Levi-Civita connection

Definition Let (M,g) be a Riemannian manifold. An affine connection ∇ on M is metric (for g) if g is parallel for ∇, that is $0 = \nabla g$ or more explicitly:

$$X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad X,Y,Z \in \mathfrak{X}(M).$$

Theorem (Fundamental theorem of Riemannian geometry)
Let (M,g) be a Riemannian manifold. Then there is a unique metric affine connection ∇ on M which is torsion-free.

Definition The unique metric and torsion-free connection for g is called the Levi-Civita connection.

Proof of Fundamental theorem of Riemannian geometry:
Let $X, Y, Z \in \mathfrak{X}(M)$ and let us first assume that ∇ is a metric and torsion-free connection for g. Then we have

$$
\begin{aligned}
&(1) \quad X g(Y,Z) = \underline{g(\nabla_X Y, Z)} + \cancel{g(Y, \nabla_X Z)}\\
&(2) \quad Y g(X,Z) = \cancel{g(\nabla_Y X, Z)} + \cancel{g(X, \nabla_Y Z)}\\
&(3) \quad -Z g(X,Y) = -\cancel{g(\nabla_Z X, Y)} - \cancel{g(X, \nabla_Y Z)}
\end{aligned}
$$

Since we assume that also ∇ is torsion-free, then

$$
\begin{aligned}
&(4) \quad -g([X,Z],Y) = -\cancel{g(\nabla_X Z, Y)} + \cancel{g(\nabla_Z X, Y)}\\
&(5) \quad -g([Y,Z],X) = -\cancel{g(\nabla_Y Z, X)} + \cancel{g(\nabla_Z Y, X)}\\
&(6) \quad g([X,Y],Z) = \underline{g(\nabla_X Y, Z)} - \cancel{g(\nabla_Y X, Z)}.
\end{aligned}
$$

Now if we add these formulas together, we obtain

$$
\begin{aligned}
(KF) \quad 2 g(\nabla_X Y, Z) = {} & X g(Y,Z) + Y g(X,Z) - Z g(X,Y)\\
& - g([X,Z],Y) - g([Y,Z],X) + g([X,Y],Z).
\end{aligned}
$$

This is so-called Koszul formula. Note that by non-degeneracy of $g_x : T_xM \times T_xM \to \mathbb{R}$, $\nabla_X Y$ is completely determined by the right hand side since we require that (KF) holds for every $Z \in \mathfrak{X}(M)$.
Now we will verify that (KF) defines the Levi-Civita. Let $F(X,Y,Z) \in \mathcal{C}^\infty(M)$ be the right hand side of (KF). So we can view F as a map

$$F: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{C}^\infty(M).$$

Note that $F(X,Y,Z_1 + Z_2) = F(X,Y,Z_1) + F(X,Y,Z_2)$ and that if $f \in \mathcal{C}^\infty(M)$, then

$$
\begin{aligned}
F(X,Y,fZ) = {} & X g(Y,fZ) + Y g(X,fZ) - fZ g(X,Y)\\
& - g([X,fZ],Y) - g([Y,fZ],X) + g([X,Y],fZ) =\\
= {} & f F(X,Y,Z) + \big[\cancel{(Xf)} g(Y,Z) + \cancel{(Yf)} g(X,Z)\\
& - (Xf) \cancel{g(Z,Y)} - (Yf) \cancel{g(Z,X)} \big]\\
= {} & f F(X,Y,Z).
\end{aligned}
$$

This shows, using nondegeneracy of g, that given $X, Y \in \mathfrak{X}(M)$, there is a unique vector field $V \in \mathfrak{X}(M)$ such that $g(V, Z) = \frac{1}{2} F(X, Y, Z)$. We define $D_X Y := V$.

But we still have to verify that this assignment defines an affine connection, that is (AC1) - (AC3) hold and that this connection is metric and torsion-free.

It is clear that (AC2) holds and that $F(X_1 + X_2, Y, Z) = F(X_1, Y, Z) + F(X_2, Y, Z)$. Let $f \in C^\infty(M)$, then we have

$$F(fX, Y, Z) = f F(X, Y, Z) + \cancel{(Yf)}\, g(X, Z) - (Zf)\, \cancel{g}(X, Y) + (Zf)\, \cancel{g}(X, Y) - (Yf)\, \cancel{g(X}, Z) = f F(X, Y, Z).$$

This shows that $D_{fX} Y = f D_X Y$ and so (AC1) holds.

Finally, we have

$$F(X, fY, Z) = f F(X, Y, Z) + (Xf)\, g(Y, Z) - \cancel{(Zf)\, g}(X, Y) + (Zf)\, g(\cancel{Y, X}) + (Xf)\, g(Y, Z) = f F(X, Y, Z) + 2 (Xf)\, g(Y, Z).$$

This shows that $D_X(fY) = f D_X Y + (Xf) Y$ and so also (AC3) holds.

Now we verify that D is torsion-free. Using non-degeneracy of g it is enough to show that

$$\begin{aligned} 0 = 2g(D_X Y - D_Y X - [X, Y], Z) &= 2g(D_X Y, Z) - 2g(D_Y X, Z) \\ -2g([X, Y], Z) &= -g([X, \cancel{Z}], Y) - g([\cancel{Y}, Z], X) \\ &+ g([\cancel{X}, Y], Z) + g([Y, \cancel{Z]}, X) + g([\cancel{X}, Z], Y) - g([\cancel{Y}, X], Z) \\ &-2g([\cancel{X}, Y], Z) = 0. \end{aligned}$$

Finally, $Dg = 0$ since we have

$$\begin{aligned} 2g(D_X Y, Z) + 2g(Y, D_X Z) &= 2g(D_X Y, Z) + 2g(D_X Z, Y) \\ = \; & X g(Y, Z) + Y \cancel{g(X, Z)} - Z \cancel{g}(X, Y) + X g(Z, Y) + Z \cancel{g(X}, Y) - Y \cancel{g(X}, Z) \\ & - g([\cancel{X}, Z], Y) - g([Y, \cancel{Z}], X) + g([X, \cancel{Y}], Z) \\ & - g([X, \cancel{Y}], Z) - g([\cancel{Z}, Y], X) + g([X, \cancel{Z}], Y) \\ = \; & 2 X g(Y, Z), \end{aligned}$$

which shows $Dg = 0$. $\square$

Local formula for Christoffel symbols

Let $\varphi: U \to \mathbb{R}^m$ be a chart on M with coordinate functions $x_1, \dots, x_m$ and associated vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$. Then

$$g(x) = \sum_{i,j=1}^{m} g_{ij}(x)\, dx_i \otimes dx_j$$

where $g_{ij}(x) = g_{ji}(x)$ are smooth functions of $x_1, \dots, x_m$. Note that by definition

$$g_{ij}(x) = g_x\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right), \quad x \in U.$$

Now the matrix $(g_{ij}(x))_{i=1,\dots,m}^{j=1,\dots,m}$ is positive definite for every $x \in U$, hence it is regular. Let $(g^{ij}(x))_{i=1,\dots,m}^{j=1,\dots,m}$ be the inverse matrix to $(g_{ij}(x))_{i=1,\dots,m}^{j=1,\dots,m}$.

We have $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ for every $i,j = 1,\dots,m$.

So Koszul formula gives for every fixed i,j,k:

$$2\, g_x\left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial x_i} g_x\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) + \frac{\partial}{\partial x_j} g_x\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}\right) - \frac{\partial}{\partial x_k} g_x\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

$$= \frac{\partial}{\partial x_i} g_{jk}(x) + \frac{\partial}{\partial x_j} g_{ik}(x) - \frac{\partial}{\partial x_k} g_{ij}(x).$$

Christoffel symbols are determined by

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{l=1}^{m} \Gamma_{ij}^{l}(x) \frac{\partial}{\partial x_l}.$$

Hence, we get

$$2\, \Gamma_{ij}^{l}(x)\, g_{lk}(x) = \frac{\partial}{\partial x_i} g_{jk}(x) + \frac{\partial}{\partial x_j} g_{ik}(x) - \frac{\partial}{\partial x_k} g_{ij}(x).$$

Using the inverse matrix, i.e. $\sum_{k=1}^{m} g^{pk}(x)\, g_{kl}(x) = \delta_{pl}$, we have

$$\text{(LC)} \qquad \Gamma_{ij}^{l}(x) = \frac{1}{2} \sum_{k=1}^{m} g^{lk}(x)\left(\frac{\partial}{\partial x_i} g_{jk}(x) + \frac{\partial}{\partial x_j} g_{ik}(x) - \frac{\partial}{\partial x_k} g_{ij}(x)\right).$$

Example Let g be the Euclidean metric on $\mathbb{R}^m$, that is

$$g_x = \sum_{i=1}^{m} dx_i \otimes dx_i.$$

Now $g_{ij}(x) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$. We see that the Christoffel symbols of the Levi-Civita connection of g vanish everywhere. Hence, the Levi-Civita connection of g is the flat metric on $\mathbb{R}^m$.

Parallel vector fields along curves

If $\gamma: [a,b] \to M$ is smooth and $X: [a,b] \to TM$ is a smooth vector field along γ, then recall that X is called parallel along γ with respect to an affine connection ∇ on M if

$$\frac{dX}{dt}(t) = (\nabla_{\gamma'(t)} X)(t) = 0 \quad \text{on } [a,b].$$

Now let g be a Riemannian metric on M and ∇ be its Levi-Civita connection. If X is parallel along γ, then note that

$$\frac{d}{dt}\left(g_{\gamma(t)}(X(t), X(t))\right)(t) = \gamma'(t)\left(g_{\gamma(t)}(X(t), X(t))\right)(t)$$

$$= g_{\gamma(t)}\left((\nabla_{\gamma'(t)} X)(t), X(t)\right) + g_{\gamma(t)}\left(X(t), (\nabla_{\gamma'(t)} X)(t)\right)$$

$$= 2\, g_{\gamma(t)}\left((\nabla_{\gamma'(t)} X)(t), X(t)\right) = 0.$$

We see that the norm (w.r. to g) of parallel vector field along curve γ is constant on $[a,b]$.

If $\mathbb{Y}: [a,b] \to TM$ is another parallel vector field along γ, then similarly we find that

$$\frac{d}{dt}\left(g_{\gamma(t)}(\mathbb{X}(t), \mathbb{Y}(t))\right)(t) = \gamma'(t)\left(g_{\gamma(t)}(\mathbb{X}(t), \mathbb{Y}(t))\right)(t)$$

$$= g_{\gamma(t)}\left((\nabla_{\gamma'(t)}\mathbb{X})(t), \mathbb{Y}(t)\right) + g_{\gamma(t)}\left(\mathbb{X}(t), (\nabla_{\gamma'(t)}\mathbb{Y})(t)\right) = 0.$$

(Here we are using that $g_{\gamma(t)}(\mathbb{X}(t), \mathbb{Y}(t))$ is a smooth function on the image γ and that the derivative of this function in the direction $\gamma'(t)$ depends only on values of this function on the image of γ.)

We see that the angle between $\mathbb{X}(t)$, $\mathbb{Y}(t)$, defined by

$$\cos\varphi = \frac{g_{\gamma(t)}(\mathbb{X}(t), \mathbb{Y}(t))}{\|\mathbb{X}(t)\|_{\gamma(t)}\, \|\mathbb{Y}(t)\|_{\gamma(t)}}$$

is constant on $[a,b]$.

If γ is a geodesic for ∇, then in particular $\|\gamma'(t)\|_{\gamma(t)} = c$ is constant and so the length of γ is

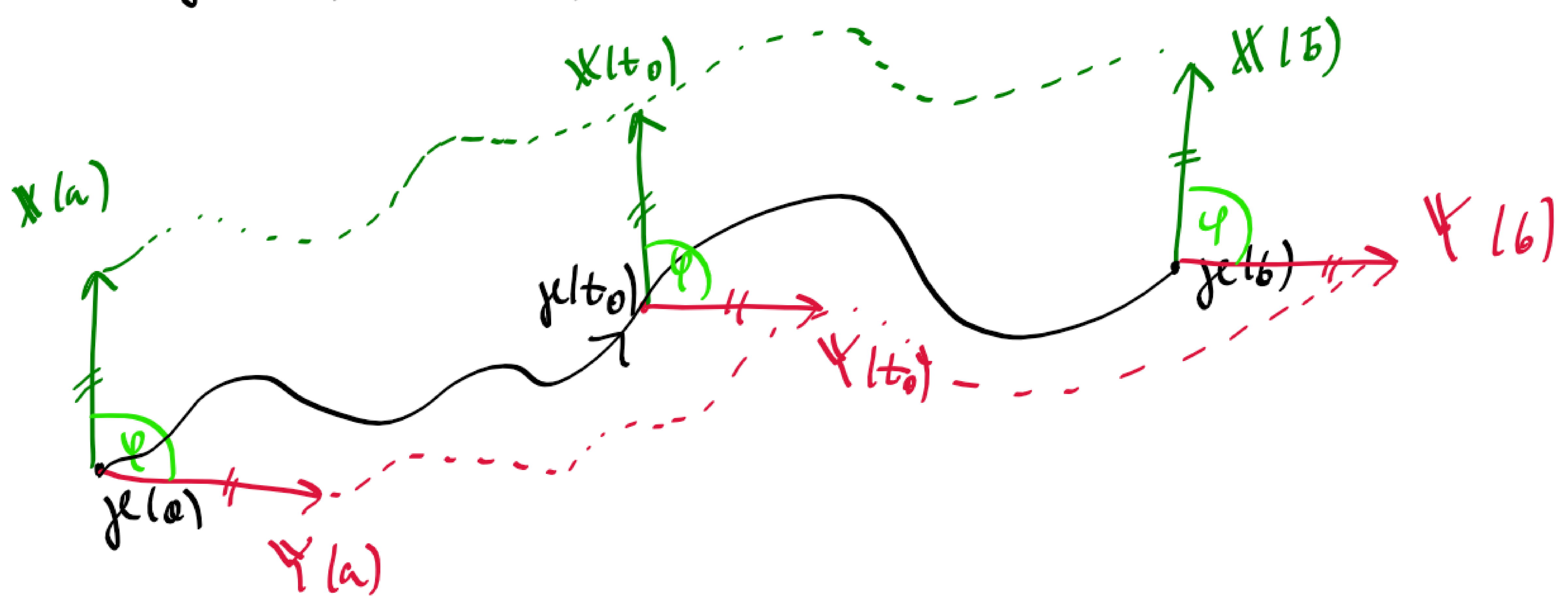
$$L(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)}\, dt = c\int_a^b dt = c(b-a).$$

So $L(\gamma)$ depends only on the length of interval $[a,b]$ and the value $\|\gamma'(t)\|_{\gamma(t)}$ at one point.

Example If ∇ is the flat connection on $\mathbb{R}^m$ which is also the Levi-Civita connection for the Euclidean metric, then we know that a vector field

$$\mathbb{X}(t) = \sum_{i=1}^m a_i(t)\frac{\partial}{\partial x_i}$$

is parallel along γ iff the functions $a_1(t), \dots, a_m(t)$ are constant. We see that indeed $\|\mathbb{X}(t)\|$ does not depend on t and the angle of two parallel vector fields is constant.



Examples •) The Euclidean metric on $\mathbb{R}^2$ in the polar coordinates $\Phi: (0,+\infty)\times(-\pi,\pi) \to \mathbb{R}^2$, $\Phi(r,\varphi) = (r\cos\varphi, r\sin\varphi)$ is given by

$$(\Phi^{-1})^*(dx_1\otimes dx_1 + dx_2\otimes dx_2) = (\Phi^{-1})^*dx_1 \otimes (\Phi^{-1})^*dx_1$$
$$+ (\Phi^{-1})^*dx_2 \otimes (\Phi^{-1})^*dx_2 = d(r\cos\varphi)\otimes d(r\cos\varphi) +$$
$$+ d(r\sin\varphi)\otimes d(r\sin\varphi) = (\cos\varphi\, dr - r\sin\varphi\, d\varphi)\otimes(\cos\varphi\, dr - r\sin\varphi\, d\varphi)$$
$$+ (\sin\varphi\, dr + r\cos\varphi\, d\varphi)\otimes(\sin\varphi\, dr + r\cos\varphi\, d\varphi) =$$
$$= (\cos^2\varphi + \sin^2\varphi)\, dr\otimes dr + r^2(\sin^2\varphi + \cos^2\varphi)\, d\varphi\otimes d\varphi$$

$$= dr\otimes dr + r^2\, d\varphi\otimes d\varphi,$$

or in the matrix notation it is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{with} \quad g_{rr}=1,\ g_{r\varphi}=0=g_{\varphi r},\ g_{\varphi\varphi}=r^2$$

with respect to the basis $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}\}$. The Christoffel symbols of the flat connection in polar coordinates are thus

$$\Gamma^r_{rr} = \frac{1}{2}\Big(g^{rr}\big(\tfrac{\partial}{\partial r} g_{rr} + \tfrac{\partial}{\partial r} g_{rr} - \tfrac{\partial}{\partial r} g_{rr}\big) +$$

$$+ g^{r\varphi}\big(\tfrac{\partial}{\partial r} g_{r\varphi} + \tfrac{\partial}{\partial r} g_{r\varphi} - \tfrac{\partial}{\partial \varphi} g_{rr}\big)\Big) = 0.$$

$$\Gamma^\varphi_{rr} = \frac{1}{2}\Big(g^{\varphi r}\big(\tfrac{\partial}{\partial r} g_{rr} + \tfrac{\partial}{\partial r} g_{rr} - \tfrac{\partial}{\partial r} g_{rr}\big)$$

$$+ g^{\varphi\varphi}\big(\tfrac{\partial}{\partial r} g_{\varphi r} + \tfrac{\partial}{\partial r} g_{\varphi r} - \tfrac{\partial}{\partial \varphi} g_{rr}\big)\Big) = 0.$$

This agrees with previous calculation in week 7.

•) Levi-Civita connection of the round metric on S^m.
We have shown that in the chart $\varphi_N : S^m \setminus \{N\} \to \mathbb{R}^m$ the round metric on S^m takes the form

$$(\varphi_N^{-1})^* \Big(\sum_{i=1}^{m+1} du_i\otimes du_i\Big)(x) = \frac{1}{1+\|x\|^2}\sum_{j=1}^{m} dx_j\otimes dx_j, \text{ that}$$

$$g_{ij}(x) = \frac{\delta_{ij}}{1+\|x\|^2} \qquad \text{where} \quad \|x\|^2 = x_1^2+\dots+x_m^2,\ x=(x_1,\dots,x_m).$$

Hence

$$\Gamma^k_{ij}(x) = \frac{1}{2}\sum_{l=1}^{m}(1+\|x\|^2)\,\delta^{kl}\Big(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial l} g_{ij}\Big)(x)$$

$$= -\sum_{l=1}^{m}\frac{1}{1+\|x\|^2}\,\delta^{kl}\big(x_i\delta_{jl} + x_j\delta_{il} - x_l\delta_{ij}\big)$$

$$= -\frac{1}{1+\|x\|^2}\big(x_i\delta_{kj} + x_j\delta_{ik} - x_k\delta_{ij}\big)$$

•) Levi-Civita connection of the round metric g_{S^2} on S^2 in spherical coordinates.
Consider $\Phi : (-\pi,\pi)\times(-\frac{\pi}{2},\frac{\pi}{2}) \longrightarrow S^2$

$$\Phi(\varphi,\theta) = (\cos\varphi\cos\theta,\ \sin\varphi\cos\theta,\ \sin\theta).$$

We have that

$$(\Phi^{-1})^* g_{S^2} = (\Phi^{-1})^*\sum_{i=1}^{3} dx_i\otimes dx_i = d(\cos\varphi\cos\theta)\otimes d(\cos\varphi\cos\theta)$$

$$+ d(\sin\varphi\cos\theta)\otimes d(\sin\varphi\cos\theta) + d(\sin\theta)\otimes d(\sin\theta) =$$

$$= (-\sin\varphi\cos\theta\, d\varphi - \cos\varphi\sin\theta\, d\theta)\otimes(-\sin\varphi\cos\theta\, d\varphi - \cos\varphi\sin\theta\, d\theta)$$

$$+ (\cos\varphi\cos\theta\, d\varphi - \sin\varphi\sin\theta\, d\theta)\otimes(\cos\varphi\cos\theta\, d\varphi - \sin\varphi\sin\theta\, d\theta)$$

$$+ (\cos\theta\, d\theta)\otimes(\cos\theta\, d\theta)$$

$$= (\sin^2\varphi\cos^2\theta + \cos^2\varphi\cos^2\theta)\, d\varphi\otimes d\varphi$$

$$+ (\cos^2\theta + \cos^2\varphi\sin^2\theta + \sin^2\varphi\sin^2\theta)\, d\theta\otimes d\theta$$

$$= \cos^2\theta\, d\varphi\otimes d\varphi + d\theta\otimes d\theta$$

Hence, the round metric in matrix notation is

$$\begin{pmatrix} \cos^2\theta & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad g_{\varphi\varphi} = \cos^2\theta, \; g_{\varphi\theta} = 0 = g_{\theta\varphi}, \; g_{\theta\theta} = 1.$$

So $g^{\varphi\varphi} = \cos^{-2}\theta, \; g^{\varphi\theta} = g^{\theta\varphi} = 0, \; g^{\theta\theta} = 1$ and so

$$\Gamma^{\varphi}_{\varphi\varphi} = \tfrac{1}{2}\Big(g^{\varphi\varphi}\big(\tfrac{\partial}{\partial\varphi} g_{\varphi\varphi} + \tfrac{\partial}{\partial\varphi} g_{\varphi\varphi} - \tfrac{\partial}{\partial\varphi} g_{\varphi\varphi}\big) + g^{\varphi\theta}\big(\tfrac{\partial}{\partial\varphi} g_{\varphi\theta} + \tfrac{\partial}{\partial\varphi} g_{\varphi\theta} - \tfrac{\partial}{\partial\theta} g_{\varphi\varphi}\big)\Big) = 0,$$

$$\Gamma^{\theta}_{\varphi\varphi} = \tfrac{1}{2}\Big(g^{\theta\varphi}\big(\tfrac{\partial}{\partial\varphi} g_{\varphi\varphi} + \tfrac{\partial}{\partial\varphi} g_{\varphi\varphi} - \tfrac{\partial}{\partial\varphi} g_{\varphi\varphi}\big) + g^{\theta\theta}\big(\tfrac{\partial}{\partial\varphi} g_{\varphi\theta} + \tfrac{\partial}{\partial\varphi} g_{\varphi\theta} - \tfrac{\partial}{\partial\theta} g_{\varphi\varphi}\big)\Big)$$
$$= -\tfrac{1}{2}(-2\cos\theta\sin\theta) = \cos\theta\sin\theta,$$

$$\Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = \tfrac{1}{2}\Big(g^{\varphi\varphi}\big(\tfrac{\partial}{\partial\theta} g_{\varphi\varphi} + \tfrac{\partial}{\partial\varphi} g_{\theta\varphi} - \tfrac{\partial}{\partial\varphi} g_{\theta\varphi}\big) + g^{\varphi\theta}\big(\tfrac{\partial}{\partial\theta} g_{\varphi\varphi} + \tfrac{\partial}{\partial\varphi} g_{\theta\varphi} - \tfrac{\partial}{\partial\theta} g_{\theta\varphi}\big)\Big)$$
$$= \tfrac{1}{2}\cos^{-2}\theta(-2\cos\theta\sin\theta) = -\tan\theta,$$

$$\Gamma^{\theta}_{\theta\varphi} = \Gamma^{\theta}_{\varphi\theta} = \tfrac{1}{2}\Big(g^{\theta\varphi}\big(\tfrac{\partial}{\partial\theta} g_{\varphi\varphi} + \tfrac{\partial}{\partial\varphi} g_{\theta\varphi} - \tfrac{\partial}{\partial\varphi} g_{\theta\varphi}\big) + g^{\theta\theta}\big(\tfrac{\partial}{\partial\theta} g_{\varphi\theta} + \tfrac{\partial}{\partial\varphi} g_{\theta\theta} - \tfrac{\partial}{\partial\theta} g_{\theta\varphi}\big)\Big) = 0,$$

$$\Gamma^{\varphi}_{\theta\theta} = \tfrac{1}{2}\Big(g^{\varphi\varphi}\big(\tfrac{\partial}{\partial\theta} g_{\theta\varphi} + \tfrac{\partial}{\partial\theta} g_{\theta\varphi} - \tfrac{\partial}{\partial\varphi} g_{\theta\theta}\big) + g^{\varphi\theta}\big(\tfrac{\partial}{\partial\theta} g_{\theta\theta} + \tfrac{\partial}{\partial\theta} g_{\theta\theta} - \tfrac{\partial}{\partial\theta} g_{\theta\theta}\big)\Big) = 0,$$

$$\Gamma^{\theta}_{\theta\theta} = \tfrac{1}{2}\Big(g^{\theta\varphi}\big(\tfrac{\partial}{\partial\theta} g_{\theta\varphi} + \tfrac{\partial}{\partial\theta} g_{\theta\varphi} - \tfrac{\partial}{\partial\varphi} g_{\theta\theta}\big) + g^{\theta\theta}\big(\tfrac{\partial}{\partial\theta} g_{\theta\theta} + \tfrac{\partial}{\partial\theta} g_{\theta\theta} - \tfrac{\partial}{\partial\theta} g_{\theta\theta}\big)\Big) = 0.$$

So a curve $\gamma : [a,b] \to S^2$ which is contained in the image of Φ is a geodesic iff

$$\Phi^{-1} \circ \gamma(t) = \tilde{\gamma}(t) = (\varphi(t), \psi(t))$$

satisfies

$$\varphi'' + \Gamma^{\varphi}_{\varphi\varphi}\varphi'\varphi' + \Gamma^{\varphi}_{\varphi\theta}\varphi'\theta' + \Gamma^{\varphi}_{\theta\varphi}\theta'\varphi' + \Gamma^{\varphi}_{\theta\theta}\theta'\theta' = 0$$
$$\theta'' + \Gamma^{\theta}_{\varphi\varphi}\varphi'\varphi' + \Gamma^{\theta}_{\varphi\theta}\varphi'\theta' + \Gamma^{\theta}_{\theta\varphi}\theta'\varphi' + \Gamma^{\theta}_{\theta\theta}\theta'\theta' = 0$$

$$(GE_\sigma S^2) \quad \boxed{\begin{aligned} &\varphi'' - 2\tan\theta\,\varphi'\theta' = 0 \\ &\theta'' + \cos\theta\sin\theta\,\varphi'\varphi' = 0 \end{aligned}}$$

Note that $\varphi' = 0$, $\theta(t) = t$ are geodesics and that also $\theta(t) = 0$ and $\varphi(t) = t$ is a geodesic. To find an explicit solutions of the geodesics equations $(GE_\sigma S^2)$ is extremely difficult if not even impossible, however we will now show

Theorem Let M be a 2-dimensional surface in $\mathbb{R}^3$, that is M is a 2-dimensional manifold with embedding $\iota : M \hookrightarrow \mathbb{R}^3$ (so that ι is an immersion and an homeomorphism onto its image). Let $g_M = \iota^* g$ be the induced Riemannian metric on M where $g = \langle -,- \rangle$ is the Euclidean metric on $\mathbb{R}^3$.

Then a curve $\gamma : [a,b] \to S^2$ is a geodesic for the Levi-Civita connection ∇ of g_M iff for every $t \in [a,b]$:

$$\gamma''(t) \in (T_{\gamma(t)} M)^\perp \subseteq T_{\gamma(t)} \mathbb{R}^3 = \mathbb{R}^3 .$$

Proof: Let $\varphi : U \to \mathbb{R}^2$ be a chart on M. Then $\varphi^{-1} : \varphi(U) \to M \subseteq \mathbb{R}^3$ is smooth. (Here we view $M \subseteq \mathbb{R}^3$ via ι). We have

$$(\varphi^{-1})^* g_M(x) = \sum_{i,j=1}^{2} g_{ij}(x)\, dx_i \otimes dx_j$$

where

$$g_{ij}(x) = g_{\varphi^{-1}(x)}\left(\frac{\partial \varphi^{-1}}{\partial x_i}(\varphi(x)), \frac{\partial \varphi^{-1}}{\partial x_j}(\varphi(x))\right) = \left\langle \frac{\partial \varphi^{-1}}{\partial x_i}(\varphi(x)), \frac{\partial \varphi^{-1}}{\partial x_j}(\varphi(x))\right\rangle, \quad i,j = 1,2.$$

Hence,

$$\frac{\partial}{\partial x_i} g_{jk}(x) + \frac{\partial}{\partial x_j} g_{ik}(x) - \frac{\partial}{\partial x_k} g_{ij}(x) =$$

$$= \left(\frac{\partial}{\partial x_i}\left\langle \frac{\partial \varphi^{-1}}{\partial x_j}, \frac{\partial \varphi^{-1}}{\partial x_k}\right\rangle\right)(\varphi(x)) + \left(\frac{\partial}{\partial x_j}\left\langle \frac{\partial \varphi^{-1}}{\partial x_i}, \frac{\partial \varphi^{-1}}{\partial x_k}\right\rangle\right)(\varphi(x))$$

$$-\left(\frac{\partial}{\partial x_k}\left\langle \frac{\partial \varphi^{-1}}{\partial x_i}, \frac{\partial \varphi^{-1}}{\partial x_j}\right\rangle\right)(\varphi(x)) = (**)$$

$$\left|\left(\frac{\partial}{\partial x_i}\left\langle \frac{\partial \varphi^{-1}}{\partial x_j}, \frac{\partial \varphi^{-1}}{\partial x_k}\right\rangle\right)(\varphi(x)) = \left\langle \frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_j}, \frac{\partial \varphi^{-1}}{\partial x_k}\right\rangle(\varphi(x)) + \left\langle \frac{\partial \varphi^{-1}}{\partial x_j}, \frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_k}\right\rangle(\varphi(x))\right|$$

$$(**) = 2\left(\frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_j}, \frac{\partial \varphi^{-1}}{\partial x_k}\right)(\varphi(x)) \text{ for any } i,j,k = 1,2.$$

Now the curve γ is a geodesic iff

$$g_{\gamma(t)}(\nabla_{\gamma'(t)} \gamma'(t), X(t)) = 0$$

for any vector field $X : [a,b) \to TM$ along γ and this is equivalent to

$$\sum_{k=1}^{2} (g_{\ell k})_{\gamma(t)}\left(\tilde{\gamma}''_k(t) + \sum_{i,j=1}^{2} \Gamma^k_{ij}(\gamma(t))\, \tilde{\gamma}_i'(t)\, \tilde{\gamma}_j'(t)\right) = 0, \quad \ell = 1,..,m,$$

where $\varphi \circ \gamma = \tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2)$. And so using $(**)$, we get

$$\sum_{k=1}^{m} (g_{k\ell})_{\gamma(t)}\left(\tilde{\gamma}''_k(t) + \sum_{i,j,w=1}^{2} g^{kw}(\gamma(t))\left\langle \frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_j}, \frac{\partial \varphi^{-1}}{\partial x_w}\right\rangle(\gamma(t))\ \tilde{\gamma}_i'(t)\, \tilde{\gamma}_j'(t)\right) = 0$$

$$\sum_{k=1}^{2}\left\langle \frac{\partial \varphi^{-1}}{\partial x_k}, \frac{\partial \varphi^{-1}}{\partial x_\ell}\right\rangle(\tilde{\gamma}(t))\, \tilde{\gamma}''_k(t) + \sum_{i,j=1}^{2}\left\langle \frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_j}, \frac{\partial \varphi^{-1}}{\partial x_\ell}\right\rangle(\tilde{\gamma}(t))\ \tilde{\gamma}_i'(t)\, \tilde{\gamma}_j'(t) = 0.$$

This is equivalent to the fact that $\frac{\partial \varphi^{-1}}{\partial x_1}(\tilde{\gamma}(t)), \frac{\partial \varphi^{-1}}{\partial x_2}(\tilde{\gamma}(t))$ are orthogonal to

$$\sum_{k=1}^{2} \frac{\partial \varphi^{-1}}{\partial x_k}(\tilde{\gamma}(t))\, \tilde{\gamma}''_k(t) + \sum_{i,j=1}^{2} \frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_j}(\tilde{\gamma}(t))\, \tilde{\gamma}_i'(t)\, \tilde{\gamma}_j'(t) = \frac{d^2(\varphi^{-1}\circ \tilde{\gamma})}{dt^2}(t) = \frac{d^2 \gamma}{dt^2}(t)$$

But $T_{\gamma(t)} M$ is spanned by $\frac{\partial \varphi^{-1}}{\partial x_1}(\tilde{\gamma}(t)), \frac{\partial \varphi^{-1}}{\partial x_2}(\tilde{\gamma}(t))$. $\square$

Example (Geodesics on sphere S^2)

By previous Theorem, a curve $\gamma : [a,b] \longrightarrow S^2$ is a geodesic for the Levi-Civita connection of g_{S^2} iff

$$(\mathrm{G} \circ S^2) \qquad \gamma''(t) = \frac{d^2\gamma}{dt^2}(t) \perp T_{\gamma(t)} S^2 \text{ for every } t \in [a,b].$$

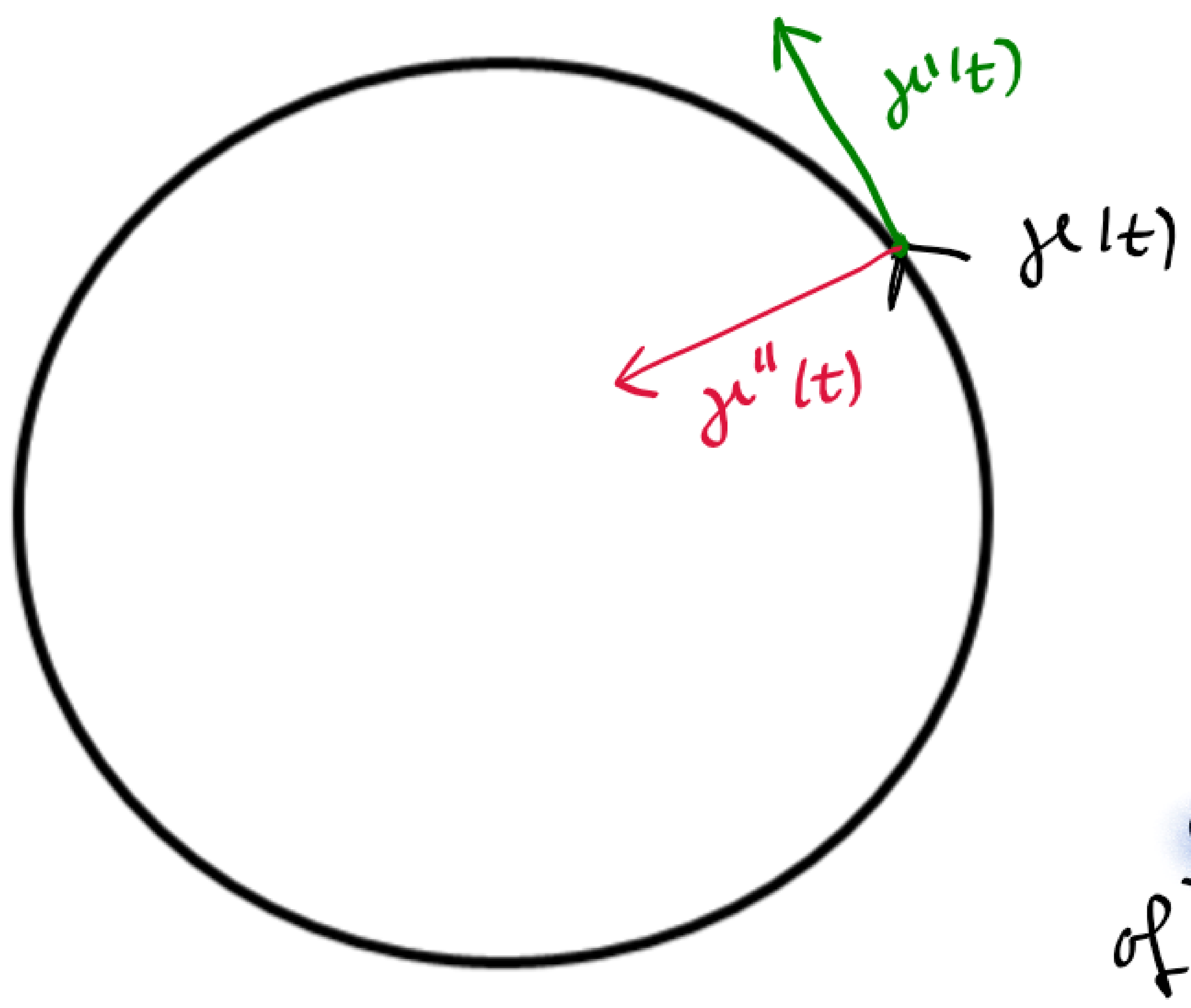
But we know that $T_{\gamma(t)} S^2$ is the orthogonal complement to $\gamma(t) \in S^2$ and so $(\mathrm{G} \circ S^2)$ is equivalent to

$$(\mathrm{G} \circ S^2)' \qquad \gamma''(t) \text{ is a multiple of } \gamma(t) \text{ for every } t \in [a,b].$$

Now for $\gamma(t) = (\cos(at), \sin(at), 0)$, $t \in [0, 2\pi]$, $a > 0$, we have

$$\gamma''(t) = -a^2 (\cos(at), \sin(at), 0)$$

and so γ is the unique geodesic on S^2 on $[0, 2\pi]$ that satisfies the initial condition $\gamma(0) = (1,0,0)$, $\gamma'(t) = (0,a,0)$.



$z = 0$

Clearly, γ is a parametrization of the equator.

Now if $A \in O(3)$ is an orthogonal transformation of $\mathbb{R}^3$, that is $A : \mathbb{R}^3 \to \mathbb{R}^3$ is linear and $\langle Au, Av \rangle = \langle u, v \rangle$, $u, v \in \mathbb{R}^3$, then $Ax \in S^2$ if $x \in S^2$ and $A \circ \gamma$ is a geodesic. It follows that

Proposition Any geodesic on S^2 is a parametrization of a great circle (i.e. intersection of S^2 with a 2-dimensional subspace of $\mathbb{R}^3$) by a curve with constant velocity.