Levi-Civita connection

Theorem Given a Riemannian manifold (M,g), there is a unique affine connection ∇ on M which metric $(\nabla g = 0)$ and torsion-free.

Sketch of proof: In the beginning we assumed that there is a metric and torsion-free connection:

- $X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad (\equiv \nabla g = 0)$
- $\nabla_X Y - \nabla_Y X = [X,Y]$

holds for every $X, Y, Z \in \mathfrak{X}(M)$. Then we summed up these equations to obtain Koszul formula:

$$2g(\nabla_X Y, Z) = X g(Y,Z) + Y g(X,Z) - Z g(X,Y)$$
$$- g([X,Z],Y) - g([Y,Z],X) + g([X,Y],Z) := T(X,Y,Z)$$

Then we checked that the right hand side is for fixed $X, Y \in \mathfrak{X}(M)$ a differential form $\mathfrak{X}(M) \ni Z \longmapsto T(X,Y,Z) \in \mathcal{C}^\infty(M)$ on M, and so by non-degeneracy of g, there is a unique vector field $V \in \mathfrak{X}(M)$ such that

$$2g(V,Z) = T(X,Y,Z) \text{ holds for every } Z \in \mathfrak{X}(M).$$

This defines a map $\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$, $(X,Y) \longmapsto V$ and we verified that this map satisfies (AC 1) – (AC3). And this proves uniqueness of the Levi-Civita connection. Then we have checked that this connection is also torsion-free and metric. (This was the idea behind the proof.) $\square$

Definition The unique affine connection from the previous Theorem is called the Levi-Civita connection.

Local formula for Christoffel symbols of Levi-Civita connection

Let $\varphi: U \to \mathbb{R}^m$ be a chart on a Riemannian manifold (M,g) with coordinate functions $x_1, \dots, x_m$ and associated vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$ on U. Let us assume that in these coordinates the Riemannian metric g is given by

$$g(x) = \sum_{i,j=1}^{m} g_{ij}(x)\, dx_i \otimes dx_j$$

where $g_{ij}(x)$ are smooth functions of $x_1, \dots, x_m$ that are defined by
$$g_{ij}(x) = g_x\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right), \quad i,j = 1, \dots, m, \quad x \in U.$$

Also recall that Christoffell symbols of the Levi-Civita connection ∇ in the chart φ are defined by

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{m} \Gamma_{ij}^{k}(x) \frac{\partial}{\partial x_k}, \quad i,j = 1,\dots,m, \; x \in \mathcal{U}.$$

Again $\Gamma_{ij}^k(x)$ are smooth functions of $x_1,\dots,x_m$.

Now we can use the Koszul formula to calculate the Cristoffel symbols of ∇. We know that the coordinate vector fields commute, that $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ on $\mathcal{U}$ for every $i,j = 1,\dots,m$.

And so we have that

$$= 2 g_x\left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = 2 g_x\left(\sum_{k=1}^{m} \Gamma_{ij}^k(x) \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) = 2 \sum_{k=1}^{m} g_{kl}(x) \Gamma_{ij}^k(x)$$

$$\left(\begin{aligned} 2g(\nabla_X Y, Z) = {} & X g(Y,Z) + Y g(X,Z) - Z g(X,Y) \\ & - g([X,Z],Y) - g([Y,Z],X) + g([X,Y],Z) \end{aligned} \right)$$

$$= \frac{\partial}{\partial x_i} g_x\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l}\right) + \frac{\partial}{\partial x_j} g_x\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_l}\right) - \frac{\partial}{\partial x_l} g_x\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

$$= \frac{\partial}{\partial x_i} g_{jl}(x) + \frac{\partial}{\partial x_j} g_{il}(x) - \frac{\partial}{\partial x_l} g_{ij}(x).$$

So we obtain

$$2 \sum_{k=1}^{m} g_{kl}(x) \Gamma_{ij}^k(x) = \frac{\partial}{\partial x_i} g_{jl}(x) + \frac{\partial}{\partial x_j} g_{il}(x) - \frac{\partial}{\partial x_l} g_{ij}(x).$$

Let $(g^{uv}(x))_{u=1,\dots,m}^{v=1,\dots,m}$ be the inverse matrix to $(g_{ij}(x))_{i=1,\dots,m}^{j=1,\dots,m}$

(the existence of the inverse matrix is guaranteed by the fact that $(g_{ij}(x))_{i=1,\dots,m}^{j=1,\dots,m}$ is positive definite for every $x \in \mathcal{U}$)

so that
$$\sum_{i=1}^{m} g^{ki}(x) g_{ij}(x) = \delta_{kj} = \begin{cases} 1 & k=j \\ 0 & k \neq j \end{cases}$$

$$\sum_{j=1}^{m} g_{ij}(x) g^{jk}(x) = \delta_{ik} = \begin{cases} 1 & i=k \\ 0 & i \neq k. \end{cases}$$

Hence,
$$\sum_{k,l=1}^{m} g_{kl}(x) \Gamma_{ij}^k(x) g^{lu}(x) = \delta_{ku} \Gamma_{ij}^k(x) = \Gamma_{ij}^u(x)$$

$$2\Gamma_{ij}^u(x) = \sum_{l=1}^{m} g^{lu}(x) \left(\frac{\partial}{\partial x_i} g_{jl}(x) + \frac{\partial}{\partial x_j} g_{il}(x) - \frac{\partial}{\partial x_l} g_{ij}(x) \right).$$

Final formula is

$$\Gamma_{ij}^{k}(x) = \frac{1}{2}\sum_{\ell=1}^{m} g^{\ell k}(x)\left(\frac{\partial}{\partial x_i} g_{j\ell}(x) + \frac{\partial}{\partial x_j} g_{i\ell}(x) - \frac{\partial}{\partial x_\ell} g_{ij}(x)\right).$$

Example (HW 6) Calculate $\Gamma_{r\varphi}^{r}, \Gamma_{r\varphi}^{\varphi}, \Gamma_{\varphi\varphi}^{r}, \Gamma_{\varphi\varphi}^{\varphi}$ for the flat affine connection on $\mathbb{R}^2$ in polar coordinates.

We know that the flat affine connection is the Levi-Civita connection for the Euclidean metric on $\mathbb{R}^m$ (this is the Riemannian metric $g = \sum_{i=1}^{m} dx_i \otimes dx_i$). Now it is enough to calculate the Euclidean metric on $\mathbb{R}^2$ in polar coordinates, that is,

$(\Phi^{-1})^* g$ where $\Phi : (0,+\infty)\times(-\pi,\pi) \longrightarrow \mathbb{R}^2$, $\Phi(r,\varphi) = (\underbrace{r\cos\varphi}_{x_1}, \underbrace{r\sin\varphi}_{x_2})$.

$$(\Phi^{-1})^* g = (\Phi^{-1})^*(dx_1\otimes dx_1 + dx_2\otimes dx_2) =$$

$$= ((\Phi^{-1})^* dx_1)\otimes((\Phi^{-1})^* dx_1) + ((\Phi^{-1})^* dx_2)\otimes((\Phi^{-1})^* dx_2)$$

$$= d(r\cos\varphi)\otimes d(r\cos\varphi) + d(r\sin\varphi)\otimes d(r\sin\varphi)$$

$$= (\cos\varphi\, dr - r\sin\varphi\, d\varphi)\otimes(\cos\varphi\, dr - r\sin\varphi\, d\varphi)$$

$$+ (\sin\varphi\, dr + r\cos\varphi\, d\varphi)\otimes(\sin\varphi\, dr + r\cos\varphi\, d\varphi)$$

$$= (\cos^2\varphi + \sin^2\varphi)\, dr\otimes dr + (-r\cos\varphi\sin\varphi + \sin\varphi\, r\cos\varphi)\, dr\otimes d\varphi$$

$$+ (-r\sin\varphi\cos\varphi + r\cos\varphi\sin\varphi)\, d\varphi\otimes dr +$$

$$+ (r^2\sin^2\varphi + r^2\cos^2\varphi)\, d\varphi\otimes d\varphi = 1\cdot dr\otimes dr + r^2\, d\varphi\otimes d\varphi$$

$g_{rr} = 1$, $g_{r\varphi} = g_{\varphi r} = 0$, $g_{\varphi\varphi} = r^2$, $\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

$g^{rr} = 1$, $g^{r\varphi} = g^{r\varphi} = 0$, $g^{\varphi\varphi} = r^{-2}$, $\begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$

$$\left[\Gamma_{ij}^{k}(x) = \frac{1}{2}\sum_{\ell=1}^{m} g^{\ell k}(x)\left(\frac{\partial}{\partial x_i} g_{j\ell}(x) + \frac{\partial}{\partial x_j} g_{i\ell}(x) - \frac{\partial}{\partial x_\ell} g_{ij}(x)\right).\right]$$

$$\Gamma_{r\varphi}^{r} = \frac{1}{2} g^{rr}\Big(\frac{\partial}{\partial r}\underbrace{g_{\varphi r}}_{=0} + \frac{\partial}{\partial\varphi}\underbrace{g_{rr}}_{=1} - \frac{\partial}{\partial r}\underbrace{g_{r\varphi}}_{=0}\Big) + \frac{1}{2}\underbrace{g^{r\varphi}}_{=0}\Big(\frac{\partial}{\partial r} g_{\varphi\varphi} + \frac{\partial}{\partial\varphi} g_{r\varphi} - \frac{\partial}{\partial\varphi} g_{r\varphi}\Big)$$

$$= 1(0) = 0.$$

$$\Gamma_{\varphi r}^{\varphi} = \frac{1}{2}\underbrace{g^{\varphi r}}_{=0}\Big(\frac{\partial}{\partial\varphi} g_{rr} + \frac{\partial}{\partial r} g_{\varphi r} - \frac{\partial}{\partial r} g_{\varphi r}\Big) + \frac{1}{2} g^{\varphi\varphi}\Big(\frac{\partial}{\partial\varphi}\underbrace{g_{r\varphi}}_{=0} + \frac{\partial}{\partial r}\underbrace{g_{\varphi\varphi}}_{=r^2} - \frac{\partial}{\partial\varphi}\underbrace{g_{\varphi r}}_{=0}\Big)$$

$$= \frac{1}{2} r^{-2}(2r) = \frac{1}{r}.$$

Parallel vector fields along curves for Levi-Civita connection

If $\gamma : [a,b] \to M$ is smooth, then a vector field $X : [a,b] \to TM$ with $X(t) \in T_{\gamma(t)}M$ is called parallel along γ for an affine connection ∇ if

$$\frac{dX}{dt}(t) = (\nabla_{\gamma'(t)} X)(t) = 0 \quad \text{for every } t \in [a,b].$$

Let us now assume that ∇ is the Levi-Civita connection for a Riemannian metric g on M. Then we have that

$$\frac{d}{dt}\left(g_{\gamma(t)}(X(t), X(t))\right)(t) = \gamma'(t)\Big[\underbrace{g_{\gamma(t)}(X(t), X(t))}_{\text{function on the image of } \gamma}\Big] =$$

↑ differentiating in the direction of $\gamma'(t)$

$$= g_{\gamma(t)}\left((\nabla_{\gamma'(t)} X)(t), X(t)\right) + g_{\gamma(t)}\left(X(t), (\nabla_{\gamma'(t)} X)(t)\right)$$

$$= 2 g_{\gamma(t)}\left((\nabla_{\gamma'(t)} X)(t), X(t)\right) = 0 \quad \text{if } X \text{ is parallel along } \gamma.$$

This calculation shows that the norm of a parallel vector field along a curve is constant (w.r.t. to g).

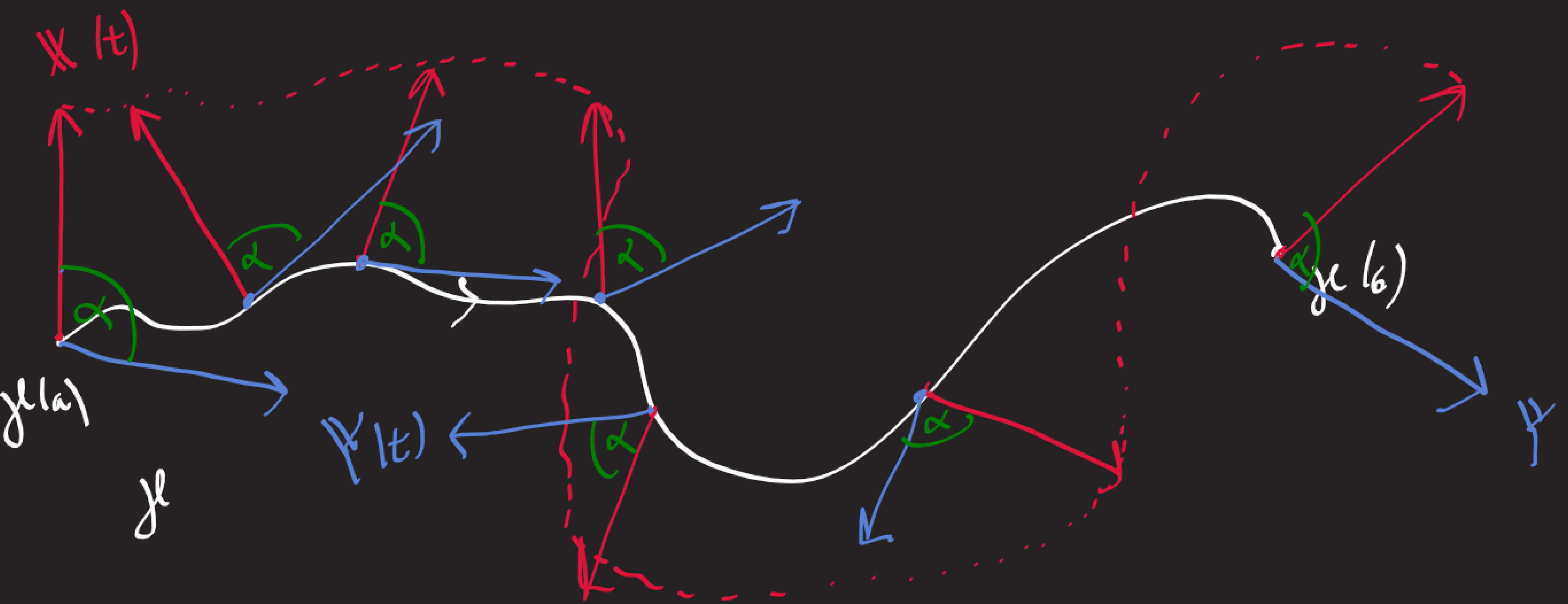
If $Y : [a,b] \to TM$ is another parallel vector field along γ for ∇, then

$$\frac{d}{dt}\left(g_{\gamma(t)}(X(t), Y(t))\right)(t) = \gamma'(t)\left[g_{\gamma(t)}(X(t), Y(t))\right]$$

$$= g_{\gamma(t)}\left((\nabla_{\gamma'(t)} X)(t), Y(t)\right) + g_{\gamma(t)}\left(X(t), (\nabla_{\gamma'(t)} Y)(t)\right) = 0.$$

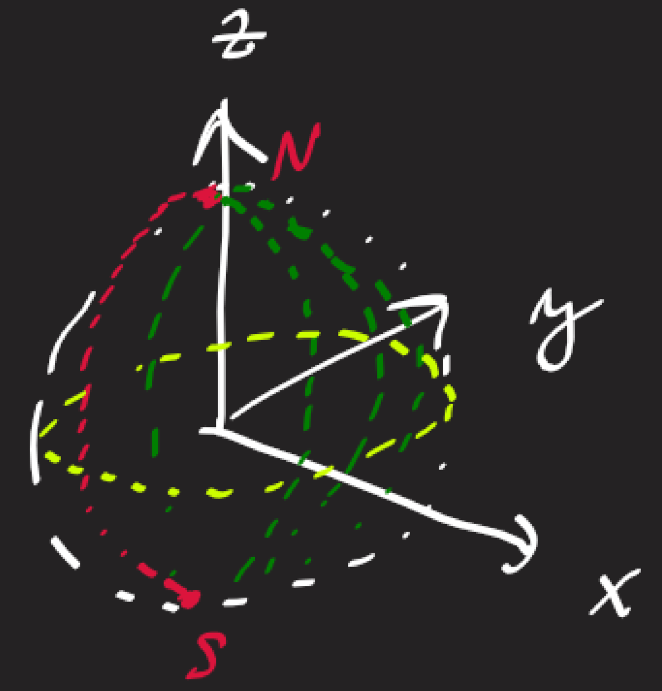
This calculation shows that the angle $\alpha(t)$ between $X(t), Y(t)$ is constant since

$$\cos \alpha(t) = \frac{g_{\gamma(t)}(X(t), Y(t))}{\sqrt{g_{\gamma(t)}(X(t), X(t))}\sqrt{g_{\gamma(t)}(Y(t), Y(t))}}.$$



Example We consider $S^2 \subseteq \mathbb{R}^3$ with its round Riemannian metric.
If $g = \sum_{i=1}^{3} dx_i \otimes dx_i$ is the Euclidean metric on $\mathbb{R}^3$ and $\iota : S^2 \hookrightarrow \mathbb{R}^3$ is the embedding, then the round metric $g_{S^2} = \iota^* g$. Let

$$\Phi : (-\pi,\pi) \times (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \longrightarrow \mathbb{R}^3,$$
$$\Phi(\varphi,\theta) = (\cos\varphi \cos\theta, \sin\varphi\cos\theta, \sin\theta).$$



The image of Φ is $\mathcal{U} := S^2 \setminus \{\text{meridian } 180°\}$
The inverse map $\Phi^{-1} : \mathcal{U} \longrightarrow (-\pi,\pi) \times (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \subseteq \mathbb{R}^2$
is a chart on S^2.

$$\Phi^* g_{S^2} = \Phi^* \Big(\sum_{i=1}^{3} dx_i \otimes dx_i \Big)$$
$$= \cos^2\theta \, d\varphi \otimes d\varphi + d\theta \otimes d\theta$$

$$g_{\varphi\varphi} = \cos^2\theta, \quad g_{\varphi\theta} = g_{\theta\varphi} = 0, \quad g_{\theta\theta} = 1, \qquad \begin{pmatrix} \cos^2\theta & 0 \\ 0 & 1 \end{pmatrix}$$

$$g^{\varphi\varphi} = \cos^{-2}\theta, \quad g^{\varphi\theta} = g^{\theta\varphi} = 0, \quad g^{\theta\theta} = 1, \qquad \begin{pmatrix} \cos^{-2}\theta & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Gamma^{\varphi}_{\varphi\varphi} = \tfrac{1}{2} g^{\varphi\varphi}\Big(\tfrac{\partial}{\partial\varphi} g_{\varphi\varphi} + \tfrac{\partial}{\partial\varphi} g_{\varphi\varphi} - \tfrac{\partial}{\partial\varphi} g_{\varphi\varphi} \Big) + \tfrac{1}{2} \underset{0}{\underset{\shortparallel}{g^{\varphi\theta}}} \Big(\tfrac{\partial}{\partial\varphi} g_{\varphi\theta} + \tfrac{\partial}{\partial\varphi} g_{\varphi\theta} - \tfrac{\partial}{\partial\theta} g_{\varphi\varphi} \Big) = 0$$

$$\Gamma^{\theta}_{\theta\theta} = \Gamma^{\theta}_{\theta\varphi} = \Gamma^{\theta}_{\theta\varphi} = 0$$

$$\Gamma^{\theta}_{\varphi\varphi} = \tfrac{1}{2} g^{\theta\varphi}\Big(\tfrac{\partial}{\partial\varphi} g_{\varphi\varphi} + \tfrac{\partial}{\partial\varphi} g_{\varphi\varphi} - \tfrac{\partial}{\partial\varphi} g_{\varphi\varphi} \Big) + \tfrac{1}{2} g^{\theta\theta}\Big(\tfrac{\partial}{\partial\varphi} g_{\varphi\theta} + \tfrac{\partial}{\partial\varphi} g_{\varphi\theta} - \tfrac{\partial}{\partial\theta} g_{\varphi\varphi} \Big)$$
$$= \tfrac{1}{2}\Big(-\tfrac{\partial}{\partial\theta} \cos^2\theta \Big) = -\tfrac{1}{2} 2\cos\theta\sin\theta = \cos\theta \sin\theta$$

$$\Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = \tfrac{1}{2} g^{\varphi\varphi}\Big(\tfrac{\partial}{\partial\varphi} g_{\theta\varphi} + \tfrac{\partial}{\partial\theta} g_{\varphi\varphi} - \tfrac{\partial}{\partial\varphi} g_{\varphi\theta} \Big) + \tfrac{1}{2} g^{\varphi\theta}\Big(\tfrac{\partial}{\partial\varphi} g_{\theta\theta} + \tfrac{\partial}{\partial\theta} g_{\varphi\theta} - \tfrac{\partial}{\partial\theta} g_{\varphi\theta} \Big)$$
$$= \tfrac{1}{2}\cos^{-2}\theta \Big(\tfrac{\partial}{\partial\theta} \cos^2\theta \Big) = \tfrac{1}{2}\cos^{-2}\theta\,(-2\cos\theta\sin\theta) = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

Geodesic equations (general form $\gamma''_k(t) + \sum_{k=1}^{m} \Gamma^{k}_{ij}(\gamma(t))\, \gamma'_i(t)\, \gamma'_j(t) = 0, \; k = 1,\dots,m$)

$$\Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = -\tan\theta, \quad \Gamma^{\theta}_{\varphi\varphi} = \cos\theta\sin\theta$$
$$(\varphi(t), \theta(t)) = \gamma(t)$$

- $\varphi'' + \cancel{\Gamma^{\varphi}_{\varphi\varphi}}\,\varphi'\varphi' + \Gamma^{\varphi}_{\varphi\theta}\,\varphi'\theta' + \Gamma^{\varphi}_{\theta\varphi}\,\theta'\varphi' + \cancel{\Gamma^{\varphi}_{\theta\theta}\,\theta'\theta'} = 0$
- $\theta'' + \Gamma^{\theta}_{\varphi\varphi}\,\varphi'\varphi' + \cancel{\Gamma^{\theta}_{\varphi\theta}}\,\varphi'\theta' + \cancel{\Gamma^{\theta}_{\theta\varphi}}\,\theta'\varphi' + \cancel{\Gamma^{\theta}_{\theta\theta}\,\theta'\theta'} = 0$

$$\boxed{\begin{aligned} &\varphi'' - 2\tan\theta \cdot \varphi' \theta' = 0 \\ &\theta'' + \cos\theta\sin\theta\, \varphi'\varphi' = 0 \end{aligned}}$$

It is extremely difficult to find an explicit solution of this system of ODE's. We can only guess some solutions: $\varphi(t) = \text{const.}, \ \theta(t) = t$ is a solution
$\theta(t) = 0, \ \varphi(t) = t$

meridians and equator (with given param.) are geodesics on $S^2 \Leftarrow \{$

Theorem Let M be a 2-dimensional surface in $\mathbb{R}^3$, that is, M is a 2-dimensional manifold with immersion $\iota : M \hookrightarrow \mathbb{R}^3$ which is a diffeomorphism onto its image. Then a curve $\gamma : [a,b] \to M$ is a geodesic for the Levi-Civita connection of $g_M = \iota^* g$

(where g is the Euclidean metric on $\mathbb{R}^3$)

iff $\gamma''(t) \perp T_{\gamma(t)} M$ for every $t \in [a,b]$.

(Here we are using that $T_{\gamma(t)} M \subseteq T_{\gamma(t)} \mathbb{R}^3 = \mathbb{R}^3$ via $T_{\gamma(t)} \iota : T_{\gamma(t)} M \hookrightarrow T_{\gamma(t)} \mathbb{R}^3 = \mathbb{R}^3$, $\gamma : [a,b] \longrightarrow M \subseteq \mathbb{R}^3$ and $\perp$ is w.r.t. the Euclidean metric on $\mathbb{R}^3$).

Proof: Let $\varphi : \mathcal{U} \longrightarrow \mathbb{R}^2$ be a chart on M. So that

$\varphi^{-1} : \varphi(\mathcal{U}) \longrightarrow M \subseteq \mathbb{R}^3$ is smooth. Assume that

$$(\varphi^{-1})^* g_M = \sum_{i,j=1}^{2} g_{ij}(x)\, dx_i \otimes dx_j .$$

Let us for simplicity write $\langle -,- \rangle$ instead of g on $\mathbb{R}^3$.

Then by definition

$$g_{ij}(x) = g_x \left(\frac{\partial \varphi^{-1}}{\partial x_i}(\varphi(x)), \frac{\partial \varphi^{-1}}{\partial x_j}(\varphi(x)) \right) = \left\langle \frac{\partial \varphi^{-1}}{\partial x_i}(\varphi(x)), \frac{\partial \varphi^{-1}}{\partial x_j}(\varphi(x)) \right\rangle$$

for every $i,j = 1,2$ and $x \in \mathcal{U}$. $\left(\frac{\partial \varphi^{-1}}{\partial x_i}(\varphi(x)) = T_{\varphi(x)} \varphi^{-1}\left(\frac{\partial}{\partial x_i}\right) \in \mathbb{R}^3 \right)$. Hence

$$\left(\frac{\partial}{\partial x_i} g_{jk}\right)(x) + \left(\frac{\partial}{\partial x_j} g_{ik}\right)(x) - \left(\frac{\partial}{\partial x_k} g_{ij}\right)(x) = \frac{\partial}{\partial x_i} \left\langle \frac{\partial \varphi^{-1}}{\partial x_j}(\varphi(x)), \frac{\partial \varphi^{-1}}{\partial x_k}(\varphi(x)) \right\rangle$$

$$+ \frac{\partial}{\partial x_j} \left\langle \frac{\partial \varphi^{-1}}{\partial x_i}(\varphi(x)), \frac{\partial \varphi^{-1}}{\partial x_k}(\varphi(x)) \right\rangle - \frac{\partial}{\partial x_k} \left\langle \frac{\partial \varphi^{-1}}{\partial x_i}(\varphi(x)), \frac{\partial \varphi^{-1}}{\partial x_j}(\varphi(x)) \right\rangle$$

$$= \left\langle \frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_j}(\varphi(x)), \frac{\partial \varphi^{-1}}{\partial x_k}(\varphi(x)) \right\rangle + \cancel{\left\langle \frac{\partial \varphi^{-1}}{\partial x_j}(\varphi(x)), \frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_k}(\varphi(x)) \right\rangle}$$

$$+ \left\langle \frac{\partial^2 \varphi^{-1}}{\partial x_j \partial x_i}(\varphi(x)), \frac{\partial \varphi^{-1}}{\partial x_k}(\varphi(x)) \right\rangle + \cancel{\left\langle \frac{\partial \varphi^{-1}}{\partial x_i}(\varphi(x)), \frac{\partial^2 \varphi^{-1}}{\partial x_j \partial x_k}(\varphi(x)) \right\rangle}$$

$$- \cancel{\left\langle \frac{\partial^2 \varphi^{-1}}{\partial x_k \partial x_i}(\varphi(x)), \frac{\partial \varphi^{-1}}{\partial x_j}(\varphi(x)) \right\rangle} - \cancel{\left\langle \frac{\partial \varphi^{-1}}{\partial x_i}(\varphi(x)), \frac{\partial^2 \varphi^{-1}}{\partial x_k \partial x_j}(\varphi(x)) \right\rangle}$$

$$= 2 \left\langle \frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_j}(\varphi(x)), \frac{\partial \varphi^{-1}}{\partial x_k}(\varphi(x)) \right\rangle.$$

By definition, a curve $\gamma : [a,b] \to M$ is a geodesic iff

$$\nabla_{\gamma'(t)} \gamma'(t) = 0 \quad \text{for every } t \in [a,b].$$

This is equivalent to

$$(*) \quad (g_M)_{\gamma(t)} \left(\nabla_{\gamma'(t)} \gamma'(t), X(t) \right) = 0$$

for every $X : [a,b] \to TM$, $X(t) \in T_{\gamma(t)} M$. Equation $(*)$ can be written in coordinates in the following way. Let $\varphi \circ \gamma = \tilde{\gamma}$ and $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2)$. Then $(*)$ is equivalent in coordinates

- $\nabla_{\gamma'(t)} \gamma'(t)$ is given by $\tilde{\gamma}_k''(t) + \sum_{i,j=1}^{2} \Gamma^k_{ij}(\tilde{\gamma}(t)) \tilde{\gamma}_i'(t) \tilde{\gamma}_j'(t)$

and so $(*)$ is given by

$$(**) \quad \sum_{k=1}^{2} (g_{kl})(\tilde{\gamma}(t)) \left(\tilde{\gamma}_k''(t) + \sum_{i,j=1}^{2} \Gamma^k_{ij}(\tilde{\gamma}(t)) \tilde{\gamma}_i'(t) \tilde{\gamma}_j'(t) \right) = 0, \quad l = 1,2.$$

We know that

- $(g_{kl})(\tilde{\gamma}(t)) = \left\langle \frac{\partial \varphi^{-1}}{\partial x_k}(\tilde{\gamma}(t)), \frac{\partial \varphi^{-1}}{\partial x_l}(\tilde{\gamma}(t)) \right\rangle.$
- $\sum_{k=1}^{2} g_{kl}(\tilde{\gamma}(t)) \Gamma^k_{ij}(\tilde{\gamma}(t)) = \sum_{k=1}^{2} \underbrace{g_{kl}(\tilde{\gamma}(t)) \left(\frac{1}{2} \sum_{u=1}^{2} g^{uk}(\tilde{\gamma}(t)) m_{iju}(\tilde{\gamma}(t)) \right.}_{\delta_{lu}}$

$$\left(m_{iju}(\tilde{\gamma}(t)) = 2 \left\langle \frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_j}(\varphi(x)), \frac{\partial \varphi^{-1}}{\partial x_u}(\varphi(x)) \right\rangle \right)$$

$$= \frac{1}{2} m_{ijl}(\tilde{\gamma}(t)) = \left\langle \frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_j}(\tilde{\gamma}(t)), \frac{\partial \varphi^{-1}}{\partial x_l}(\tilde{\gamma}(t)) \right\rangle.$$

We get that $(**)$ can be rewritten as

$$\sum_{k=1}^{2} \left\langle \frac{\partial \varphi^{-1}}{\partial x_k}(\tilde{\gamma}(t)), \frac{\partial \varphi^{-1}}{\partial x_l}(\tilde{\gamma}(t)) \right\rangle \tilde{\gamma}_k''(t) + \sum_{i,j=1}^{2} \left\langle \frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_j}(\tilde{\gamma}(t)), \frac{\partial \varphi^{-1}}{\partial x_l}(\tilde{\gamma}(t)) \right\rangle \tilde{\gamma}_i'(t) \tilde{\gamma}_j'(t)$$

$$= \left\langle \sum_{k=1}^{2} \frac{\partial \varphi^{-1}}{\partial x_k}(\tilde{\gamma}(t)) \tilde{\gamma}_k''(t) + \sum_{i,j=1}^{2} \frac{\partial^2 \varphi^{-1}}{\partial x_i \partial x_j}(\tilde{\gamma}(t)) \tilde{\gamma}_i'(t) \tilde{\gamma}_j'(t), \frac{\partial \varphi^{-1}}{\partial x_l}(\tilde{\gamma}(t)) \right\rangle =$$

$$= \left\langle \frac{d^2}{dt^2}(\varphi^{-1} \circ \tilde{\gamma})(t), \frac{\partial \varphi^{-1}}{\partial x_l}(\tilde{\gamma}(t)) \right\rangle = \left\langle \left(\frac{d^2}{dt^2} \gamma \right)(t), \frac{\partial \varphi^{-1}}{\partial x_l}(\tilde{\gamma}(t)) \right\rangle$$

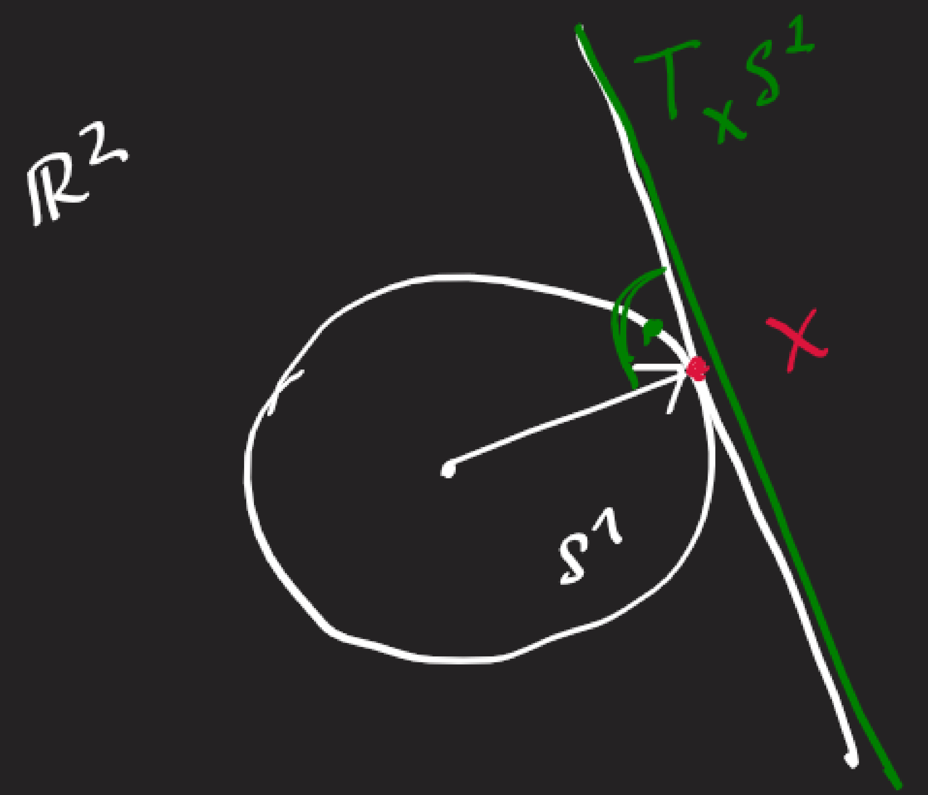
$= \left\langle \gamma''(t), \frac{\partial \varphi^{-1}}{\partial x_l}(\tilde{\gamma}(t)) \right\rangle = 0$, $l = 1,2$. But $T_{\gamma(t)} M$ is spanned by

$\frac{\partial \varphi^{-1}}{\partial x_1}(\tilde{\gamma}(t))$ and $\frac{\partial \varphi^{-1}}{\partial x_2}(\tilde{\gamma}(t))$, and so $(**)$ precisely means that $\gamma''(t)$ is orthogonal to $T_{\gamma(t)}M$ for every $t \in [a,b]$. $\square$

Remark: Note that the proof goes through for an m-dimensional manifold in $\mathbb{R}^k$ with $k > m$. Theorem is true for any m-dimensional submanifold in $\mathbb{R}^k$.

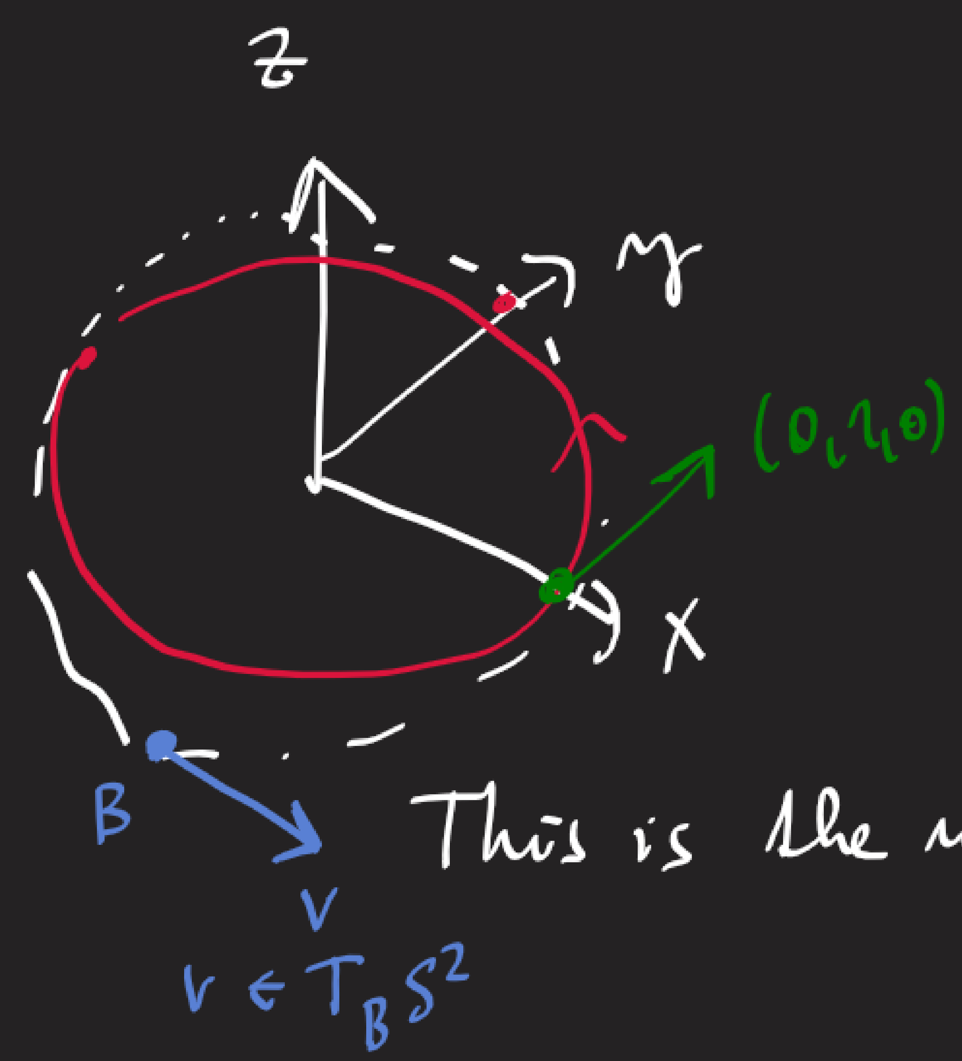
Geodesics on sphere S^2

We know from the previous Theorem that any geodesic γ on S^2 for the Levi-Civita connection of the round metric g_{S^2} is characterized by the fact that $\gamma''(t)$ is orthogonal to $T_{\gamma(t)}S^2$. But we know that $T_{\gamma(t)}S^2$ is the orthogonal complement to $\gamma(t)$ in $\mathbb{R}^3$.



And so $\gamma''(t) \perp T_{\gamma(t)}S^2 \Leftrightarrow$

$\gamma''(t)$ is proportional to $\gamma(t)$, that is, $\gamma''(t)$ is a multiple of $\gamma(t)$ for every t.



Let $\gamma : [-\pi, \pi] \longrightarrow \mathbb{R}^3$

$\gamma(t) = (\cos t, \sin t, 0)$.

$\gamma'(t) = (-\sin t, \cos t, 0)$

$\gamma''(t) = (-\cos t, -\sin t, 0) = -\gamma(t)$.

We see that γ is indeed a geodesic on S^2.

This is the unique geodesic that satisfies: $\gamma(0) = (1,0,0)$
$\gamma'(0) = (0,1,0)$.

Note that the maximal solution is rather defined on $\mathbb{R}$.

If $A \in O(3)$ is an orthogonal matrix ($A \in M_{3\times 3}(\mathbb{R}) = M(3,\mathbb{R})$, $A^T = A^{-1}$), then we know that $\langle Au, Av \rangle = \langle u, v \rangle$ for every $u, v \in \mathbb{R}^3$, and so $A \circ \gamma$ is also a geodesic on S^2 with $(A\circ\gamma)(0) = A(1,0,0)$
$(A\circ\gamma)'(0) = A(0,1,0)$.

Given $B \in S^2$ and $v \in T_B S^2$, then we can find $A \in O(3)$ such that $A(1,0,0) = B$, $v = A(0,1,0)$.

Thus all geodesics on S^2 are great circles (intersection of S^2 with a 2-dimensional vector subspace in $\mathbb{R}^3$) with a parametrization with constant velocity.