Tensor fields

Let $(M, \mathcal{A})$ be a smooth manifold. For any $x \in M$ we know that T_xM is a vector space and so $T^{k,\ell}_x M = T^{k,\ell}(T_xM)$ is the associated vector space of tensors of type (k,ℓ) on T_xM. We defined

$$T^{k,\ell}M = \bigcup_{x \in M} T^{k,\ell}_x M.$$

We observed that $T^{k,\ell}M$ is a smooth manifold and the canonical projection $\pi: T^{k,\ell}M \to M$ is smooth.

We defined a (smooth) tensor field of type (k,ℓ) on M as a smooth function

$$T: M \to T^{k,\ell}M$$

such that $\pi \circ T = \mathrm{Id}_M$. In other words, for every $x \in M$: $T(x) \in T^{k,\ell}_x M$ and coefficient functions of T depend smoothly on x. It follows that the set $\mathcal{T}^{k,\ell}(M)$ of all smooth tensor fields on M is a vector space.

We also defined $T \otimes S$, $T \odot S$, $T \wedge S$ for $T \in \mathcal{T}^{k,0}(M)$ and $\mathcal{T}^{\ell,0}(M)$.

Pullback of a tensor field

__Definition__ Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be smooth manifolds and $\Phi: M \to N$ be smooth. If $T \in \mathcal{T}^{k,0}(N)$, where $k \geq 0$, then

$$\Phi^* T(x)(v_1, \dots, v_k) = T_{\Phi(x)}(T_x\Phi(v_1), \dots, T_x\Phi(v_k)) \in \mathcal{T}^{k,0}(M)$$

where $x \in M$, $v_1, \dots, v_k \in T_xM$, is called the pullback of T along Φ.

Let us verify that Φ^*T is indeed a smooth vector field on M. For that we have to verify two things:

1) $\Phi^*T(x): \underbrace{T_xM \times \dots \times T_xM}_{k \text{ copies}} \to \mathbb{R}$

is multilinear. But this easily follows from the fact that $T_{\Phi(x)}$ is multilinear and $T_x\Phi$ is linear.

2) The coefficients of Φ^*T depend smoothly on coordinates on M. Let us fix some chart $\varphi : U \to \mathbb{R}^m$ around x with coordinate functions $x_1, \dots, x_m$ and let $\psi : V \to \mathbb{R}^n$ is a chart around $\Phi(x)$ with coordinate functions $y_1, \dots, y_n$. Then on V we have

$$T(y) = \sum_{i_1, \dots, i_k = 1}^{n} \underbrace{T_{i_1 \dots i_k}(y)}_{} \, dy_{i_1} \otimes \dots \otimes dy_{i_k}$$

coefficient function of T in the chart ψ.

Then on $\Phi^{-1}(V) \cap U$ we have

$$\Phi^* T(x) = \sum_{i_1, \dots, i_k = 1}^{n} T_{i_1 \dots i_k}(\Phi(x)) \, \Phi^* dy_{i_1} \otimes \dots \otimes \Phi^* dy_{i_k} = (*)$$

$$\Phi^* dy_i = d(\Phi_i) = \sum_{j=1}^{m} \frac{\partial \Phi_i}{\partial x_j} dx_j$$

$\Phi = (\Phi_1, \dots, \Phi_n)$, where $\Phi_i = (\psi \circ \Phi \circ \varphi^{-1})_i$ the i-th component of a smooth map from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^n$

$$\begin{tikzcd} & U \arrow[r, "\Phi"] \arrow[d, shift right, "\varphi"'] & V \arrow[d, "\psi"] \\ \mathbb{R}^m \supseteq & \varphi(U) \arrow[r] \arrow[u, shift right, "\varphi^{-1}"'] & \psi(V) \subseteq \mathbb{R}^n \end{tikzcd}$$

$$(*) = \sum_{i_1 \dots i_k = 1}^{n} T_{i_1 \dots i_k}(\Phi(x)) \left[\sum_{l_1 = 1}^{m} \frac{\partial \Phi_{i_1}}{\partial x_{l_1}} dx_{l_1} \right] \otimes \dots \otimes \left[\sum_{l_k = 1}^{m} \frac{\partial \Phi_{i_k}}{\partial x_{l_k}} dx_{l_k} \right]$$

$$= \sum_{i_1 \dots i_k = 1}^{n} \sum_{l_1 \dots l_k = 1}^{m} \underbrace{T_{i_1 \dots i_k}(\Phi(x)) \frac{\partial \Phi_{i_1}}{\partial x_{l_1}} \cdots \frac{\partial \Phi_{i_k}}{\partial x_{l_k}}}_{\text{are smooth}} dx_{l_1} \otimes \dots \otimes dx_{l_k}.$$

✓

<u>Lemma</u> Let $\Phi : M \to N$ be smooth, then the map

- $\Phi^* : \mathcal{T}^{k,0}(N) \to \mathcal{T}^{k,0}(M)$ is linear, that $\Phi^*(T+S) = \Phi^*T + \Phi^*S$, $\Phi^*(\lambda T) = \lambda \Phi^* T$ and
- $\Phi^*(T \otimes S) = (\Phi^*T) \otimes (\Phi^* S)$ where $T \in \mathcal{T}^{k,0}(M)$, $S \in \mathcal{T}^{l,0}(M)$ (and similarly for $\odot$ and $\wedge$)

Examples •) $f:\mathbb{R}^2\to\mathbb{R}^2$, $f(x,y)=(\underbrace{x^2y}_{f_1},\underbrace{y^2+x}_{f_2})$

$$\theta = dx + y\,dy \in \Omega^1(\mathbb{R}^2) = \mathcal{T}^{1,0}(\mathbb{R}^2$$

$$f^*\theta = f^*(dx+ydy) = f^*(dx) + f^*(y\,dy)$$

$$= d(x^2y) + f^*(y)\cdot f^*(dy) = \circledast$$

$$\left(\text{if } g:\mathbb{R}^2\to\mathbb{R} \text{ is smooth, then } dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right)$$

$$\circledast = \frac{\partial(x^2y)}{\partial x}dx + \frac{\partial(x^2y)}{\partial y}dy + (y^2+x)\,d(y^2+x)$$

$$= \overbrace{2xy\,dx + x^2\,dy}^{f^*dx} + (y^2+x)(\overbrace{dx + 2y\,dy}^{f^*(dy)})$$

$$= (2xy + y^2 + x)\,dx + (x^2 + 2y(y^2+x))\,dy$$

$$= (2xy + y^2 + x)\,dx + (x^2 + 2y^3 + 2yx)\,dy.$$

•) $f:\mathbb{R}^2\to\mathbb{R}^2$, $f(x,y)=(x^2y, y^2+x)$

Lebesgue measure on $\mathbb{R}^2$ (can be viewed as) is equal to the tensorfield $dx\wedge dy$

(here we will need that $dx\wedge dx = 0 = dy\wedge dy$
$dx\wedge dy = -dy\wedge dx$)

also called the exterior product or wedge

$$f^*(dx\wedge dy) = (f^*dx)\wedge(f^*dy) = (2xy\,dx + x^2dy)\wedge(dx+2ydy)$$

$$= \cancel{2xy\,dx\wedge dx} + 2xy\,2y\,dx\wedge dy + x^2\,dy\wedge dx + \cancel{x^2\,2y\,dy\wedge dy}$$

$$= (2xy\,2y - x^2)\,dx\wedge dy = \underbrace{(4xy^2 - x^2)}_{}\,dx\wedge dy$$

$$|\operatorname{Jac} f| = 4xy^2 - x^2$$

$$\operatorname{Jac} f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix} = \begin{pmatrix} 2xy & x^2 \\ 1 & 2y\end{pmatrix}$$

Lie derivative of tensor fields

Let $X\in\mathfrak{X}(M)$, recall the flow Fl_t^X at time t is a smooth map defined on an open subset of M

and that $Fl^{X} : I(X) \longrightarrow M$ is smooth where $I(X)$ is an open neighbourhood of $\{0\} \times M$ inside $\mathbb{R} \times M$. Also recall

$$Fl^{X}_{t+s} = Fl^{X}_{t} \circ Fl^{X}_{s}$$

as in Theorem from week 4.

<u>Definition</u> Let $X \in \mathfrak{X}(M)$ and $T \in \mathcal{T}^{k,0}(M)$. Then the Lie derivative of T along X is defined as

$$\mathcal{L}_{X} T = \frac{d}{dt}\left((Fl^{X}_{t})^{*} T \right)\Big|_{t=0} \in \mathcal{T}^{k,0}(M).$$

<u>Theorem</u> Let $X \in \mathfrak{X}(M)$ and $T \in \mathcal{T}^{k,0}(M)$, $S \in \mathcal{T}^{l,0}(M)$. Then:

1) $\mathcal{L}_{X} : \mathcal{T}^{k,0}(M) \longrightarrow \mathcal{T}^{k,0}(M)$ is linear.

2) $\mathcal{L}_{X}(T \otimes S) = (\mathcal{L}_{X} T) \otimes S + T \otimes (\mathcal{L}_{X} S)$

(Leibniz property of Lie derivative, similarly $\odot$ and $\wedge$)

3) If $\theta \in \Omega^{1}(M)$ and $Y \in \mathfrak{X}(M)$, then

$$\underbrace{(\underset{\in \Omega^{1}(M)}{\mathcal{L}_{X}\theta})(Y)}_{\in C^{\infty}(M)} = X(\underbrace{\theta(Y)}_{\in C^{\infty}(M)}) - \underbrace{\theta(\overbrace{\mathcal{L}_{X} Y}^{\in \mathfrak{X}(M)})}_{\in C^{\infty}(M)}.$$

4) If $\varphi : U \to \mathbb{R}^{m}$ is a chart on M with coordinate functions $x_1, \dots, x_m$, and if

$$X = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i} \quad \text{and} \quad \theta = \sum_{i=1}^{m} \alpha_i(x)\, dx_i \quad \text{on } U,$$

then

$$(\mathcal{L}_{X}\theta)(x) = \sum_{j=1}^{m} \left(\sum_{i=1}^{m} \left(a_i \frac{\partial}{\partial x_i} \alpha_j + \alpha_i \frac{\partial}{\partial x_j} a_i \right) \right)(x)\, dx_j.$$

Proof: 3) Since the left and the right hand sides define a smooth 1-forms on M, it is enough to verify that these 1-forms agree in any chart. Let $\varphi: U \to \mathbb{R}^m$ be as in statement of 4). Let us assume that

$$Y = \sum_{i=1}^{m} b_i(x) \frac{\partial}{\partial x_i} \quad \text{where } b_i(x) \text{ are smooth.}$$

Let us write $\Phi_t = Fl^X_t$ and let

$\Phi_t = ((\Phi_t)_1, \dots, (\Phi_t)_m)$ so that

$(\Phi_t)_i$ is the i-th component of Φ_t in the coordinates $x_1, \dots, x_m$. Let us fix $x \in U$.

| Notation: we sometimes write θ_x instead $\theta(x) \in T^*_x M$. |

$$(\mathcal{L}_X \theta)(Y)(x) = \frac{d}{dt} (\Phi_t^* \theta)_x (Y(x)) \Big|_{t=0}$$

$$= \frac{d}{dt} \left(\theta_{\Phi_t(x)} (T_x \Phi_t (Y(x))) \right) \Big|_{t=0}$$

$$= \frac{d}{dt} \left(\sum_{i=1}^{m} (\alpha_i(\Phi_t(x)) dx_i) \underbrace{\left(\sum_{j=1}^{m} \sum_{l=1}^{m} \frac{\partial (\Phi_t)_j}{\partial x_l}(x) \left(b_l(x) \frac{\partial}{\partial x_j}\right)\right)}_{\delta_{ij}} \right) \Big|_{t=0}$$

$$= \frac{d}{dt} \left(\sum_{i=1}^{m} \alpha_i(\Phi_t(x)) \sum_{l=1}^{m} \left(\frac{\partial (\Phi_t)_i}{\partial x_l} b_l \right)(x) \right) \Big|_{t=0}$$

$$\frac{d}{dt} \frac{\partial (\Phi_t)_i}{\partial x_l} = \frac{\partial}{\partial x_l} \frac{d}{dt} (\Phi_t)_i = \frac{\partial a_i}{\partial x_l}$$

$$= \left(\sum_{i=1}^{m} \sum_{j=1}^{m} a_j(\Phi_t(x)) \frac{\partial \alpha_i}{\partial x_j} \sum_{l=1}^{m} \underbrace{\left(\frac{\partial (\Phi_t)_i}{\partial x_l} b_l \right)}_{\delta_{il}} (x) \right.$$

$$\left. + \sum_{i=1}^{m} \alpha_i(\Phi_t(x)) \sum_{l=1}^{m} \frac{\partial a_i}{\partial x_l} b_l(x) \right) \Big|_{t=0}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \left(a_j(x) \frac{\partial \alpha_i}{\partial x_j}(x)\, b_i(x) + \alpha_i(x) \frac{\partial a_i}{\partial x_j} b_j \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \left(a_j(x) \frac{\partial \alpha_i}{\partial x_j}(x)\, b_i(x) + \alpha_i(x) \frac{\partial a_i}{\partial x_j} b_j \right)$$

$$= \sum_{i=1}^{m} \Big(\underbrace{\sum_{j=1}^{m} \Big(a_j(x) \frac{\partial \alpha_i}{\partial x_j}(x) + \alpha_j(x) \frac{\partial a_j}{\partial x_i}(x) \Big)}_{B_i(x)} \Big) b_i$$

$$\Rightarrow \mathcal{L}_X \theta = \sum_{i=1}^{m} B_i(x)\, dx_i$$

This is the proof of the 4) point, the point 3 is then just rewriting this formula using the formula $\mathcal{L}_X Y$. $\square$

Introduction to Riemannian geometry

<u>Definition</u> A Riemannian metric (tensor field) on $(M, \mathcal{A})$ is $g \in \mathcal{T}^{2,0}(M)$ such that for every $x \in M$:

(RM1)
$$g_x : T_xM \times T_xM \longrightarrow \mathbb{R}$$
is symmetric, i.e. $g_x(u,v) = g_x(v,u)$ where $u, v \in T_xM$, and

(RM2) g_x is positive definite, that is,
$\|u\|_x^2 = g_x(u,u) \geq 0$ and $=0$ holds iff $u = 0$.

Tensor field g is also called a Riemannian structure on M.

<u>Definition</u> A Riemannian manifold is a pair (M, g) where M is a smooth manifold and g is a Riemannian metric on M.

Local description of Riem. tensor

Let $\varphi : U \to \mathbb{R}^m$ be a chart on M with coordinate function $x_1, \dots, x_m$. Then we know that

$B^\varphi_x = \{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \}$ is a basis of $T_x M$ and $(B^\varphi_x)^* = \{ dx_1, \dots, dx_m \}$ is the dual basis. Then we may write

$$g(x) = \sum_{i,j=1}^{m} \underbrace{g_{ij}(x)}_{\text{smooth functions of } x_1, \dots, x_m} dx_i \otimes dx_j .$$

(RM1) is (on $\mathcal{U}$) equivalent to

(RM1)' $\quad g_{ij}(x) = g_{ji}(x)$ for every x

(RM2) is (on $\mathcal{U}$) equivalent to the fact that

(RM2)' $\quad \left(g_{ij}(x) \right)_{i=1,\dots,m}^{j=1,\dots,m} \in M_{m \times m}(\mathbb{R}) = M(m, \mathbb{R})$

is positive definite. Note that the matrix in (RM2)' is the matrix of $g(x)$ w.r. to the basis B^φ_x.

<u>Theorem</u> Let $\Phi : M \to N$ be a smooth map of manifolds and assume that g is a Riemannian metric on N. If Φ is an immersion, that is, for every $x \in M$ the map $T_x \Phi : T_x M \to T_{\Phi(x)} N$ is linear, then $\Phi^* g$ is a Riemannian metric on M.

Proof: ad (RM1) Let $u, v \in T_x M$ at $x \in M$. Then we have that

$$\Phi^* g_x (u, v) = g_{\Phi(x)} (T_x \Phi(u), T_x \Phi(v)) =$$
$$= g_{\Phi(x)} (T_x \Phi(v), T_x \Phi(u)) = \Phi^* g_x (v, u).$$

ad (RM2) $\quad \Phi^* g_x (u, u) = g_{\Phi(x)} (T_x \Phi(u), T_x \Phi(u)) \geq 0$

and $= 0$ iff $T_x \Phi(u) = 0$ iff $u = 0$. $\square$

<u>Definition</u> Let (M, g_M) and (N, g_N) be two Riemannian manifolds. Then a smooth $\Phi: M \to N$ is called a (local) isometry if $\Phi^* g_N = g_M$.

<u>Definition</u> Let $\gamma: [a,b] \to M$ be a piecewise smooth curve on M, then the length of γ (w. r. to a Riemannian metric g on M) is defined as

$$L(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)}\, dt$$

where $\|\gamma'(t)\|_{\gamma(t)} = \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}$.

(γ is piecewise smooth if $\exists$ $a = t_0 < t_1 < \ldots < t_n = b$ such that $\gamma|_{[t_i, t_{i+1}]}$ is smooth (even we can assume just C^1) for every $i = 0, \ldots, n-1$.)

<u>Remark</u> Metric structure on M. If $x, y \in M$, then we can define their distance as

$$d(x,y) = \inf_\gamma \{ L(\gamma) \mid \gamma: [a,b] \to M \text{ is piecewise smooth}, \gamma(a) = x, \gamma(b) = y \}.$$

Then d is a metric on M and (M, d) is a metric space (if we allow that d can attain infite value $+\infty$ if M is connected). The induced metric topology coincides with the given topology on M.

<u>Examples</u> 1) Euclidean structure on $\mathbb{R}^m$

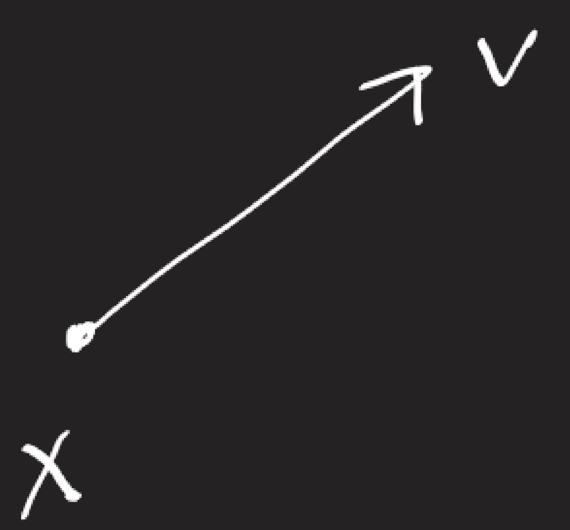
on $\mathbb{R}^m$ with the standard coordinates $x_1, \ldots, x_m$ consider

$$g = g_{st} = \sum_{i=1}^m dx_i \otimes dx_i, \qquad [g]_{B_x^{Id}} = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ \vdots & & \ddots & \\ & & & 1 \end{pmatrix}$$

$B_x^{Id} = \left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \right\}$ the unit matrix

$v = (v_1, \dots, v_m) \in \mathbb{R}^m = T_x \mathbb{R}^m \qquad x \in \mathbb{R}$

$$g_x(v, v) = \sum_{i=1}^{m} v_i^2$$



$v = (v_1, \dots, v_m)$

curve $\gamma : [a, b] \longrightarrow \mathbb{R}^m$

$$L(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)}\, dt = \int_a^b \|\gamma'(t)\|\, dt$$

$$\|\gamma'(t)\|_{\gamma(t)} = \underbrace{\|\gamma'(t)\|}_{\text{Euclidean norm}} = \sqrt{\sum_{i=1}^{m} \gamma_i'(t)^2}$$

the distance of two points x, y in $\mathbb{R}^m$ is equal to the Euclidean distance

$$d(x, y) = \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2}$$

since it can be shown that the shortest path connecting x and y is the line segment with endpoints x and y

<u>Isometries of Euclidean</u> structure on $\mathbb{R}^m$

Any isometry of g_{st} is of the form

$$\mathbb{R}^m \ni x \longmapsto Ax + b$$

where $b \in \mathbb{R}^m$ is fixed and $A \in O(m)$ is an orthonormal transformation

Note that any isometry of $\mathbb{R}^m$ is an affine map (it maps lines to lines)

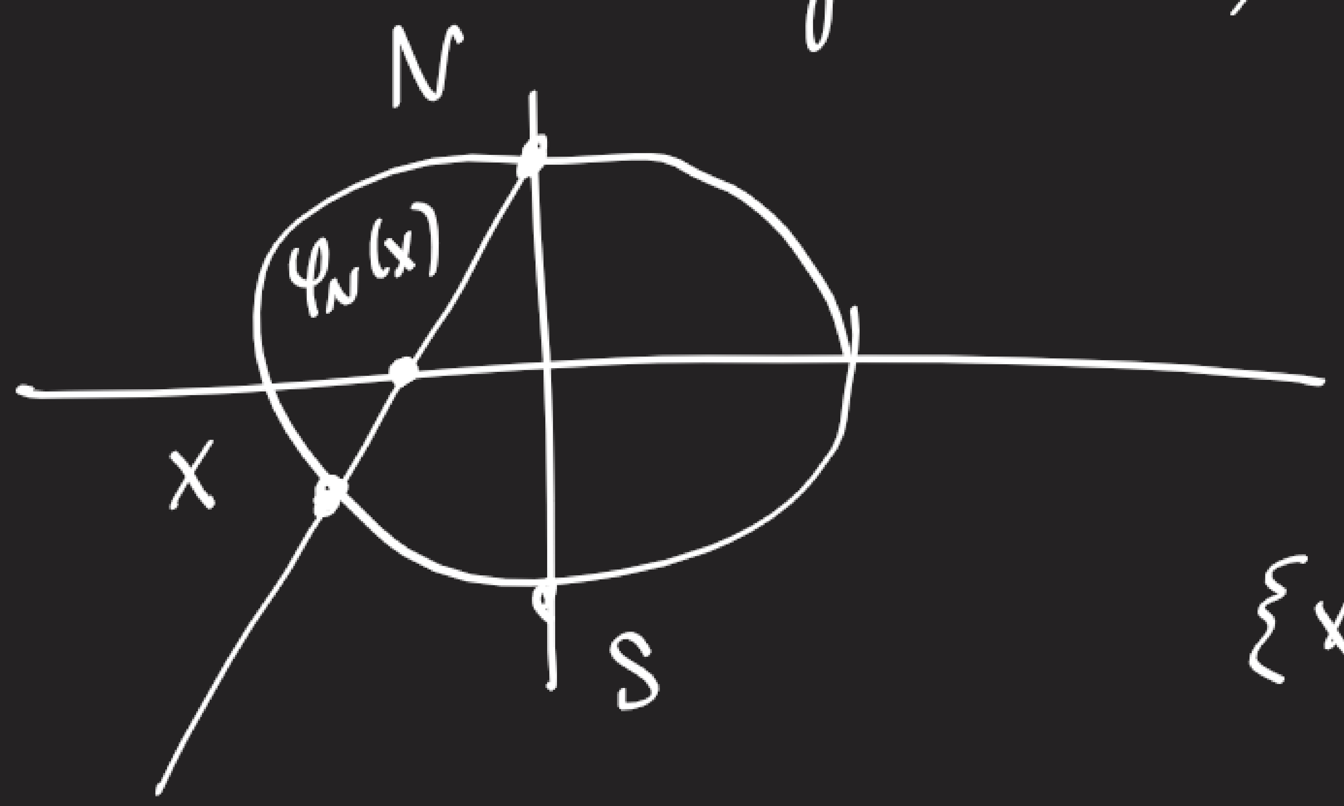
2) Standard (or round) Riemannian metric on S^m

S^m be the unit sphere in $\mathbb{R}^{m+1}$

$$S^m = \{ x \in \mathbb{R}^{m+1} ; \|x\| = 1 \}$$

S^m is a smooth manifold of dim m with atlas

$\mathcal{A} = \{ \varphi_N, \varphi_S$ -- are the stereographic projection from the north pole N and the south pole S $\}$



$\mathbb{R}^m = \{x_{m+1} = 0\}$

$\varphi_N : S^m \setminus \{N\} \to \mathbb{R}^m$

$$\varphi_N(x_1, \dots, x_m) = \frac{1}{1 - x_{m+1}} (x_1, \dots, x_m)$$

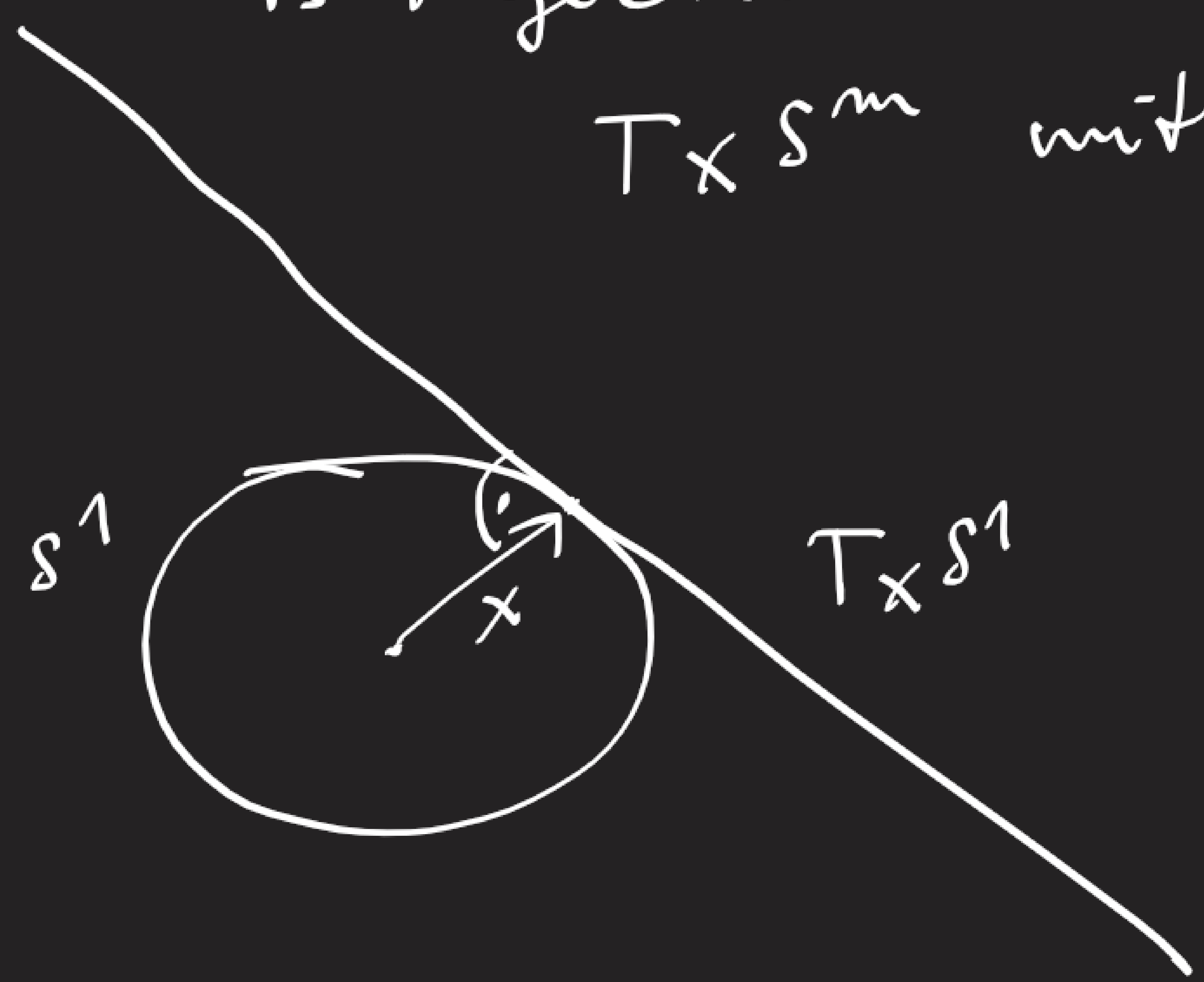
since $\varphi_N^{-1} : \mathbb{R}^m \to S^m \xhookrightarrow{\imath} \mathbb{R}^{m+1}$
($\imath$ is the canonical inclusion) is smooth

$$\varphi_N^{-1}(x) = \left(2x_1, \dots, 2x_m, \sum_{i=1}^m x_i^2 - 1\right) \Big/ \underbrace{\left(1 + \sum_{i=1}^m x_i^2\right)}_{\theta(x)}$$

it follows that

$$\imath : S^m \hookrightarrow \mathbb{R}^{m+1} \quad \text{is smooth}$$

it is clear that $\imath$ is an immersion
(by def. $T_x\imath : T_x S^m \hookrightarrow T_x \mathbb{R}^{m+1} = \mathbb{R}^{m+1}$, $x \in S^m$
is injective and it identifies $T_x S^m$ with $x^\perp$ in $\mathbb{R}^{m+1}$)



hence $\imath^* g_{st}$ is a Riemannian metric on S^m, denote it by g_{S^m}, called the round metric on S^m

how does g_{sm} look like in the chart φ_N

let us for a moment write the coordinates on $\mathbb{R}^{m+1}$ as $u_1, \dots, u_{m+1}$

$$(\varphi_N^{-1})^* g_{sm} = (\varphi_N^{-1})^* \left(\sum_{i=1}^{m+1} du_i \otimes du_i \right)$$

$$\varphi_N^{-1}(x_1, \dots, x_m) = \left(2x_1, \dots, 2x_m, \sum_{i=1}^{m} x_i^2 - 1 \right) \Big/ \overbrace{\left(1 + \sum_{i=1}^{m} x_i^2 \right)}^{\Theta(x)}$$

·) $(\varphi_N^{-1})^* du_i = d\left(\frac{2x_i}{\Theta(x)} \right)$ $\quad i = 1, \dots, m$

··) $(\varphi_N^{-1})^* du_{m+1} = d\left(\frac{\sum_i x_i^2 - 1}{1 + \sum_i x_i^2} \right)$ $\quad d\Theta = 2 \sum_{i=1}^{m} x_i \, dx_i$

ad ·) $d\left(\frac{2x_i}{\Theta(x)} \right) = \frac{2dx_i}{\Theta(x)} - \frac{2x_i \, d\Theta}{\Theta^2(x)}$

ad ··) $d\left(\frac{\Theta(x) - 2}{\Theta(x)} \right) = d\left(1 - \frac{2}{\Theta(x)} \right) = -2 \frac{d\Theta}{\Theta(x)^2}$

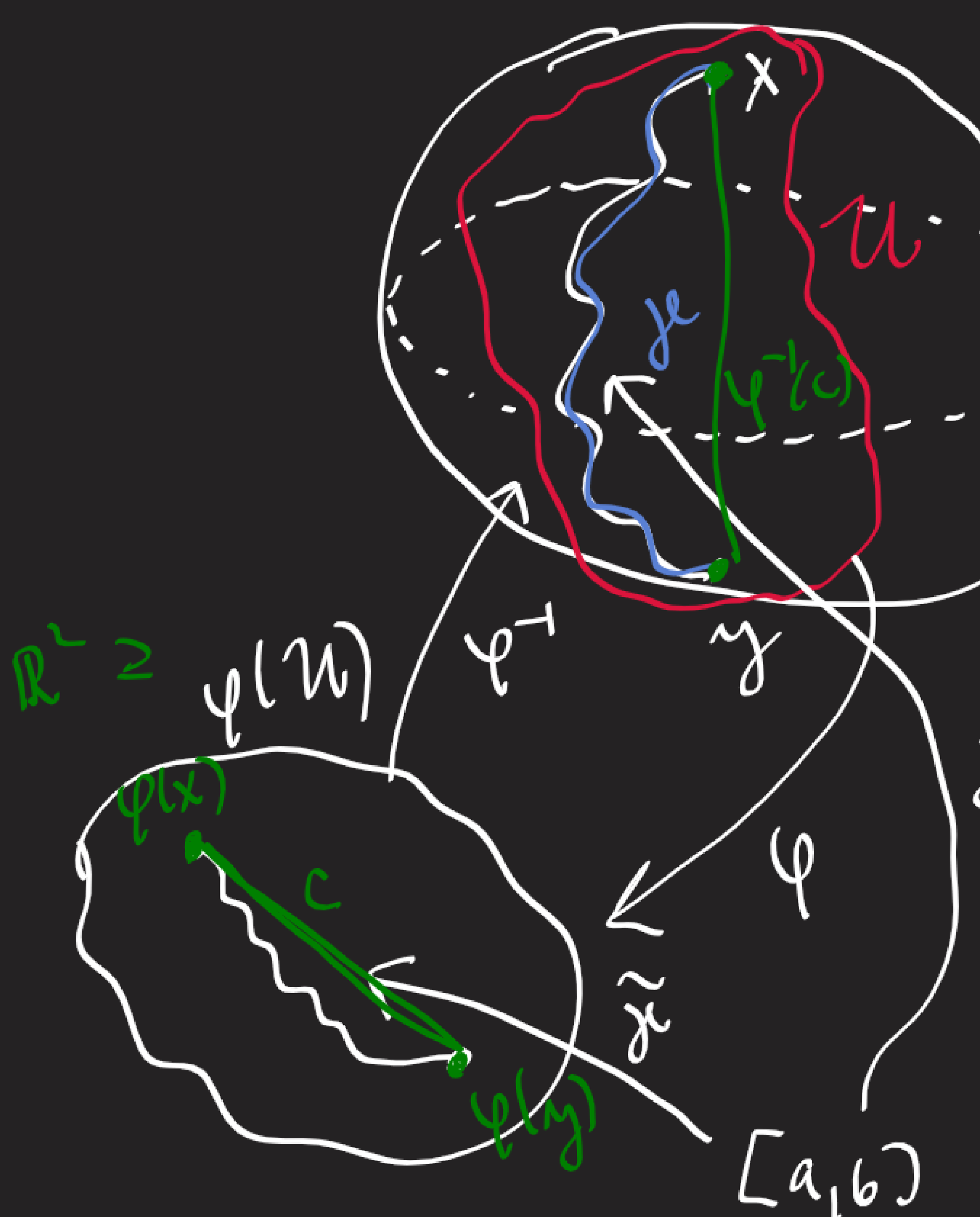
$$(\varphi_N^{-1})^* g_{st} = \sum_{i=1}^{m} \left(\frac{2dx_i \otimes 2dx_i}{\Theta^2(x)} - \frac{2x_i \, d\Theta \otimes 2dx_i + 2dx_i \otimes 2x_i \, d\Theta}{\Theta^3(x)} + \frac{4 x_i \, d\Theta \otimes x_i \, d\Theta}{\Theta^4(x)} \right) + 4 \frac{d\Theta \otimes d\Theta}{\Theta^4(x)}$$

$$= \sum_{i=1}^{m} \frac{4 \, dx_i \otimes dx_i}{\Theta^2(x)} - 4 \cancel{\frac{d\Theta \otimes d\Theta}{\Theta^3(x)}} + \frac{4 (\Theta(x) - 1) \, d\Theta \otimes d\Theta}{\Theta^4(x)} + \cancel{4 \frac{d\Theta \otimes d\Theta}{\Theta^4(x)}}$$

$$= 4 \sum_{i=1}^{m} \frac{dx_i \otimes dx_i}{\Theta^2(x)} = \frac{4}{\Theta^2(x)} g_{st}$$

Euclidean metric on $\mathbb{R}^m$ or the standard Riemannian metric on $\mathbb{R}^m$

Question Find shortest paths on S^m ?



$S^2 \subseteq \mathbb{R}^3$

this would be solved if we can find a chart $\varphi : \mathcal{U} \to \mathbb{R}^2$ such that

$$(A): (\varphi^{-1})^* g_{S^2} = g_{st}\big|_{\varphi^{-1}(\mathcal{U})},$$

then it holds that whenever we are given a curve γ whose image is contained in $\mathcal{U}$, then there is a unique curve $\tilde{\gamma}$ such

$$\gamma = \varphi^{-1} \circ \tilde{\gamma}$$

If (A) holds, then $L(\gamma) = L(\tilde{\gamma})$.

And so the shortest curve connecting x and y is necessarily the φ^{-1} image of the line segment connecting $\varphi(x)$ and $\varphi(y)$

(Gauss 1828, Theorema egregium)

There are no charts on S^m with the property as in A.

Proof is by showing that the curvature $g_{S^m} \neq 0$ while the curvature of $g_{st} = 0$.