Tensor fields on manifolds

Tensors

Let V be a real vector space of dimension n with basis $M = \{e_1, .., e_n\}$. Let
$$V^* = \{ f: V \to \mathbb{R} \mid f \text{ is linear} \}$$
be the dual space of V and $M^* = \{\varepsilon_1, .., \varepsilon_n\}$ be the dual basis to M, that is
$$\varepsilon_j(e_i) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Definition Assume that $V_1, .., V_k$ are finite dim. real vector spaces. A map
$$f: V_1 \times \dots \times V_k \longrightarrow \mathbb{R}$$
is called multilinear if for every $v_i, v_i' \in V_i$ where $i = 1, .., k$ and $\lambda \in \mathbb{R}$:

- $f(v_1, .., v_{i-1}, v_i + v_i', v_{i+1}, .., v_k) = f(v_1, .., v_{i-1}, v_i, v_{i+1}, .., v_k) + f(v_1, .., v_{i-1}, v_i', v_{i+1}, .. v_k)$ and
- $f(v_1, .., \lambda v_i, .., v_k) = \lambda f(v_1, .., v_i, .., v_k)$

Definition Let $k, l \in \mathbb{N} \cup \{0\}$. A tensor of type (k, l) on V is a multilinear map
$$\underbrace{V \times \dots \times V}_{k \text{ copies}} \times \underbrace{V^* \times \dots \times V^*}_{l \text{ copies}} \longrightarrow \mathbb{R}.$$
We denote by $T^{k,l}(V)$ the set of tensors of type (k, l) on V.

Lemma a) $T^{k,l}(V)$ is a vector space with operations $T, S \in T^{k,l}(V)$, $\lambda \in \mathbb{R}$:

- $(T+S)(v_1, .., \alpha_l) = T(v_1, .., \alpha_l) + S(v_1, .., \alpha_l)$ and
- $(\lambda T)(v_1, .., \alpha_l) = \lambda T(v_1, .., \alpha_l)$

where $v_1, .., v_k \in V$, $\alpha_1, .., \alpha_l \in V$

b) If $\alpha_1, .., \alpha_k \in V^*$ and $v_1, .., v_l \in V$ then,
$$\alpha_1 \otimes \dots \otimes \alpha_k \otimes v_1 \otimes \dots \otimes v_l : V \times \dots \times V \times V^* \times \dots \times V^* \longrightarrow \mathbb{R}$$
$$(\alpha_1 \otimes \dots \otimes \alpha_k \otimes v_1 \otimes \dots \otimes v_l)(u_1, .., u_k, \beta_1, .., \beta_l) =$$
$$= \alpha_1(u_1) \cdot \alpha_2(u_2) \cdot \dots \dots \cdot \beta_l(v_l)$$ where $u_1, .., u_k \in V$, $\beta_1, .., \beta_l \in V^*$ is a tensor of type (k, l) on V.

c) If $M = \{e_1, .., e_n\}$ is a basis of V and $M^* = \{\varepsilon_1, .., \varepsilon_n\}$ is the dual basis, then
$$M^{k,l} = \{ \varepsilon_{i_1} \otimes \dots \otimes \varepsilon_{i_k} \otimes e_{j_1} \otimes \dots \otimes e_{j_l} : i_1, .., i_k, j_1, .., j_l \in \{1, .., n\} \}$$
is a basis of $T^{k,l}(V)$. Hence $\dim T^{k,l}(V) = n^{k+l}$.

Proof is left as an exercise. $\square$

Examples: • $T^{1,0}(V) \simeq V^*$ and $T^{0,1}(V) \simeq V$.

•) $T^{1,1}(V) \simeq \mathrm{End}(V) = \{\varphi : V \to V \mid \varphi \text{ is } \mathbb{R}\text{-linear}\}$

If $v \in V$ and $T \in T^{1,1}(V)$ are fixed, then the map $V^* \to \mathbb{R}$, $\alpha \mapsto T(v, \alpha)$ is linear and so there is unique $v_T \in V$ such that $T(v, \alpha) = \alpha(v_T)$. It is easy to verify that the map $\varphi_T : V \to V$, $\varphi_T(v) = v_T$ is linear and that

$$T^{1,1}(V) \ni T \longmapsto \varphi_T \in \mathrm{End}(V)$$

is linear. Since it is clearly injective and $\dim T^{1,1}(V) = n^2 = \dim \mathrm{End}(V)$, it is a linear isomorphism. Also note that if

$$T = \sum_{i,j=1}^{m} T^i_j \, \varepsilon_j \otimes e_i \quad \text{, then} \quad [\varphi_T]_M = (\varphi_{ij})^{j=1,\dots,m}_{i=1,\dots,n} \quad \text{where } \varphi_{ij} = T^i_j .$$

•) $T^{0,2}(V)$ is by definition the vector space of bilinear maps $V \times V \to \mathbb{R}$.

If $$T = \sum_{i,j=1}^{m} T_{ij} \, \varepsilon_i \otimes \varepsilon_j \quad \text{, then} \quad [T]_M = (T_{ij}) \in M(n, \mathbb{R})$$

is the matrix of T with respect to M.

Coefficients of tensors and change of basis

Let $M' = \{e_1', \dots, e_n'\}$ be another basis of V and

$$[Id]^M_{M'} = (a_{ij}) \in GL(n, \mathbb{R})$$

be the change of basis matrix from M to M', that is

$$e_j = \sum_{i=1}^{m} a_{ij} \, e_i' .$$

Let $(M')^* = \{\varepsilon_1', \dots, \varepsilon_n'\}$ be the dual basis to M', then

$$[Id]^{M^*}_{(M')^*} = \left[\left([Id]^M_{M'}\right)^{-1}\right]^T = \left([Id]^{M'}_{M}\right)^T = (b_{ij})^T \in GL(n, \mathbb{R}).$$

Let us consider the associated tensors of type $(1,1)$

$$\Lambda = \sum_{i,j} \Lambda^i_j \, \varepsilon_j \otimes e_i \quad \text{and} \quad \Lambda^{-1} = \sum_{i,j} (\Lambda^{-1})^i_j \, \varepsilon_j \otimes e_i$$

where $\Lambda^i_j := a_{ij}$ and $(\Lambda^{-1})^i_j := b_{ij}$. We will from now on write $\varepsilon^i := \varepsilon_i$, $\varepsilon'^i = \varepsilon'_i$, $i = 1, \dots, n$. Note that

$$e_j = \sum_{i=1}^{m} \Lambda^i_j \, e_i' , \quad e_j' = \sum_{i=1}^{m} (\Lambda^{-1})^i_j \, e_i , \quad \varepsilon^i = \sum_{j=1}^{m} (\Lambda^{-1})^i_j \, \varepsilon'^j , \quad \varepsilon'^i = \sum_{j=1}^{n} \Lambda^i_j \, \varepsilon^j .$$

If $T \in T^{k,l}(V)$, then we may write

$$T = \sum_{i_1,\dots,i_k,j_1,\dots,j_l=1}^{m} T^{j_1 \dots j_l}_{i_1 \dots i_k} \, \varepsilon^{i_1} \otimes \dots \otimes \varepsilon^{i_k} \otimes e_{j_1} \otimes \dots \otimes e_{j_l} =$$

$$= \sum_{i_1 \dots j_l} \sum_{p_1} \dots \sum_{q_l} T^{j_1 \dots j_l}_{i_1 \dots i_k} (\Lambda^{-1})^{i_1}_{p_1} \dots (\Lambda^{-1})^{i_k}_{p_k} \Lambda^{q_1}_{j_1} \dots \Lambda^{q_l}_{j_l} \varepsilon'^{p_1} \otimes \dots \otimes \varepsilon'^{p_k} \otimes e'_{q_1} \otimes \dots \otimes e'_{q_l}$$

$$= \sum_{p_1,\dots,p_k,q_1,\dots,q_l=1}^{n} (T')^{q_1,\dots,q_l}_{p_1,\dots,p_k} \, \varepsilon'^{p_1} \otimes \dots \otimes \varepsilon'^{p_k} \otimes e'_{q_1} \otimes \dots \otimes e'_{q_l}$$

where

$$\text{(CCT)} \quad (T')^{q_1,\dots,q_l}_{p_1,\dots,p_k} = \sum_{\substack{i_1,\dots,i_k \\ j_1,\dots,j_l=1}}^{m} T^{j_1,\dots,j_l}_{i_1,\dots,i_k} (\Lambda^{-1})^{i_1}_{p_1} \dots (\Lambda^{-1})^{i_k}_{p_k} \Lambda^{q_1}_{j_1} \dots \Lambda^{q_l}_{j_l}$$

Operations on tensors

1) **Tensor product**

Let $T \in T^{k,0}(V)$ and $S \in T^{l,0}(V)$. Then

$T \otimes S \in T^{k+l,0}(V)$, $T \otimes S(v_1, \dots, v_{k+l}) = T(v_1, \dots, v_k) S(v_{k+1}, \dots, v_{k+l})$

where $v_i \in V$, $i = 1, \dots, k+l$

2) **Symmetrization and alternation**

Let $T \in T^{k,0}(V)$ and $S \in T^{l,0}(V)$. Then $T \odot S$ and $T \wedge S \in T^{k+l,0}(V)$ are defined by

$$(T \odot S)(v_1, \dots, v_{k+l}) = \sum_{\sigma \in S_{k+l}} \frac{1}{(k+l)!} T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) S(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

$$(T \wedge S)(v_1, \dots, v_{k+l}) \sum_{\sigma \in S_k} \frac{\operatorname{sgn} \sigma}{(k+l)!} T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) S(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

where $v_1, \dots, v_{k+l} \in V$ and S_{k+l} is the permutation group on $\{1, \dots, k+l\}$ and $\operatorname{sgn} \sigma$ is the sign of $\sigma \in S_{k+l}$.

Now $T \odot S$ is symmetric in the sense that

$$(T \odot S)(v_1, \dots, v_i, \dots, v_j, \dots, v_{k+l}) = (T \odot S)(v_1, \dots, v_j, \dots, v_i, \dots v_{k+l})$$

while $T \wedge S$ is skew-symmetric in the sense that

$$(T \wedge S)(v_1, \dots, v_i, \dots, v_j, \dots, v_{k+l}) = -(T \wedge S)(v_1, \dots, v_j, \dots, v_i, \dots, v_{k+l}).$$

Example: $V = \mathbb{R}^n$, $v_1, \dots, v_n \in \mathbb{R}$, then the map

$$(v_1, \dots, v_n) \longmapsto \det(v_1 | \dots | v_n)$$

is a skew-symmetric tensor of type $(n, 0)$. Here $(v_1 | \dots | v_n) \in M(n, \mathbb{R})$ is the matrix with columns $v_1, \dots, v_n$.

If $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$ with the dual basis $\{\varepsilon_1, \dots, \varepsilon_n\}$, then this tensor is

$$\varepsilon_1 \wedge \dots \wedge \varepsilon_n.$$

Cotangent bundle and smooth 1-forms

Let $(M, \mathcal{A})$ be a smooth manifold of dim. m where $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^m \mid \alpha \in A\}$.
If $x \in M$, we know that T_xM is a real vector space of dim. m.
Let $T_x^*M := (T_xM)^*$ be the dual vector space and put

$$T^*M = \bigcup_{x \in M} T_x^*M.$$

We sometimes write (x, α) instead of $\alpha \in T_x^*M$. Note that there is a canonical projection $\pi : T^*M \to M$, $\pi(x,\alpha) = x$.

Let $\varphi : U \to \mathbb{R}^m$ be a chart from $\mathcal{A}$ with coordinate functions $x_1, \dots, x_m$ and associated vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ on U. Recall that

$$B_x^\varphi = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \right\}$$

is a basis of T_xM for every $x \in U$. Let $(B_x^\varphi)^* = \{dx_1, \dots, dx_m\}$ be the dual basis of T_x^*M, that is

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}, \quad i,j = 1, \dots, m.$$

The map
$$\Phi : \pi^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$$
$$\Phi(x, \alpha) = (\varphi(x), [\alpha]_{(B_x^\varphi)^*})$$

is injective and its image is $\varphi(U) \times \mathbb{R}^m$. The inverse map is

$$\Phi^{-1} : \varphi(U) \times \mathbb{R}^m \to \pi^{-1}(U)$$
$$\Phi^{-1}((x_1, \dots, x_m), (q_1, \dots, q_m)) = \left(\varphi^{-1}(x_1, \dots, x_m), \sum_{i=1}^m q_i\, dx_i\right).$$

As in the case of TM, it can be shown (using CCT) that

Theorem There is a canonical smooth structure on T^*M determined by atlas
$$\mathcal{A}_{T^*M} = \{\Phi_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2m} \mid \alpha \in A\}$$
so that T^*M is a smooth manifold of dimension $2m$ and $\pi : T^*M \to M$ is smooth.

Proof is a repetition of that for TM. $\square$

Definition $(T^*M, \mathcal{A}_{T^*M})$ is called the cotangent bundle of M. A smooth 1-form on M is a smooth map
$$\alpha : M \to T^*M$$
such that $\pi \circ \alpha = \mathrm{Id}_M$. We denote by $\Omega^1(M)$ the set of 1-forms on M.

Example If $f \in \mathcal{C}^{+\infty}_{\mathcal{A}}(M)$, then $df \in \Omega^1(M)$ defined by $df(x,v) = v(f)$ where we view $v \in T_xM = D_xM$. In the coordinates on U,
$df(x) = \sum_{i=1}^m \left(\frac{\partial}{\partial x_i} f\right)(x)\, dx_i$. The 1-form is usually called the differential of f.

Tensor fields on manifolds

Put $T_x^{k,l}M = T^{k,l}(T_xM)$ where $k, l \in \mathbb{N} \cup \{0\}$ and

$$T^{k,l}M = \bigcup_{x \in M} T_x^{k,l}M.$$

We know that $T_x^{k,l}M$ is a vector space of dim m^{k+l}. Following the construction smooth atlas on TM and T^*M, it follows that also $T^{k,l}M$ is a smooth manifold of dim $m + m^{k+l}$ and that the canonical projection $\pi : T^{k,l}M \to M$ which is smooth.

Definition A. A smooth tensor field on M of type (k,l) is a smooth map $T: M \to T^{k,l}M$ such that $\pi \circ T = Id_M$. We denote by $\mathcal{T}^{k,l}(M)$ the set of all tensor fields of type (k,l) on M.

Local descriptions of tensor fields

Let $\varphi: U \to \mathbb{R}^m$ be the chart on M as above and $T \in \mathcal{T}^{k,l}(M)$. By definition, $T(x) \in T_x^{k,l}M$ for every $x \in U$. We know that

$$(B\varphi_x)_{k,l} = \{ dx_{i_1} \otimes \dots \otimes dx_{i_k} \otimes \frac{\partial}{\partial x_{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{j_l}} \mid i_1, \dots, i_k, j_1, \dots, j_l \in \{1, \dots, m\} \}$$

is a basis of $T_x^{k,l}M$ for every $x \in U$. Hence we may write

$$T(x) = \sum_{\substack{i_1,\dots,i_k \\ j_1,\dots,j_l = 1}}^{m} T^{j_1 \dots j_k}_{i_1 \dots i_k}(x)\, dx_{i_1} \otimes \dots \otimes dx_{i_k} \otimes \frac{\partial}{\partial x_{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{j_l}}, \quad x \in U,$$

where $T^{j_1 \dots j_k}_{i_1 \dots i_k}(x)$ are smooth functions of $x_1, \dots, x_m$.

Examples $\mathcal{T}^{(1,0)}(M) = \Omega^1(M)$ and $\mathcal{T}^{(0,1)}(M) = \mathfrak{X}(M)$.

Definition A'. (Alternative definition of tensor fields)

A smooth vector field on M of type (k,l) is a multilinear map

$$T: \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{k \text{ copies}} \times \underbrace{\Omega^1(M) \times \dots \times \Omega^1(M)}_{l\text{-copies}} \to C^\infty(M)$$

such that

(LOF) $T(X_1, \dots, f X_j, \dots, X_k, \theta_1, \dots, \theta_l)(x) = f(x)\, T(X_1, \dots, X_k, \theta_1, \dots, \theta_l)(x)$
$= T(X_1, \dots, X_k, \theta_1, \dots, f\theta_i, \dots \theta_l)(x)$

for every $X_1, \dots, X_k \in \mathfrak{X}(M)$, $\theta_1, \dots, \theta_l \in \Omega^1(M)$, $f \in C^\infty(M)$ and $j = 1, \dots, k$, $i = 1, \dots, l$.

Proof can be found in xxx. □

Notation We will sometimes write T_x instead of $T(x)$ for $T \in \mathcal{T}^{(k,l)}(M)$ and $x \in M$.

Operations on tensor fields

Lemma
- $\mathcal{T}^{k,l}(M)$ is a vector space with operations $(T+S)(x) = T(x) + S(x)$ and $(\lambda T)(x) = \lambda T(x)$, where $T, S \in \mathcal{T}^{k,l}(M)$ and $\lambda \in \mathbb{R}$.
- If $T \in \mathcal{T}^{k,0}(M)$ and $S \in \mathcal{T}^{l,0}(M)$, then $T \otimes S$ and $T \wedge S$ and $T \odot S$ defined by $(T \otimes S)(x) = T(x) \otimes S(x)$ and $(T \odot S)(x) = T(x) \odot S(x)$ and $(T \wedge S)(x) = T(x) \wedge S(x)$ are smooth tensor fields on M of type $(k+l, 0)$.

Definition Let $f: M \to N$ be a smooth map of manifolds and T be a tensor field of type $(k,0)$ on N. Then f^*T is a tensor field of type $(k,0)$ on M, called the pullback of T, defined by

$$(f^*T)(x)(v_1, \dots, v_k) = T_{f(x)}(T_x f(v_1), \dots, T_x f(v_k)).$$

Validity of the previous definition:

- $(f^*T)(x): T_xM \times \dots \times T_xM \to \mathbb{R}$ is multilinear since $T_{f(x)}$ is multilinear and $T_x f$ is linear.

•) $f^*T : M \to T^{(k,0)}M$ is smooth since if $\varphi : \mathcal{U} \to \mathbb{R}^m$ is a chart on M around x_0 and $\rho : \mathcal{V} \to \mathbb{R}^n$ is a chart around $f(x_0)$ with coordinate functions $y_1, \dots, y_n$ and

$$T(y) = \sum_{i_1 \dots i_k = 1}^{m} T^{i_1 \dots i_k}(y)\, dy_{i_1} \otimes \dots \otimes dy_{i_k} \quad \text{on } f(\mathcal{U}) \cap \mathcal{V}, \text{ then}$$

$$f^*T(x) = \sum_{i_1 \dots i_k = 1}^{m} T^{i_1 \dots i_k}(f(x)) \sum_{l_1 = 1}^{m} \frac{\partial f_{i_1}}{\partial x_{l_1}} dx_{l_1} \otimes \dots \otimes \sum_{l_k = 1}^{m} \frac{\partial f_{i_k}}{\partial x_{l_k}} dx_{l_k}$$

$$= \sum_{l_1 \dots l_k = 1}^{m} \sum_{i_1 \dots i_k = 1}^{m} T^{i_1 \dots i_k}(f(x)) \frac{\partial f_{i_1}}{\partial x_{l_1}} \cdots \frac{\partial f_{i_k}}{\partial x_{l_k}} dx_{l_1} \otimes \dots \otimes dx_{l_k}$$

on $\mathcal{U} \cap f^{-1}(\mathcal{V})$ where f_i is the i-th component of

$$\rho \circ f \circ \varphi^{-1} : \underset{\mathbb{R}^m}{\underset{\cap}{\varphi(f^{-1}(\mathcal{V}) \cap \mathcal{U})}} \to \underset{\mathbb{R}^n}{\underset{\cap}{\rho(f(\mathcal{U}) \cap \mathcal{V})}}.$$

We see that f^*T has smooth coefficients and so it is a smooth tensor field of type $(k,0)$ on M.

Examples 1) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (x^2+y, xzy, z^3)$

$\theta \in \Omega^1(\mathbb{R}^3)$ $\quad \theta = dz + x\,dy$, then

$$f^*\theta = d(z^3) + (x^2+y)\, d(xzy)$$
$$= 3z^2 dz^2 + (x^2+y)(zy\,dx + xz\,dy + xy\,dz)$$
$$= (x^2+y)zy\,dx + (x^2+y)(xz)\,dy + (3z^2 + xy)\,dz$$

2) f as above and $dx^3 = dx \wedge dy \wedge dz$ (Lebesgue measure on $\mathbb{R}^3$) then $f^* dx^3 = |\mathrm{Jac} f|\, dx \wedge dy \wedge dz = 3z^3(2x^2 - y^2)\, dx^3$

$$\mathrm{Jac} f = \begin{pmatrix} 2x & y & 0 \\ zy & xz & xy \\ 0 & 0 & 3z^2 \end{pmatrix} \text{ is the Jacobi matrix of } f$$

$$|\mathrm{Jac} f| = \det \mathrm{Jac} f = 3z^2(2x^2 z - y^2 z) = 3z^3(2x^2 - y^2).$$

Lemma Let $f : M \to N$ be smooth. Then
1) $f^* : \mathcal{T}^{0,k}(N) \to \mathcal{T}^{0,k}(M)$ is linear and
2) $f^*T \otimes f^*S = f^*(T \otimes S)$ when $T \in \mathcal{T}^{0,k}(M)$ and $S \in \mathcal{T}^{0,l}(M)$.

Lie derivative of tensor fields

Definition Let Fl^X be the flow of $X \in \mathfrak{X}(M)$ and $T \in \mathcal{T}^{k,0}(M)$. Then the Lie derivative of T along X is defined as

$$\mathcal{L}_X T = \frac{d}{dt}\left((Fl_t^X)^* T\right)\Big|_{t=0} \in \mathcal{T}^{k,0} M.$$

Note that $((Fl_t^{X})^* T)_x(v) = T_{Fl_t^X(x)}(T Fl_t^X(v))$.

Theorem Let $X \in \mathfrak{X}(M)$. Then

1) $\mathcal{L}_X : \mathcal{T}^{k,0}(M) \longrightarrow \mathcal{T}^{k,0}(M)$ is linear.

2) $\mathcal{L}_X(T \otimes S) = \mathcal{L}_X T \otimes S + T \otimes \mathcal{L}_X S$ where $T \in \mathcal{T}^{k,0}M$ and $S \in \mathcal{T}^{l,0}M$.

3) $(\mathcal{L}_X \theta)(Y) = X(\theta(Y)) - \theta(\mathcal{L}_X(Y))$ where $\theta \in \Omega^1(M)$ and $Y \in \mathfrak{X}(M)$

4) If $X = \sum_{i=1}^m a_i(x) \frac{\partial}{\partial x_i}$ and $\theta = \sum_{i=1}^m \alpha_i(x) dx_i$ in a chart U, then

$$\mathcal{L}_X \theta = \sum_{j=1}^m \left[\sum_{i=1}^m \left(a_i(x) \frac{\partial \alpha_j}{\partial x_i} + \alpha_i(x) \frac{\partial a_i}{\partial x_j}(x) \right) \right] dx_j \quad \text{on } U.$$

Proof: ad 1) Follows from the fact that $(Fl_t^X)^* : \mathcal{T}^{k,0}M \to \mathcal{T}^{k,0}M$ is linear.

ad 2) Follows from Leibniz rule for ordinary functions.

ad 3) Let us fix an open chart $\varphi : U \to \mathbb{R}^m$ on M assume that Fl^X is defined for every $t \in J$ and $x \in U$ where J is an open interval containing 0. Assume that

$$X = \sum_{i=1}^m a_i(x) \frac{\partial}{\partial x_i},\ Y = \sum_{i=1}^m b_i(x) \frac{\partial}{\partial x_i} \text{ and } \theta = \sum_{i=1}^m \alpha_i(x)\, dx_i \text{ on } U.$$

Let $(\Phi_t)_i$ be the i-th component of Fl^X in the coordinates on U. Let us fix $x \in U$ and consider the flow $t \mapsto Fl_t^X(x) = (\Phi_t)(x)$. Then

$$(\mathcal{L}_X \theta)_x(Y(x)) = \frac{d}{dt}\left(\sum_{i=1}^m \alpha_i(\Phi_t(x)) \sum_{j=1}^m \frac{\partial (\Phi_t)_i}{\partial x_j}(x) b_j(x) \right)\Big|_{t=0}$$

$$= \sum_{i=1}^m \sum_{l=1}^m a_l(x) \frac{\partial \alpha_i}{\partial x_l}(x)\, \delta_{ij}\, b_j(x)$$

$$+ \sum_{i=1}^m \alpha_i(x) \sum_{j=1}^m \frac{d}{dt} \frac{\partial (\Phi_t)_i}{\partial x_j}\Big|_{t=0} b_j(x) \Big)$$

$$= \sum_{i,j=1}^m a_i(x) \frac{\partial \alpha_j}{\partial x_i} b_j + \sum_{i=1}^m \alpha_i(x) \sum_{j=1}^m \frac{\partial a_i}{\partial x_j}(x) b_j(x)$$

$$= \sum_{i,j=1}^m a_i(x) \frac{\partial}{\partial x_i}\left(\alpha_j(x) b_j(x)\right) - \sum_{i,j=1}^m \alpha_j(x) \left(a_i(x) \frac{\partial b_j}{\partial x_i}(x) - b_i(x) \frac{\partial a_j}{\partial x_i}(x) \right)$$

$$= (X\alpha(Y))(x) - \alpha(\mathcal{L}_X Y)(x).$$

ad 4) Follows from the proof of 3). □

Remark: The formula 4) can be written invariantly as

(Cartan formula) $\mathcal{L}_X \theta = i_X d\theta + d i_X \theta$.

Here d is the de Rham differential and i_X is the insertion operator determined by X.