dots, x_m), \dots, \Phi_m(t, x_1, \dots, x_m)\big)$

where Φ_i denotes the i-th component of $\Phi_t = Fl^X_t$.

Let us now fix a point $x_0 = (x_1, \dots, x_m)$ and let
$t \longmapsto (x_1(t), \dots, x_m(t)) = c_x(t) = \Phi(t,x) = \Phi_t(x)$
be the integral curve of X at x. Then we have
$$\frac{d}{dt}(x_1(t), \dots, x_m(t))\Big|_{t=0} = \frac{d}{dt}\big(\Phi_i(t,x)\big)_{t=0} = a_i(x), \quad i = 1, \dots, m.$$

The induced flow on TM in coordinates $(x_1, \dots, x_m, p_1, \dots, p_m)$ on $\pi^{-1}(U)$ is

$$(t, x_1, \dots, x_m, p_1, \dots, p_m) \longmapsto \Big((\Phi_t)_1(x_1, \dots, x_m), \dots, (\Phi_t)_m(x_1, \dots, x_m),$$
$$\sum_{j=1}^{m} p_j \frac{\partial (\Phi_t)_1}{\partial x_j}(x_1, \dots, x_m), \dots, \sum_{j=1}^{m} p_j \frac{\partial (\Phi_t)_m}{\partial x_j}(x_1, \dots, x_m) \Big)$$

Let us now fix $x_1, \dots, x_m, p_1, \dots, p_m$. Then $c_{\tilde{X}}(x_1, \dots, p_m)$ is

$$t \mapsto \left(\Phi_1(t, x_1, \dots, x_m), \dots \sum_{j=1}^{m} p_j \frac{\partial \Phi_m}{\partial x_j}(t, x_1, \dots, x_m) \right)$$

Now if we differentiate and evaluate at $t=0$, we get for $k=1, \dots, m$

$$\frac{d}{dt}\left(\sum_{j=1}^{m} p_j \frac{\partial (\Phi_t)_k}{\partial x_j}(x_1, \dots, x_m) \right) \Bigg|_{t=0} =$$

$$= \sum_{j=1}^{m} p_j \frac{\partial^2 \Phi}{\partial t \partial x_j}(0, x_1, \dots, x_m) = \sum_{j=1}^{m} p_j \frac{\partial^2 \Phi}{\partial x_j \partial t}(0, x_1, \dots, x_m)$$

$$= \sum_{j=1}^{m} p_j \frac{\partial}{\partial x_j}\left(\frac{\partial \Phi}{\partial t}(x_1, \dots, x_m) \right) =$$

$$= \sum_{j=1}^{m} p_j \frac{\partial a_i}{\partial x_j}(x_1, \dots, x_m). \qquad \text{Hence}$$

$$\tilde{X} = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \left(\sum_{i=1}^{m} p_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial p_j}.$$

Now if $Y = \sum_{i=1}^{m} b_i(x) \frac{\partial}{\partial x_i}$ on U, then inserting $b_i(x)$ instead of p_i and differentiating with respect to t at $t=0$, we get that $\mathcal{L}_X Y = [X, Y]$. $\square$