Riemannian metric

Definition A Riemannian metric (tensor) on a smooth manifold $(M, \mathcal{A})$ is a smooth tensor field $g \in \mathcal{T}^{(2,0)}(M)$ such that for every $x \in M$ and $u, v \in T_x M$

(RM1) $g_x : T_x M \times T_x M \to \mathbb{R}$ is symmetric, that is $g_x(u,v) = g_x(v,u)$ and

(RM2) g_x is positive definite, that is $g_x(v,v) \geq 0$ and $g_x(v,v) = 0 \Leftrightarrow v = 0$.

g is also sometimes called a Riemannian structure on M.

Definition A Riemannian manifold is a pair (M, g) where M is a smooth manifold (with its atlas) and g is a Riemannian metric on M.

Notation We will write only M instead of $(M, \mathcal{A})$ if we do not need atlas on it explicitly.

Local description of Riemannian metric

Let $\varphi : U \to \mathbb{R}^m$ be a chart on M with coordinate functions $x_1, \dots, x_m$. Then we know that

$$(B_x^\varphi)^* = \{dx_1, \dots, dx_m\}$$

is a basis of $T_x^* M$ at every $x \in U$. In these coordinates, a Riemannian metric is given by

$$g(x) = \sum_{i,j=1}^{m} g_{ij}(x)\, dx_i \otimes dx_j$$

where $g_{ij}(x)$ are smooth functions of $x_1, \dots, x_m$. The condition (RM1) is then (on U) equivalent to

(RM1)' $\quad g_{ij}(x) = g_{ji}(x), \quad i,j = 1, \dots, m, \ x \in U$

and (RM2) is (on U) equivalent to the fact that the matrix

(RM2)' $\quad (g_{ij}(x))_{i=1,\dots,m}^{j=1,\dots,m} \in M(m, \mathbb{R})$

is for every x positive definite. Note that the matrix in (RM2)' is the matrix of g_x w. r. to the basis of $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$ of $T_x M$.

Theorem Let M be a smooth manifold, (N, g) be a Riemannian manifold and $\Phi : M \to N$ be an immersion, that is, Φ is smooth and for every $x \in M$, $T_x\Phi : T_x M \to T_{\Phi(x)} N$ is injective. Then $\Phi^* g$ is a Riemannian metric on M.

Proof: We know that $\Phi^* g \in \mathcal{T}^{(2,0)} M$. For $u, v \in T_x M$ and $x \in M$ we have $(\Phi^* g)_x(u,v) = g_{\Phi(x)}(T_x\Phi(u), T_x\Phi(v)) =$

$= g_{\Phi(x)}(T_x\Phi(v), T_x\Phi(u)) = (\Phi^* g)_x(v,u)$ and so (RM1) holds.
Moreover, $(\Phi^* g)_x(u) = g_{\Phi(x)}(T_x\Phi(u), T_x\Phi(u)) \geq 0$ and $= 0$ holds iff $T_x\Phi(u) = 0$. But since $T_x\Phi$ is injection, $T_x\Phi(u) = 0$ iff $u = 0$. Hence (RM2) holds as well. $\square$

Definition Let (M, g_M) and (N, g_N) be Riemannian manifolds. A smooth map $\Phi : M \to N$ is a (local) isometry if $\Phi^* g_N = g_M$.

Definition Let $\gamma : [a,b] \to M$ be a piecewise smooth curve on a Riemannian manifold (M,g). The length of γ is defined as

$$L(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)} \, dt$$

where $\|u\|_x = g_x(u,u)$ is the norm on $T_x M$ induced by g_x.

Remark •) The lenght of a curve does not change under reparametrization, that is if $g : [c,d] \to [a,b]$ is 1-1 and $g' > 0$ on $[c,d]$, then $L(\gamma) = L(\gamma \circ g)$.

•) Assume that (M,g) is a Riemannian manifold. If $x, x' \in M$ we define their "distance" by

$$d(x,x') = \inf \{ L(\gamma) \mid \gamma : [a,b] \to M \text{ is piecewise smooth}, \ \gamma(a) = x, \gamma(b) = x' \}.$$

It can be verified that (M, d) is a metric space (if we allow that d can attain $+\infty$ if M is not connected) and that the induced (metric) topology is the topology τ_M on M.

Examples 1) Standard Riemannian (or Euclidean) structure on $\mathbb{R}^m$.
The standard Riemannian (or Euclidean) metric on $\mathbb{R}^m$ is (in the standart coordinates on $\mathbb{R}^m$) given by

$$g(x) \equiv g_{st}(x) = \sum_{i=1}^m dx_i \otimes dx_i .$$

The length of a piecewise smooth curve $\gamma : [a\ b] \to \mathbb{R}^m$ is given by

$$\int_a^b \|\gamma'(t)\| \, dt$$

where $\| \ \|$ is the Euclidean norm on $\mathbb{R}^m$.

Facts •) The induced metric d on $\mathbb{R}^m$ is the Euclidean distance as the shortest curve connecting two points $x, x' \in \mathbb{R}^m$ is the line segment with endpoints x and x' (this can be proved by calculus of variations).

•) Any local isometry $\mathbb{R}^m \to \mathbb{R}^m$ is of the form
$$\mathbb{R}^m \ni x \longmapsto Ax + b$$
where $A \in O(m)$ is an orthogonal transformation of $\mathbb{R}^m$ and $b \in \mathbb{R}^m$

is fixed. (Try if you can prove this claim.)

2) Standard Riemannian structure on $S^m \subseteq \mathbb{R}^{m+1}$.
Let S^m be the unit sphere in $\mathbb{R}^{m+1}$ with its standard structure (from week 2). Let

$$\iota : S^m \hookrightarrow \mathbb{R}^{m+1}$$

be the canonical inclusion. As the inverses to the stereographic projections φ_N and φ_N from the north and south poles are immersions

$$\mathbb{R}^m \longrightarrow \mathbb{R}^{m+1},$$

it follows that ι is smooth and

$$T_x\iota : T_x S^m \longrightarrow T_x \mathbb{R}^{m+1} \simeq \mathbb{R}^{m+1}$$

is for any fixed $x \in S^m$ injective. Note that $T_x\iota$ realizes $T_x S^m$ as an m-dimensional subspace of $\mathbb{R}^{m+1}$. Since

$$f : \mathbb{R}^{m+1} \longrightarrow \mathbb{R},\ f(x_1,\dots,x_{m+1}) = \|x\|^2$$

restricts to S^m as the constant function 1, we have that

$$0 = \frac{d}{dt} f(\gamma(t))\Big|_{t=0} = \sum_{i=1}^{m+1} \frac{\partial f}{\partial x_i}(x)\cdot \gamma_i'(0) = 2\sum_{i=1}^{m+1} x_i \gamma_i'(0)$$

for any smooth curve $\gamma : (-\varepsilon,\varepsilon) \longrightarrow S^m$, $\gamma(0) = x = (x_1,\dots,x_{m+1})$, $\varepsilon > 0$.
This shows that $T_x S^m$ is the orthogonal complement to $x \in S^m$ in $\mathbb{R}^{m+1}$.

As ι is immersion, $\iota^* g_{st}$ is the standard (or round) Riemannian metric on S^m, denoted by $g = g_{S^m}$. In the chart

$$\varphi_N : \mathbb{R}^m \longrightarrow S^m,\ \varphi_N(x) = \frac{1}{1-x_{m+1}}(x_1,\dots,x_m),\ \text{with inverse}$$

$$\varphi_N^{-1}(x_1,\dots,x_m) = \Big(2x_1,\dots,2x_m, \sum_{i=1}^m x_i^2 - 1\Big) \Big/ \Big(1+\sum_{i=1}^m x_i^2\Big),$$

the round metric on S^m takes the form
(here we denote for a moment standard coordinates on $\mathbb{R}^{m+1}$ by $u_1,\dots,u_{m+1}$)

$$(\varphi_N^{-1})^*\Big(\sum_{i=1}^{m+1} du_i \otimes du_i\Big) = \sum_{i=1}^{m+1} (\varphi^{-1})^* du_i \otimes (\varphi^{-1})^* du_i$$

$$= \sum_{i=1}^m d\Big(\frac{2x_i}{1+\sum_{j=1}^m x_j^2}\Big) \otimes d\Big(\frac{2x_i}{1+\sum_{j=1}^m x_j^2}\Big) + d\Big(\frac{\sum_1^m x_j^2 - 1}{1+\sum_{j=1}^m x_j^2}\Big) \otimes d\Big(\frac{\sum_1^m x_j^2 - 1}{1+\sum_1^m x_j^2}\Big) = (*)$$

$$d\Big(\frac{2x_i}{1+\sum_1^m x_j^2}\Big) = \frac{2\,dx_i}{1+\sum_1^m x_j^2} - 2\sum_{j=1}^m \frac{2x_i x_j\,dx_j}{(1+\sum_1^m x_l^2)^2} = \frac{2\,dx_i}{\theta} - \frac{2x_i\,d\theta}{\theta^2}$$

$$d\Big(\frac{\sum_1^m x_j^2 - 1}{1+\sum_1^m x_j^2}\Big) = d\Big(\frac{\sum_1^m x_j^2 + 1 - 2}{1+\sum_1^m x_j^2}\Big) = d\Big(1 - \frac{2}{1+\sum_1^m x_j^2}\Big) = 2\sum_{j=1}^m \frac{2x_j\,dx_j}{(1+\sum_1^m x_l^2)^2}$$

$$= 2\frac{d\theta}{\theta^2} \quad \text{where } \theta = 1+\sum_{i=1}^m x_i^2 \text{ and } d\theta = \sum_{i=1}^m 2x_i\,dx_i$$

$$(*) = 4\sum_{i=1}^m \frac{dx_i\otimes dx_i}{\theta^2} - 4\sum_{i=1}^m \frac{x_i\,dx_i\otimes d\theta + x_i\,d\theta\otimes dx_i}{\theta^3} + \sum_{i=1}^m \frac{4x_i^2\,d\theta\otimes d\theta}{\theta^4} + 4\frac{d\theta\otimes d\theta}{\theta^2}$$

$$= 4 \sum_{i=1}^{m} \frac{dx_i \otimes dx_i}{\theta^2} - 4 \frac{d\theta \otimes d\theta}{\theta^3} + 4 \frac{(\theta - 1)\, d\theta \otimes d\theta}{\theta^4} + 4 \frac{d\theta \otimes d\theta}{\theta^4}$$

$$= 4 \sum_{i=1}^{m} \frac{dx_i \otimes dx_i}{\theta^2} = \frac{4}{1 + \sum_{i=1}^{m} x_i^2} \sum_{j=1}^{m} dx_j \otimes dx_j = \frac{4}{1 + \sum_{i=1}^{m} x_i^2}\, g_{st} .$$

Note that g_{S^m} is (in the chart φ_N) a scalar multiple of the standard Riemannian metric on $\mathbb{R}^m$?

<u>Question:</u> How do shortest paths on S^m look like ?