Short recapitulation from last lecture

Let M be a smooth manifold. Then a Riemannian metric g on M is a smooth tensor field of type $(2,0)$, i.e. $g \in \mathcal{T}^{2,0}(M)$, such that for every $x \in M$:

$$g_x = g(x) : T_xM \times T_xM \longrightarrow \mathbb{R}$$

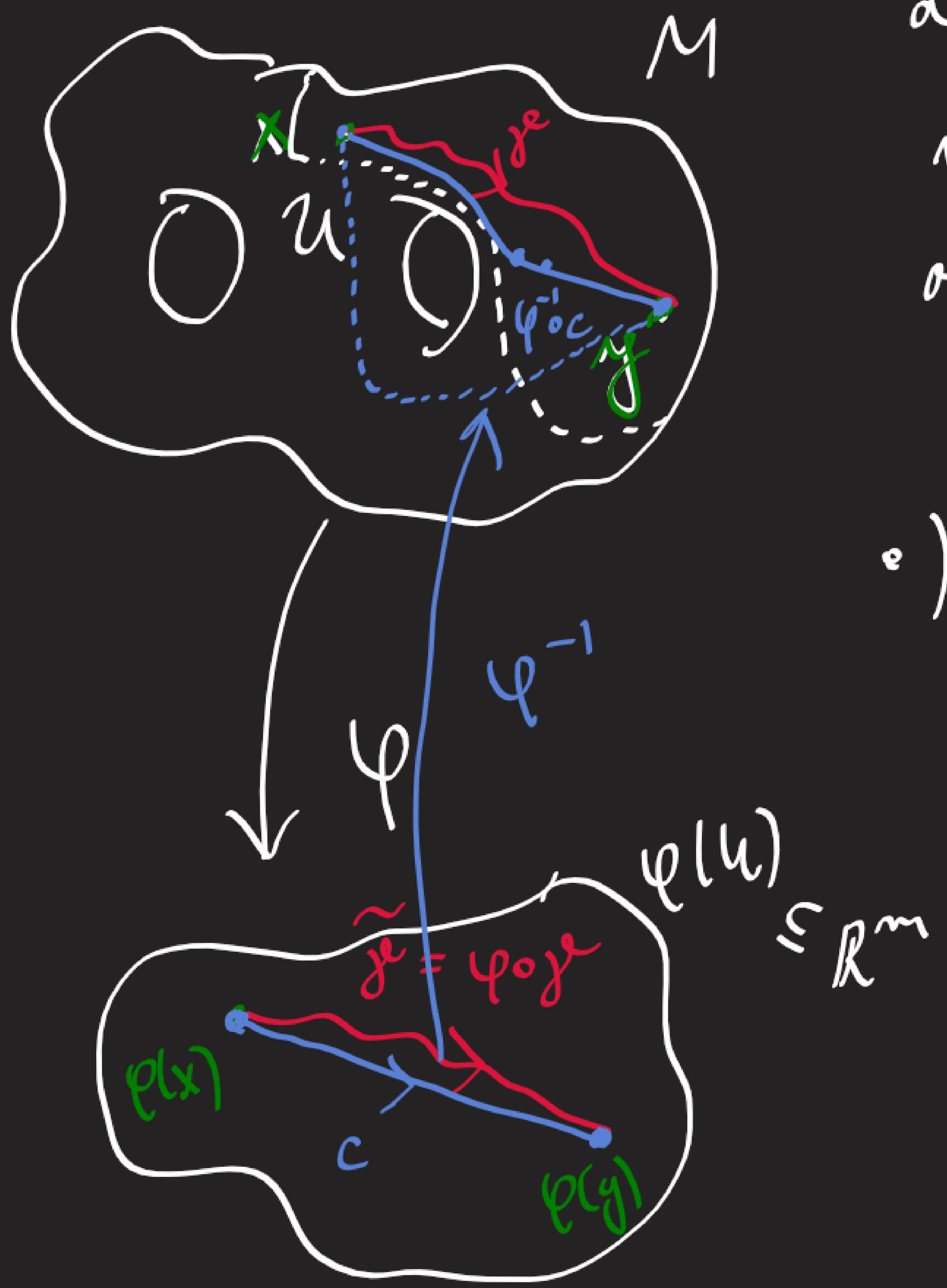
is an inner product on the vector space T_xM.

(M,g) is called a Riemannian manifold. If $\gamma : [a,b] \to M$ is a curve, then

$$L(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)}\, dt,$$

where $\|v\|_x = \sqrt{g_x(v,v)}$ for $x \in M$, $v \in T_xM$, is independent of reparametrization of γ and is defined as the length of γ.

Natural problem: given $x, y \in M$, find a shortest curve on M which connects x and y. This is in general a difficult problem, however, if we are able to find a chart $\varphi : U \to \mathbb{R}^m$ on M such that:



- $x, y \in U$ and
- $\tilde{g} = (\varphi^{-1})^* g$ is the Euclidean metric $\sum_{i=1}^m dx_i \otimes dx_i$ on $\mathbb{R}^m$ (restricted to $\varphi(U)$),

then (at least locally), the distance minimazing curves in U would be the φ images of line segments in $\varphi(U)$.

$$L(\gamma) = L(\tilde{\gamma}) = \int_a^b \|\tilde{\gamma}'(t)\|_{\tilde{\gamma}(t)}\, dt$$

$$\|\tilde{\gamma}'(t)\|_{\tilde{\gamma}(t)} = \sqrt{\tilde{g}_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t), \tilde{\gamma}'(t))}.$$

The existence of charts with the property as in the second point is obstructed by non-vanishing of the curvature of Riemannian metric g. Our next goal is to define this curvature.

Affine connection

Definition An affine connection on manifold M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

such that (we write $\nabla_X Y := \nabla(X, Y)$):

(AF1) $$\nabla_{(f_1 X_1 + f_2 X_2)} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$$

where $f_1, f_2 \in \mathcal{C}^\infty(M)$, $X_1, X_2, Y \in \mathfrak{X}(M)$.

(AF2) $$\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$$

where $X, Y_1, Y_2 \in \mathfrak{X}(M)$.

(AF3) $$\nabla_X (fY) = (Xf)Y + f \nabla_X Y \qquad \text{(Leibniz rule)}$$

where $f \in \mathcal{C}^\infty(M)$, $X, Y \in \mathfrak{X}(M)$.

Local description

Let $\varphi : U \to \mathbb{R}^m$ be a chart on M with coordinate function $x_1, \dots, x_m$ and vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$ on U. Let $X, Y \in \mathfrak{X}(M)$ so that

$$X(x) = \sum_{i=1}^m a_i(x) \frac{\partial}{\partial x_i} \quad \text{and} \quad Y(x) = \sum_{i=1}^m b_i(x) \frac{\partial}{\partial x_i}.$$

(a_i are smooth functions $x_1, \dots, x_m$. the same is true for b_j, $i = 1, \dots, m$, $j = 1, \dots, m$)

$x_i = \pi_i \circ \varphi$

$\pi_i : \mathbb{R}^m \to \mathbb{R}$ projection on the i-th component

$$\nabla_X Y = \nabla_{\sum_{i=1}^m a_i(x) \frac{\partial}{\partial x_i}} \left(\sum_{j=1}^m b_j(x) \frac{\partial}{\partial x_j} \right) =$$

$$\overset{(AF1)}{=} \sum_{i=1}^m a_i(x) \nabla_{\frac{\partial}{\partial x_i}} \left(\sum_{j=1}^m b_j(x) \frac{\partial}{\partial x_j} \right)$$

$$\overset{(AF2)}{=} \sum_{i=1}^m \sum_{j=1}^m a_i(x) \nabla_{\frac{\partial}{\partial x_i}} \left(b_j(x) \frac{\partial}{\partial x_j} \right)$$

$$\overset{(AF3)}{=} \sum_{i=1}^m \sum_{j=1}^m a_i(x) \left(\left(\frac{\partial}{\partial x_i} b_j \right)(x) \frac{\partial}{\partial x_j} + b_j(x) \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right)$$

$$= \sum_{j=1}^m \sum_{i=1}^m a_i(x) \left(\frac{\partial}{\partial x_i} b_j \right)(x) \frac{\partial}{\partial x_j} + \sum_{i,j=1}^m a_i(x) b_j(x) \sum_{k=1}^m \Gamma_{ij}^k(x) \frac{\partial}{\partial x_k}$$

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^m \Gamma_{ij}^k(x) \frac{\partial}{\partial x_k}$$

$\Gamma_{ij}^k(x)$ smooth functions of $x_1, \dots, x_m$

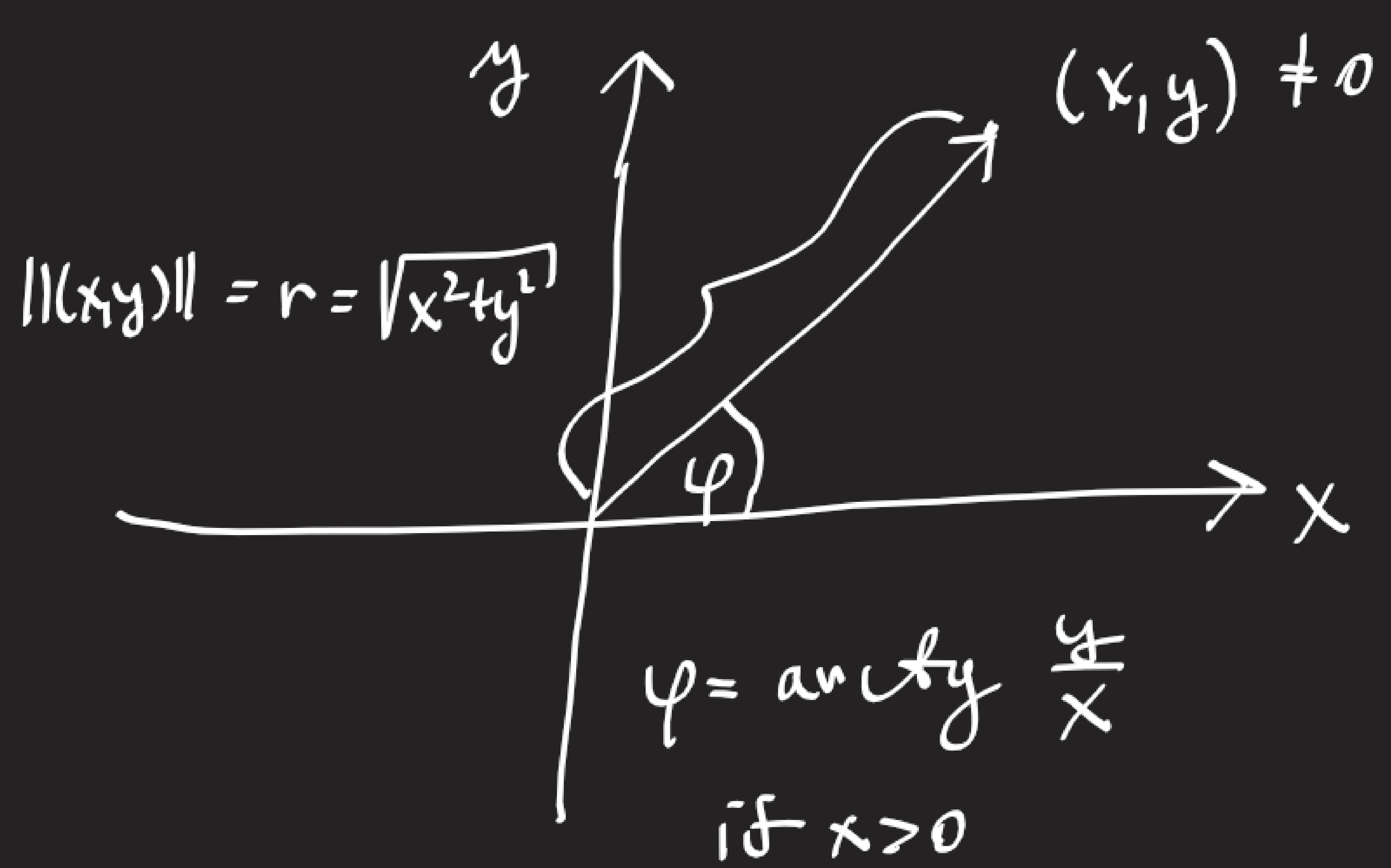
We see that ∇ is on $\mathcal{U}$ completely determined by functions $\Gamma^k_{ij}(x)$, these are so called <u>Christoffel symbols</u> of ∇ in the chart φ. From this computation also follows that $\nabla_X Y|_{\mathcal{U}}$ depends only $X|_{\mathcal{U}}$ and $Y|_{\mathcal{U}}$.

Functions $\Gamma_{ij}^{k}(x)$ do <u>not</u> define a tensor field of type $(2,1)$ on M. Transformation law for $\Gamma^k_{ij}(x)$ (in different charts) is rather complicated and it involves also partial derivatives of coefficients of Jacobi matrices of transition functions.

<u>Example</u> Flat affine connection on $\mathbb{R}^m$

$$\Gamma^k_{ij}(x) = 0 \quad \text{so that} \quad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0 \quad \text{for every } i,j = 1,\dots,m.$$

Let us compute Christoffel symbols for $m=2$ in polar coordinates



$$\Phi : (0,+\infty) \times (-\pi,\pi) \longrightarrow \mathbb{R}^2$$

$$\Phi(r,\varphi) = (\underbrace{r\cos\varphi}_{x}, \underbrace{r\sin\varphi}_{y})$$

Let us compute

$$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = \underline{\Gamma^r_{rr}} \frac{\partial}{\partial r} + \underline{\Gamma^\varphi_{rr}} \frac{\partial}{\partial \varphi}$$

$$\nabla_{\frac{\partial}{\partial \varphi}} \frac{\partial}{\partial r} = \underline{\Gamma^\varphi_{\varphi r}} \frac{\partial}{\partial \varphi} + \underline{\Gamma^r_{\varphi r}} \frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} = \cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \varphi} = -r\sin\varphi \frac{\partial}{\partial x} + r\cos\varphi \frac{\partial}{\partial y}$$

$$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = \nabla_{\cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y}} \left(\cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y} \right)$$

$$= \cos\varphi \, \nabla_{\frac{\partial}{\partial x}} \left(\cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y} \right) + \sin\varphi \, \nabla_{\frac{\partial}{\partial y}} \left(\cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y} \right)$$

$$= \cos\varphi \left(\frac{\partial}{\partial x}\cos\varphi \right) \frac{\partial}{\partial x} + \cos\varphi \left(\frac{\partial}{\partial x}\sin\varphi \right) \frac{\partial}{\partial y} + \sin\varphi \left(\frac{\partial}{\partial y}\cos\varphi \right) \frac{\partial}{\partial x} + \sin\varphi \left(\frac{\partial}{\partial y}\sin\varphi \right) \frac{\partial}{\partial y}$$

$$= \cos\varphi\left(\frac{\partial}{\partial x}\cos\varphi\right)\frac{\partial}{\partial x} + \cos\varphi\left(\frac{\partial}{\partial x}\sin\varphi\right)\frac{\partial}{\partial y} + \sin\varphi\left(\frac{\partial}{\partial y}\cos\varphi\right)\frac{\partial}{\partial x} + \sin\varphi\left(\frac{\partial}{\partial y}\sin\varphi\right)\frac{\partial}{\partial y}$$

$$\varphi = \operatorname{arctg}\frac{y}{x} \quad | \, x>0$$

$$\frac{\partial}{\partial x}\cos\varphi = \frac{\partial}{\partial x}\cos\left(\operatorname{arctg}\frac{y}{x}\right) = -\sin\varphi \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

$$= -\sin\varphi \frac{-y}{x^2+y^2} = \sin\varphi\frac{r\sin\varphi}{r^2} = \frac{\sin^2\varphi}{r}$$

$$\frac{\partial}{\partial y}\cos\varphi = \frac{\partial}{\partial y}\cos\left(\operatorname{arctg}\frac{y}{x}\right) = -\sin\varphi\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = -\sin\varphi\frac{x}{x^2+y^2}$$

$$= -\sin\varphi\frac{r\cos\varphi}{r^2}$$

$$= \cos\varphi\cdot\frac{\cancel{\sin^2\varphi}}{r}\frac{\partial}{\partial x} - \cos\varphi\cos\varphi\frac{r\cancel{\sin\varphi}}{r^2}\frac{\partial}{\partial y}$$

$$+ \cancel{\sin\varphi}\left(-\sin\varphi\frac{\cos\varphi}{r}\right)\frac{\partial}{\partial x} + \cancel{\sin\varphi}\cos\varphi\left(\frac{\cos\varphi}{r^2}\right)\frac{\partial}{\partial y} = 0$$

$$\boxed{\Gamma_{rr}^{r} = \Gamma_{rr}^{\varphi} = 0}$$

$$\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial r} = -\sin\varphi\frac{\partial}{\partial x} + \cos\varphi\frac{\partial}{\partial y} = \frac{1}{r}\frac{\partial}{\partial\varphi}$$

$$\boxed{\Gamma_{\varphi r}^{\varphi} = \frac{1}{r}, \quad \Gamma_{\varphi r}^{r} = 0}$$

HW: Calculate $\Gamma_{r\varphi}^{\varphi}, \Gamma_{r\varphi}^{r}, \Gamma_{\varphi\varphi}^{r}, \Gamma_{\varphi\varphi}^{\varphi}$.

<u>Theorem</u> Let $\mathbb{X}, \mathbb{Y} \in \mathfrak{X}(M)$ and ∇ be an affine connection on M. Then

$(T_xM \ni)\ \nabla_{\mathbb{X}}\mathbb{Y}(x),\ x\in M,$

depends only on $\mathbb{X}(x)$ and on the values of $\mathbb{Y}$ on any curve γ such that $\gamma(0)=x$, $[\gamma] = \mathbb{X}(x) \in T_xM$.

<u>Proof.</u> We know that $\nabla_{\mathbb{Y}}\mathbb{X}(x)$ can be calculated as

$$\sum_{j=1}^{m}\sum_{i=1}^{m} a_i(x)\left(\frac{\partial}{\partial x_i}b_j\right)(x)\frac{\partial}{\partial x_j} + \sum_{i,j=1}^{m} a_i(x)\,b_j(x)\sum_{k=1}^{m}\Gamma_{ij}^{k}(x)\frac{\partial}{\partial x_k}.$$

We need $a_i(x)$, $b_j(x)$ and also $\sum_{i=1}^{m} a_i(x)\left(\frac{\partial}{\partial x_i}b_j\right)(x)$ for every $j=1,\dots,m$.

Note that $\sum_{i=1}^{m} a_i(x)\left(\frac{\partial}{\partial x_i}b_j\right)(x) = \mathbb{X}(x)(b_j)$ where we use that

$X(x) \in T_x M = D_x M$. And

$$X(x)\, b_j(x) = \frac{d}{dt}\left(b_j(\gamma(t))\right)\Big|_{t=0}$$

where γ is any curve on M with $\gamma(0)=x$ whose equivalence class $[\gamma] \in T_x M$ coincides with $X(x)$. $\square$

<u>Definition</u> Let ∇ be an affine connection on M. Let $\gamma : [a,b] \to M$ be a smooth curve and $X : [a,b] \to TM$ be a smooth vector field along γ (defined on the image γ), that is $X(t) \in T_{\gamma(t)} M$. Then

$$\frac{dX}{dt}(t) = \left(\nabla_{\gamma'(t)} X\right)(t) \in T_{\gamma(t)} M$$

is called the covariant derivative of X along γ (at time t).
The vector field X is called parallel if $\frac{dX}{dt}(t) = 0$ on $[a,b]$.
The curve γ is a geodesic for ∇ if γ' is parallel.

<u>Local description</u>

Let $\varphi : U \to \mathbb{R}^m$ be a chart, $X = \sum_{i=1}^m a_i(x) \frac{\partial}{\partial x_i}$ and $\Gamma_{ij}^k(x)$ are the Cristoffel symbols for ∇ in the chart φ. If $\tilde\gamma(t) = (\tilde\gamma_1(t), \dots, \tilde\gamma_m(t))$ where $\tilde\gamma = \varphi \circ \gamma : [a,b] \to \mathbb{R}^m$.
Then X is parallel along γ if

$$\text{(ODE1)}\quad \frac{da_k}{dt}(\tilde\gamma(t)) + \sum_{i,j=1}^m \Gamma_{ij}^k(\tilde\gamma(t))\, \gamma_i'(t)\, a_j(\tilde\gamma(t)) = 0 \quad \text{for every } k=1,\dots,m.$$

Note that this is a system of k first order ODE's for the unknown functions $a_k(\tilde\gamma(t))$, $k=1,\dots,m$. Note that this is a system of linear ODE's, hence the space of solutions of this ODE's is a vector space. By Picard-Lindelöff theorem, we know that given any $t_0 \in [a,b]$ and $(a_1,\dots,a_m) \in \mathbb{R}^m$, then the system (ODE1) has a unique solution (which is defined on $[a,b]$) that satisfies the initial condition

$$a_i(\tilde\gamma(t_0)) = a_i, \quad i=1,\dots,m.$$

The curve γ is a geodesic, in that case $X(\tilde\gamma(t)) = \sum_{i=1}^m \tilde\gamma_i'(t) \frac{\partial}{\partial x_i}$

and so we get

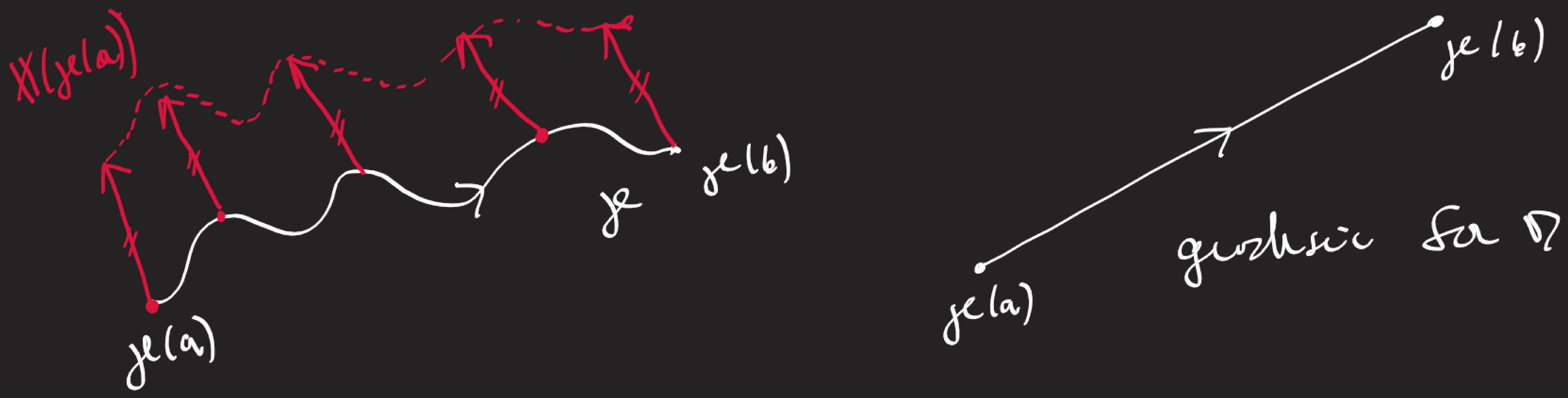
$$\text{(ODE2)} \quad \frac{d\tilde{\gamma}_k''}{dt}(t) + \sum_{i,j=1}^{m} \Gamma_{ij}^{k}(\tilde{\gamma}(t))\, \tilde{\gamma}_i'(t)\, \tilde{\gamma}_j'(t) = 0 \quad , k = 1,\dots,m.$$

Note that this is a system of k second order ODE's for the functions $(\tilde{\gamma}_1, \dots, \tilde{\gamma}_m)$

Example Let ∇ be the flat connection on $\mathbb{R}^m$ with $\Gamma_{ij}^k(x) = 0$. Then a vector field $X = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i}$ is parallel along a curve γ iff $a_i(\gamma(t))$ are constant on the domain of γ for every $i = 1,\dots,m$. We also see that a curve γ is a geodesic for ∇ iff

$$\frac{d\gamma_i''}{dt}(t) = 0 \quad \text{for every } i = 1,\dots,m.$$

This means that γ_i' is constant and this is equivalent to the fact that γ is a parametrization of line segment with constant velocity (w.r. to Euclidean metric).



Tensor fields (revisited)

Let us recall that a tensor field T of type $(k,1)$ on manifold M can be viewed as a multilinear map

$$T: \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{k \text{ copies}} \times \Omega^1(M) \longrightarrow \mathcal{C}^\infty(M)$$

(multilinearity means T is linear in every argument) and which satisfies

$$\text{(LOF)} \quad T(X_1, \dots, fX_i, \dots, \theta) = f\, T(X_1, \dots, X_k, \theta) = T(X_1, \dots, X_k, f\theta)$$

where $X_1, \dots, X_k \in \mathfrak{X}(M)$, $\theta \in \Omega^1(M)$, $f \in \mathcal{C}^\infty(M)$.

Alternatively, $T \in \mathcal{T}^{k,1}(M)$ can be viewed as the following object. Let $x \in M$, $v_1, \dots, v_k \in T_x M$, $\alpha \in T_x^* M$. Then since the map

$$T_x^* M \ni \alpha \longmapsto T_x(v_1, \dots, v_k, \alpha) \in \mathbb{R}$$

is linear, and hence there is a unique $w \in T_x M$ such that

$$\alpha \longmapsto T_x(v_1, \dots, v_k, \alpha) = \alpha(w).$$

Now if $X_1, \dots, X_k$ are smooth vector fields on M and $\theta \in \Omega^1(M)$, then there is a unique $Y \in \mathfrak{X}(M)$ such that

$$\Omega^1(M) \ni \theta \longmapsto T(X_1, \dots, X_k, \theta) = \theta(Y) \in C^\infty(M).$$

It is easy to see that the map

$$(X_1, \dots, X_k) \longmapsto Y$$

is linear in every argument and

$$(X_1, \dots, f X_i, \dots X_k) \longmapsto f Y \text{ where } f \in C^\infty(M).$$

Hence, we can view $T \in \mathcal{T}^{k,1}(M)$ as a map

$$T : \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{k\text{-tuple}} \longrightarrow \mathfrak{X}(M)$$

such that

$(LOF)'$ $\quad T(X_1, \dots, f X_i, \dots, X_k) = f\, T(X_1, \dots, X_k)$ for every $i = 1, \dots, k$, $X_j \in \mathfrak{X}(M)$ for $j = 1, \dots, k$ and $f \in C^\infty(M)$.

Covariant derivative of tensor fields

Theorem Let D be an affine connection on M, $\omega \in \Omega^1(M)$. Then

$$D\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$$

$$D\omega(X, Y) = X(\omega(Y)) - \omega(D_X Y), \quad X, Y \in \mathfrak{X}(M)$$

is a tensor field of type $(2,0)$.

Proof: Linearity of $D\omega$ in both arguments is left as an exercise, that is $D\omega(\alpha_1 X_1 + \alpha_2 X_2, Y) = \alpha_1 D\omega(X_1, Y) + \alpha_2 D\omega(X_2, Y)$ and $D\omega(X, \alpha_1 Y_1 + \alpha_2 Y_2) = \alpha_1 D\omega(X, Y_1) + \alpha_2 D\omega(X, Y_2)$ where

$\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}, \mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y} \in \mathfrak{X}(M), \quad \alpha_1, \alpha_2 \in \mathbb{R}.$

We also have to verify that $D\omega$ is linear over functions, that is, if $f \in \mathcal{C}^\infty(M)$, $\mathbb{X}, \mathbb{Y} \in \mathfrak{X}(M)$, then

$$D\omega(f\mathbb{X}, \mathbb{Y}) \overset{?}{=} f\,(D\omega(\mathbb{X},\mathbb{Y})) = D\omega(\mathbb{X}, f\mathbb{Y}).$$

We have

$$\begin{aligned} D\omega(f\mathbb{X},\mathbb{Y}) &= f\mathbb{X}(\omega(\mathbb{Y})) - \omega(D_{f\mathbb{X}}\mathbb{Y}) \\ &= f\left(\mathbb{X}(\omega(\mathbb{Y})) - \omega(D_{\mathbb{X}}\mathbb{Y})\right) = f\,(D\omega(\mathbb{X},\mathbb{Y})). \end{aligned}$$

$$\begin{aligned} D\omega(\mathbb{X}, f\mathbb{Y}) &= \mathbb{X}(\omega(f\mathbb{Y})) - \omega(D_{\mathbb{X}}(f\mathbb{Y})) \\ &= \mathbb{X}(f\,\omega(\mathbb{Y})) - \omega((\mathbb{X}f)\mathbb{Y} + f D_{\mathbb{X}}\mathbb{Y}) \\ &= \cancel{(\mathbb{X}f)\omega(\mathbb{Y})} + f\,\mathbb{X}(\omega(\mathbb{Y})) - \cancel{(\mathbb{X}f)\omega(\mathbb{Y})} - f\,\omega(D_{\mathbb{X}}\mathbb{Y}) \\ &= f\,(D_\omega(\mathbb{X},\mathbb{Y})). \qquad \square \end{aligned}$$

<u>Definition</u> $D\omega$ is called the covariant derivative of ω by D and we will write $(D_{\mathbb{X}}\omega)(\mathbb{Y}) := (D\omega)(\mathbb{X},\mathbb{Y})$.

<u>Theorem</u> Let $T \in \mathcal{T}^{k,l}(M)$ and D be an affine connection on M. Then

$$DT : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k+1 \text{ vector fields}} \times \overbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}^{l \text{ copies}} \to \mathcal{C}^\infty(M)$$

$$\begin{aligned} DT(\mathbb{X}_1, \ldots, \mathbb{X}_{k+1}, \theta_1, \ldots, \theta_l) &= \mathbb{X}_1(T(\mathbb{X}_2, \ldots, \mathbb{X}_{k+1}, \theta_1, \ldots, \theta_l)) \\ &- T(D_{\mathbb{X}_1}\mathbb{X}_2, \ldots, \mathbb{X}_{k+1}, \theta_1, \ldots, \theta_l) - T(\mathbb{X}_2, D_{\mathbb{X}_1}\mathbb{X}_3, \ldots, \theta_l) - \ldots \\ &\ldots - T(\ldots, \theta_{l-1}, D_{\mathbb{X}_1}\theta_l) \end{aligned}$$

is a tensor field of type $(k+1, l)$ on M.

Here $D_{\mathbb{X}}\theta$ is the differential 1-form $\mathbb{Y} \mapsto (D_{\mathbb{X}}\theta)(\mathbb{Y})$

Proof: Is just a repetition of the proof of the previous theorem. $\square$

Torsion of affine connection

Theorem Let ∇ be an affine connection M. Then

$$T = T^{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

is a tensor field of type $(2,1)$ on M.

Proof: Verify that T is linear in both arguments and that (LOF)' holds for T. Δ

Definition The tensor field T^{∇} is called the torsion (field) of the affine connection ∇. The affine connection is called symmetric or torsion free if $T^{\nabla} = 0$.

Local description

Let $\varphi : U \to \mathbb{R}^m$ be a chart, $X, Y \in \mathfrak{X}(M)$ with

$$X = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_{j=1}^{m} b_j(x) \frac{\partial}{\partial x_j}.$$

$$T(X,Y)(x) = (\nabla_X Y - \nabla_Y X - [X,Y])(x)$$

$$= \sum_{i,j=1}^{m} \left(a_i(x) \cancel{\left(\frac{\partial}{\partial x_i} b_j\right)(x) \frac{\partial}{\partial x_j}} + a_i(x) b_j(x) \sum_{k=1}^{m} \Gamma_{ij}^{k}(x) \frac{\partial}{\partial x_k} \right)$$

$$- \sum_{i,j=1}^{m} \left(b_i(x) \cancel{\left(\frac{\partial}{\partial x_i} a_j\right)(x) \frac{\partial}{\partial x_j}} + a_j(x) b_i(x) \sum_{k=1}^{m} \Gamma_{ij}^{k}(x) \frac{\partial}{\partial x_k} \right)$$

$$- \sum_{i,j=1}^{m} \left(a_i(x) \cancel{\left(\frac{\partial}{\partial x_i} b_j\right)(x) \frac{\partial}{\partial x_j}} - b_i(x) \cancel{\left(\frac{\partial}{\partial x_i} a_j\right)(x) \frac{\partial}{\partial x_j}} \right)$$

$$= \sum_{i,j,k} \left(a_i(x) b_j(x) \Gamma_{ij}^{k}(x) + a_j(x) b_i(x) \Gamma_{ij}^{k}(x) \right) \frac{\partial}{\partial x_k}$$

This suggests that

$$T(x) = \sum_{i,j,k=1}^{m} T_{ij}^{k}(x)\, dx_i \otimes dx_j \otimes \frac{\partial}{\partial x_k}$$

$$T_{ij}^{k}(x) = \Gamma_{ij}^{k}(x) + \Gamma_{ji}^{k}(x) \quad \left(\text{symmetrization of } \Gamma_{ij}^{k}(x) \text{ w.r.t. } i \text{ and } j. \right)$$

Example ∇ is the flat connection on $\mathbb{R}^m$, then, since $\Gamma_{ij}^{k}(x) = 0$, $T_{ij}^{k}(x) = 0$ on $\mathbb{R}^m$. Hence, the flat connection is torsion free.