Tensors

Let V be a real vector space of dim. $n \in \mathbb{N}$ with basis $M = \{e_1, \dots, e_n\}$. Let V^* be the dual vector space, that is $V^* = \{f : V \to \mathbb{R} \mid f \text{ is linear}\}$, and $M^* = \{\varepsilon_1, \dots, \varepsilon_n\}$ be the dual basis, that is

$$\varepsilon_i(e_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

f is linear:
- $f(u+v) = f(u) + f(v)$, $u, v \in V$
- $f(\lambda u) = \lambda f(u)$, $\lambda \in \mathbb{R}$, $u \in V$

I will denote vectors in V in latin and dual vectors from V^* by greek letters. $V \cong (V^*)^*$.

$\boxed{V \ni v \mapsto (\alpha \mapsto \alpha(v))}$

Definition

Let $V_1, \dots, V_k$ be f.d. real vector spaces. A map

$$f : V_1 \times \dots \times V_k \to \mathbb{R}$$

is called multilinear if

- $f(v_1, \dots, v_{i-1}, v_i + v_i', v_{i+1}, \dots, v_k) = f(v_1, \dots, v_k) + f(v_1, \dots, v_{i-1}, v_i', v_{i+1}, \dots, v_k)$
- $f(v_1, \dots, v_{i-1}, \lambda v_i, v_{i+1}, \dots, v_k) = \lambda f(v_1, \dots, v_k)$

whenever $v_i, v_i' \in V_i$, $i = 1, \dots, k$ and $\lambda \in \mathbb{R}$. (In other words, f is linear in each component.)

Remark $V_1 \times \dots \times V_k$ is also a vector space with

- $(v_1, \dots, v_k) + (v_1', \dots, v_k') = (v_1 + v_1', \dots, v_k + v_k')$
- $\lambda(v_1, \dots, v_k) = (\lambda v_1, \dots, \lambda v_k)$

where $v_i, v_i' \in V_i$, $i = 1, \dots, k$, $\lambda \in \mathbb{R}$. Note that the concept of multilinear map on $V_1 \times \dots \times V_k$ differs from the concept of linear map $V_1 \times \dots \times V_k \to \mathbb{R}$ (if $k > 1$).

Definition A tensor of type (k, l) (where $k, l \in \mathbb{N} \cup \{0\}$) on the vector space V is a multilinear map

$$\underbrace{V \times \dots \times V}_{k \text{ copies}} \times \underbrace{V^* \times \dots \times V^*}_{l \text{ copies}} \to \mathbb{R}.$$

I will denote $T^{k,l}(V)$ the set of all tensors of type (k, l) on V.

Lemma 1) $T^{k,l}(V)$ is a vector space with operations

$\cdot)$ $(T+S)(v_1,\dots,v_k,\alpha_1,\dots,\alpha_l) = T(v_1,\dots,v_k,\alpha_1,\dots,\alpha_l) + S(v_1,\dots,v_k,\alpha_1,\dots,\alpha_l)$

$\cdot)$ $(\lambda T)(v_1,\dots,v_k,\alpha_1,\dots,\alpha_l) = \lambda T(v_1,\dots,v_k,\alpha_1,\dots,\alpha_l)$

where $\lambda\in\mathbb{R}$, $v_i\in V$, $\alpha_j\in V^*$, $T, S\in T^{k,l}(V)$, $i=1,\dots,k$, $j=1,\dots,l$.

2) If $\alpha_1,\dots,\alpha_k\in V^*$, $v_1,\dots,v_l\in V$ then the map

$$(\alpha_1\otimes\dots\otimes\alpha_k\otimes v_1\otimes\dots\otimes v_l): \underbrace{V\times\dots\times V}_{k \text{ copies}}\times\underbrace{V^*\times\dots\times V^*}_{l \text{ copies}}\longrightarrow\mathbb{R}$$

$$(\alpha_1\otimes\dots\otimes\alpha_k\otimes v_1\otimes\dots\otimes v_l)(u_1,\dots,u_k,\beta_1,\dots,\beta_l) =$$
$$= \alpha_1(u_1)\dots\alpha_k(u_k)\,\beta_1(v_1)\dots\beta_l(v_l),$$

where $u_1,\dots,u_k\in V$ and $\beta_1,\dots,\beta_l\in V^*$, is a tensor of type (k,l) on V.

3) The (ordered) set

$$\{\varepsilon_{i_1}\otimes\dots\otimes\varepsilon_{i_k}\otimes e_{j_1}\otimes\dots\otimes e_{j_l} \mid i_1,\dots,i_k,j_1,\dots,j_l\in\{1,\dots,n\}\}$$

is a basis of $T^{k,l}(V)$.

Proof is left as an exercise. $\square$

Examples: 1) $T^{1,0}(V)\cong V^*$, $T^{0,1}(V)\cong(V^*)^*\cong V$

2) $T^{1,1}(V)\cong \mathrm{End}(V) = \{\varphi: V\to V \mid \varphi \text{ is linear}\}$

$$T^{1,1}(V)\longrightarrow \mathrm{End}(V)$$

$T\in T^{1,1}(V)$, $v\in V$ be fixed

$$V^*\ni\alpha\longmapsto T(v,\alpha)\in\mathbb{R}$$

since T is linear in the second argument, then this map $V^*\to\mathbb{R}$ is linear, we know that there is a unique $v_T\in V$ such

$$\alpha\longmapsto T(v,\alpha) = \alpha(v_T)$$

since T is linear in the first argument, then $V\ni v\longmapsto v_T\in V$ is also linear, put

$$\varphi_T: V\to V,\ \varphi_T(v) := v_T,$$

it is easy to verify that the map

$$T^{1,1}(V) \longrightarrow End(V), \quad T \longmapsto \varphi_T$$

is linear, injective and (by dimensional reasons $\dim T^{1,1}(V) = n^{1+1} = n^2 = End(V)$) it is an isomorphism

basis $M = \{e_1, \dots, e_n\}$ of V, dual basis $M^* = \{\varepsilon_1, \dots, \varepsilon_n\}$ of V^*

if $T = \sum_{i,j=1}^{n} T^i_j \varepsilon_j \otimes e_i \longmapsto \varphi_T$ is uniquely det. by $[\varphi_T]_M = (\varphi_{ij})_{i=1,\dots,n}^{j=1,\dots,n}$

$T^i_j \in \mathbb{R}$

$\varphi_{ij} = T^i_j$ for every $i,j = 1, \dots, n$.

c) $T^{2,0}(V) = \{f : V \times V \longrightarrow \mathbb{R} \mid f \text{ multilinear}\}$

$= \{f : V \times V \longrightarrow \mathbb{R} \mid f \text{ bilinear}\}$

$T = \sum_{i,j=1}^{n} T_{ij}\, \varepsilon_i \otimes \varepsilon_j \longmapsto$ $f : V \times V \longrightarrow \mathbb{R}$ bilinear

$[f]_M = (T_{ij})_{i=1,\dots,n}^{j=1,\dots,n}$

$\{\varepsilon_i \otimes \varepsilon_j \mid i,j = 1, \dots, n\}$ is a basis of $T^{2,0}(V)$

Coefficients of tensors and change of basis of V

Let us assume that $M' = \{e'_1, \dots, e'_n\}$ is a different basis of V, $(M')^* = \{\varepsilon'_1, \dots, \varepsilon'_n\}$ be the dual basis

$$[Id]^M_{M'} = (a_{ij})_{i=1,\dots,n}^{j=1,\dots,n} \in M(n, \mathbb{R}) = M_{n \times n}(\mathbb{R})$$

$$e_j = \sum_{i=1}^{n} a_{ij} e'_i, \quad j = 1, \dots, n$$

$$[Id]^{M^*}_{(M')^*} = \left(\left([Id]^M_{M'}\right)^T\right)^{-1} = \left([Id]^{M'}_{M}\right)^T = \left[(b_{ij})_{i=1,\dots,n}^{j=1,\dots,n}\right]^T \in M(n, \mathbb{R})$$

$$\varepsilon_j = \sum_{i=1}^{n} b_{ji} \varepsilon'_i, \quad j = 1, \dots, n$$

$$e'_j = \sum_{i=1}^{n} b_{ij} e_i, \quad j = 1, \dots, n$$

$$\varepsilon'_j = \sum_{i=1}^{n} a_{ji} \varepsilon_i, \quad j = 1, \dots, n$$

$$M(n,\mathbb{R}) \ni (a_{ij})_{i=1,\dots,n}^{j=1,\dots,n} \longmapsto \Lambda = \sum_{i,j=1}^{n} \Lambda^{i}_{\ j}\, \varepsilon_j \otimes e_i \qquad \Lambda^{i}_{\ j} = a_{ij}$$

$$M(n,\mathbb{R}) \ni (b_{ij})_{i=1,\dots,n}^{j=1,\dots,n} \longmapsto \Lambda^{-1} = \sum_{i,j=1}^{n} (\Lambda^{-1})^{i}_{\ j}\, \varepsilon_j \otimes e_i \qquad (\Lambda^{-1})^{i}_{\ j} = b_{ij}$$

I will write from now on

$$\varepsilon^{i} := \varepsilon_i \,, \quad (\varepsilon')^{i} := \varepsilon'_i \,, \quad i = 1,\dots,n.$$

We have

$$e_j = \sum_{i=1}^{n} \Lambda^{i}_{\ j}\, e_i' \,, \qquad e_j' = \sum_{i=1}^{n} (\Lambda^{-1})^{i}_{\ j}\, e_i$$

$$\varepsilon^{j} = \sum_{i=1}^{n} (\Lambda^{-1})^{j}_{\ i}\, \varepsilon'^{i} \,, \qquad \varepsilon'^{j} = \sum_{i=1}^{n} \Lambda^{j}_{\ i}\, \varepsilon^{i}$$

$$T \in T^{k,l}(V) \qquad T^{j_1 \dots j_l}_{i_1 \dots i_k} \in \mathbb{R}$$

$$T = \sum_{\substack{i_1,\dots,i_k \\ j_1,\dots,j_l = 1}}^{n} T^{j_1 \dots j_l}_{i_1 \dots i_k}\, \varepsilon^{i_1} \otimes \dots \otimes \varepsilon^{i_k} \otimes e_{j_1} \otimes \dots \otimes e_{j_l}$$

$$= \sum_{\substack{i_1,\dots,i_k \\ j_1,\dots,j_l = 1}}^{n} T^{j_1 \dots j_l}_{i_1 \dots i_k} \sum_{\substack{p_1 \dots p_k \\ q_1 \dots q_l = 1}}^{n} (\Lambda^{-1})^{i_1}_{\ p_1} \dots (\Lambda^{-1})^{i_k}_{\ p_k}\, \Lambda^{q_1}_{\ j_1} \dots \Lambda^{q_l}_{\ j_l}\, \varepsilon'^{p_1} \otimes \dots \otimes \varepsilon'^{p_k} \otimes e'_{q_1} \otimes \dots \otimes e'_{q_l}$$

$$= \sum_{\substack{q_1,\dots,q_l \\ p_1,\dots,p_k = 1}}^{n} (T')^{q_1 \dots q_l}_{p_1 \dots p_k}\, \varepsilon'^{p_1} \otimes \dots \otimes \varepsilon'^{p_k} \otimes e'_{q_1} \otimes \dots \otimes e'_{q_l}$$

$$\boxed{(T')^{q_1 \dots q_l}_{p_1 \dots p_k} = \sum_{\substack{i_1,\dots,i_k = 1 \\ j_1,\dots,j_l}} T^{j_1 \dots j_l}_{i_k \dots i_k}\, (\Lambda^{-1})^{i_1}_{\ p_1} \dots (\Lambda^{-1})^{i_k}_{\ p_k}\, \Lambda^{q_1}_{\ j_1} \dots \Lambda^{q_l}_{\ j_l}}$$

Remark: Here the only important thing is to note the coefficients of T w.r.t. M and M' can be computed using the matrices

$$[\mathrm{Id}]^{M'}_{M} \,, \quad [\mathrm{Id}]^{M}_{M'}$$

Operations on tensors

.) Tensor product

$T \in T^{k,0}(V)$, $S \in T^{l,0}(V)$

$T \otimes S \in T^{k+l,0}(V)$

$T \otimes S(v_1, \dots, v_{k+l}) = T(v_1, \dots, v_k) \cdot S(v_{k+1}, \dots, v_{k+l})$

.) Symmetrization and alternation (or skew-symmetrization)

$T \in T^{k,0}(V)$, $S \in T^{l,0}(V)$

..) $T \odot S \in T^{k+l,0}(V)$, $T \odot S(v_1, \dots, v_{k+l})$

$$= \sum_{\sigma \in S_{k+l}} \frac{1}{(k+l)!} T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \, S(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

S_{k+l}.. permutation group on $\{1, \dots, k+l\}$

$\sigma \in S_{k+l}$ permutation

$v_1, \dots, v_{k+l} \in V$

$T \odot S$ is symmetric in the sense that

$$T \odot S(v_1, \dots, v_i, \dots, v_j, \dots, v_{k+l}) = T \odot S(v_1, \dots, v_j, \dots, v_i, \dots, v_{k+l})$$

..) $T \wedge S \in T^{k+l,0}(V)$, $T \wedge S(v_1, \dots, v_{k+l}) =$

$$\sum_{\sigma \in S_{k+l}} \frac{\mathrm{sgn}(\sigma)}{(k+l)!} T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \, S(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

$\mathrm{sgn}(\sigma)$.. the sign of $\sigma \in S_{k+l}$

$T \wedge S$ is skew-symmetric in the sense that

$$T \wedge S(v_1, \dots, v_i, \dots, v_j, \dots, v_{k+l}) = - T \wedge S(v_1, \dots, v_j, \dots, v_i, \dots, v_{k+l})$$

<u>Example</u> $V = \mathbb{R}^n$, $v_1, \dots, v_n \in \mathbb{R}^n$

$$(v_1, \dots, v_n) \longmapsto \det(v_1 | \dots | v_n)$$

$(v_1 | \dots | v_n) \in M(n, \mathbb{R})$ whose columns are $v_1, \dots, v_n$

is a skew-symmetric tensor of type $(n, 0)$

and this tensor is equal to

$\varepsilon_1 \wedge \dots \wedge \varepsilon_n$ where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$ and $\{\varepsilon_1, \dots, \varepsilon_n\}$ is the dual basis.

Cotangent bundle

Let $(M, \mathcal{A})$ be a smooth manifold of dim m with
$\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{R}^m \mid \alpha \in A\}$

$x \in M$ is fixed, then $T_x M$ is a vector space of dim m and $T_x^* M := (T_x M)^*$ be the dual vector space.

Let $\varphi : U \to \mathbb{R}^m$ be a fixed chart on M, then we know that

$$B_x^\varphi = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \right\}$$

is a basis of $T_x M$ for every $x \in U$. Here $x_1, \dots, x_m$ are the associated coordinate functions.

Let

$$(B_x^\varphi)^* = \{ dx_1, \dots, dx_m \}$$

be the dual basis of $T_x^* M$ for every $x \in U$, that is

$$dx_i \left(\frac{\partial}{\partial x_j} \right) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Let us write $(x, \alpha) := \alpha \in T_x^* M$. Let

$$T^*M = \bigcup_{x \in M} T_x^* M \quad \text{(disjoint union)}.$$

<u>Definition</u> We call $T_x^* M$ the cotangent space at x and T^*M the cotangent bundle.

Construction of a (canonical) smooth atlas on T^*M

Note that there is a canonical projection

$$\pi : T^*M \longrightarrow M, \quad \pi(x, \alpha) = x.$$

Define

$$\Phi : \pi^{-1}(U) \longrightarrow \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$$

$$\Phi(x, \alpha) = \left(\varphi(x), [\alpha]_{(B_x^\varphi)^*} \right)$$

$[\alpha]_{(B_x^\varphi)^*}$ .. coefficients of α w.r. to $(B_x^\varphi)^*$

$\operatorname{Im} \Phi = \varphi(U) \times \mathbb{R}^m$. The inverse map is

$$\Phi^{-1}(x_1, \dots, x_m, q_1, \dots, q_m) = \left(\varphi^{-1}(x_1, \dots, x_m), \sum_{i=1}^m q_i \, dx_i \right).$$

Let $\psi: V \to \mathbb{R}^m$ be a different chart on M with coordinate functions $y_1, \dots, y_m$. Then on $U \cap V$ we have two different bases of $T_x M$, $x \in U \cap V$, which are $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$ and $\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m}\}$ and dually, we have two different bases of $T_x^* M$ $\{dx_1, \dots, dx_m\}$ and $\{dy_1, \dots, dy_m\}$. Recall that

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{m} a_{ij}(x) \frac{\partial}{\partial y_i}, \quad j = 1, \dots, m$$

$$a_{ij}(x) = \frac{\partial (\psi \circ \varphi^{-1})_i}{\partial x_j}(\varphi(x)), \quad i, j = 1, \dots, m$$

$$(a_{ij}(x))_{i=1,\dots,m}^{j=1,\dots,m} = [Id]_{B_x^\psi}^{B_x^\varphi} = \left([Id]_{(B_x^\psi)^*}^{(B_x^\varphi)^*} \right)^T$$

If we write $y_i = (\psi \circ \varphi^{-1})_i$, $i = 1, \dots, m$,

then $dy_j = \sum_{i=1}^{m} \frac{\partial y_j}{\partial x_i} dx_i = \sum_{i=1}^{m} \frac{\partial (\psi \circ \varphi^{-1})_j}{\partial x_i} dx_i = \sum_{i=1}^{m} a_{ji}(x) dx_i$

Theorem The collection

$$\mathcal{A}_{T^*M} = \{ \Phi_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2m} \mid \alpha \in A \}$$

is a smooth atlas on T^*M.

Proof is just a repetition of that for the tangent bundle. $\square$

Def We denote by $\Omega^1(M)$ the space of all smooth 1-forms on M where a smooth 1-form on M is a smooth map
$\alpha: M \to T^*M$ such that $\pi \circ \alpha = Id_M$.

Example If $f \in \mathcal{C}^\infty_{\mathbb{R}}(M)$, then $df \in \Omega^1(M)$ is defined as $df(x)(v) = v(f)$. Here $v \in T_x M = D_x M$ as a derivation at $x \in M$. In local coordinates,

$df(x) = \sum_{i=1}^{m} \left(\frac{\partial}{\partial x_i} f\right)(x)\, dx_i$. This is called the exterior derivative of f

Tensor bundle over M

Put $T^{k,l}_x M := T^{k,l}(T_x M)$ where $x \in M$ and $k, l \in \mathbb{N} \cup \{0\}$. And also put

$$T^{k,l} M = \bigcup_{x \in M} T^{k,l}_x M.$$

As before $T^{k,l} M$ has a canonical structure of smooth manifold of dim $m + m^{k+l}$. Note that there is a canonical projection $\pi : T^{k,l} M \longrightarrow M$ which is defined in an obvious way.

<u>Definition</u> We call $T^{k,l} M$ a tensor bundle of M of type (k,l). We call a smooth map $T : M \longrightarrow T^{k,l} M$ such that $\pi \circ T = Id_M$ a smooth tensor field of type (k,l) on M and we denote by $\mathcal{T}^{k,l}(M)$ the set of all smooth tensor fields of type (k,l) on M.

Alternative definition of smooth tensor fields

<u>Definition</u> A smooth tensor field of type (k,l) on M is a map

$$T : \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{k \text{ copies}} \times \underbrace{\Omega^1(M) \times \dots \times \Omega^1(M)}_{l \text{ copies}} \longrightarrow \mathcal{C}^\infty_{\mathbb{R}}(M)$$

such that

$$\text{(LOF)} \quad T(\mathbb{X}_1, \dots, f\mathbb{X}_i, \dots, \mathbb{X}_k, \alpha_1, \dots, \alpha_l)(x) = f(x)\, T(\mathbb{X}_1, \dots, \mathbb{X}_k, \alpha_1, \dots, \alpha_l)(x)$$
$$= T(\mathbb{X}_1, \dots, \mathbb{X}_k, \alpha_1, \dots, f\alpha_j, \dots, \alpha_l)(x)$$

where $\mathbb{X}_1, \dots, \mathbb{X}_k \in \mathfrak{X}(M)$, $\alpha_1, \dots, \alpha_l \in \Omega^1(M)$, $f \in \mathcal{C}^\infty_{\mathbb{R}}(M)$, $x \in M$ and $i = 1, \dots, k$ and $j = 1, \dots, l$.

((LOF) T is linear w.r. to multiplication by smooth function on M).

Operations on tensor fields

<u>Lemma</u> 1) $\mathcal{T}^{k,l}(M)$ is a vector space with operations

$$(T+S)(x) = T(x) + S(x), \quad (\lambda T)(x) = \lambda T(x)$$

where $T, S \in \mathcal{T}^{k,l}(M)$, $x \in M$, $\lambda \in \mathbb{R}$ (and we are using that $T(x) \in T^{k,l}_x M = T^{k,l}(T_x M)$).

2) If $T \in \mathcal{T}^{k,0}(M)$ and $S \in \mathcal{T}^{l,0}(M)$, then

$$T \otimes S, \; T \odot S, \; T \wedge S \in \mathcal{T}^{k+l,0}(M) \quad \text{where}$$

$$(T \otimes S)(x) = T(x) \otimes S(x), \; (T \odot S)(x) = T(x) \odot S(x) \text{ and}$$

$$(T \wedge S)(x) = T(x) \wedge S(x).$$