Primus Research Program:
The interaction of solids and fluids

Sebastian Schwarzacher

Faculty of Mathematics and Physics, Charles University, Prague

Project description

Fluid solid interaction happens in many everyday instances. For example in the trachea where the air-flow inside the windpipe interacts with the cartilage surrounding it. The project aims to **systematically develop an analysis for the related theory of partial differential equations.** The challenge is sub-summarized in the free interface between the solid and the fluid – the variable domain. Our objective is to advance the available mathematical theory in fluid-structure interaction towards the realistic case where both fluid and solid are three-dimensional objects.

B. Benešová

Our Team

- **2 Senior researchers**: S. Schwarzacher (PI) and B. Benešová (expertise: Analysis for solid mechanics).
- 4 Postdocs: G. Gravina (Calculus of variations), M. Kampschulte (Analysis for solid mechanics), G. Sperone (PDEs for fluid-mechanics) and B. She (Numerics).
- **3 PhD students:** C. Mîndrilă, J. Scherz and A. Češík.

S. Schwarzacher

Setup

We consider $\Omega = \Omega_f \cup \Omega_s$, where Ω is the complete (Eulerian) domain under investigation. The solid will be situated on Ω_s which can be a one, two or three dimensional set, and the fluid will be contained in Ω_f. The geometry of Ω_s is a *part of the solution* and variable in time.
The solid is characterized by a coordinate mapping $\eta : \omega \to \Omega$, where ω is the reference geometry that can be seen as a relaxed state or a state of minimal elastic energy $K(\eta)$ depending on various quantities that are frame-independent like the symmetric gradient or the Jacobian of the deformation.
The fluid is characterized by its velocity $v : \Omega_f \to \mathbf{R}^3$ and its pressure $p : \Omega_f \to \mathbf{R}$. In the compressible and/or heat conducting setting the pressure is usually characterized by the density and temperature of the fluid.

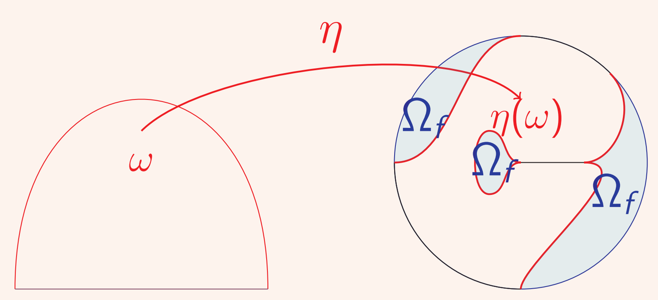


Figure: Sketch of the elastic deformation of a bulk.

PDEs

We call (η, v, p) a weak solution if (in the incompressible case) the fluid velocity satisfies: $\operatorname{div}(v) = 0$ and

the **coupled weak momentum equation** is satisfied:

$$\begin{aligned}\partial_t\Big(\rho_s \int_\omega \partial_t\eta \cdot b \,dx + \rho_f \int_{\Omega_f(t)} v\cdot\phi \,dy\Big) - \rho_s\int_\omega \partial_t\eta\cdot\partial_t b\,dx - \rho_f\int_{\Omega_f(t)} v\cdot\partial_t\phi + v\otimes v\cdot\nabla\phi\,dy \\ + \mu\int_{\Omega_f(t)} \nabla v\cdot\nabla\phi - p\phi\,dy + \langle K'(\eta), b\rangle_\omega = 0\end{aligned} \tag{1}$$

for all (b, ϕ) with $\phi(t, y) = b(t, \eta^{-1}(y))$ for all $y \in \partial\Omega_f \cap \overline{\Omega}_s$ in the distributional sense in time.

General difficulties

- Challenges of the fluid:
  - Sensitive reaction to boundary shape (which changes in time).
  - Non-uniqueness (Navier-Stokes theorem), non-smoothness in (3D) and for various models.
- Challenges for the solid:
  - In general not much regularity can be expected. In particular in the unsteady situation.
  - Self-touching implies that the Euler-Lagrange equation is only locally satisfied.
- Challenges of the interaction:
  - Mismatch of Eulerian regime (fluid) and Lagrangian regime (solid).
  - Self-touching of the domain implies that the fluid domain splits.

Completed projects & Current projects

Papers sent to a journal:

1. C. Mîndrilă, S. Schwarzacher: Existence of steady very weak solutions to Navier-Stokes equations with non-Newtonian stress tensors, (2019), *Preprint: arXiv:1911.02055*.
2. O. Saari, S. Schwarzacher: A reverse Hölder inequality for the gradient of solutions to Trudinger's equation, (2019), *Preprint: arXiv:1910.10498* .
3. F. Rindler, S. Schwarzacher, J. L. Velázquez: Two-speed solutions to non-convex rate-independent systems, (2019), *Preprint: arXiv:1907.05035* .
4. B. Muha, S. Schwarzacher: Existence and regularity for weak solutions for a fluid interacting with a non-linear shell in 3D, (2019).
5. D. Breit, A. Cianchi, L. Diening, S. Schwarzacher: Global Schauder estimates for the p-Laplace system, (2019), *Preprint: arXiv:1903.12496* .

Papers in preparation (WIP):

1. B Benešová, M. Kampschulte, S. Schwarzacher: A variational approach to fluid-structure interactions.
2. D. Breit, S. Schwarzacher: Compressible heat-conducting fluids interacting with a nonlinear-elastic shell
3. C. Mîndrilă, S. Schwarzacher: Existence of periodic solutions to fluid structure interaction.
4. S. Schwarzacher, B. She: Stability and consistency for numerical solutions to fluid-structure interactions involving compressible fluids.
5. S. Schwarzacher, M. Srocinski: Weak strong uniqueness for an elastic plate interacting with the Navier Stokes equation.

Plans for 2020/21

For the next year we aim to proceed by investigating the contact of elastic structures in order to start the research on the mathematical description of bouncing of elastic objects in viscous fluids. The benefits have impacts on all parts of the proposal:

- Global existence of solutions by excluding contact (via regularity).
- Existence of fluid-structure interaction including solid deformations in all directions and/or compression.
- Mathematical quantification of bouncing effects between elastic solids in a viscous liquid.
- The development of strategies to simulate contact and boincing in fluid-structure interaction.
- Convergence results of numerical schemes (using the developed analysis).

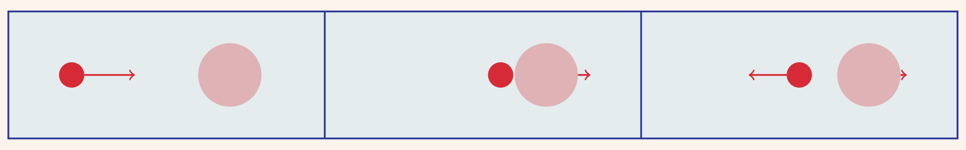
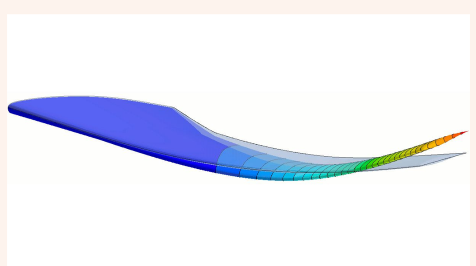


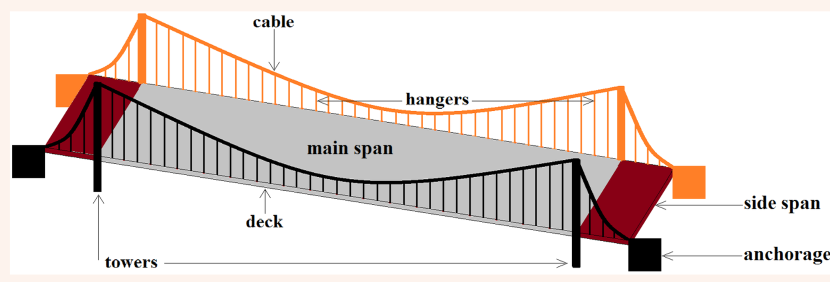
Figure: Contact and bouncing of two solid objects in a liquid

Further aims: Calculating the lift

An open problem is to give a mathematical description of the lift coming from a fluid/wind effecting structures.

Example 1: Airplane wing, Example 2: Suspension bridges:





Question 1: A fluid is blowing constantly at an airplane wing. How can the force of the wind acting on the wing be captured mathematically? What shape of the wing is (mathematically) optimizing the lift?

Question 2: A bridge can oscillate in orthogonal direction to the blowing wind. Do unsteady periodic (oscillating) solutions exist?

Challenges for future projects

- Non-linear terms involving the inverse of the determinant of the Jacobian.
- The paradox of no-contact of elastic solids in incompressible fluids.
- Low regularity of the fluid and the solid (particular in 3 dimensions).
- Compression phenomena for solid deformations and the potential vanishing of the determinant of the Jacobian.

Summary 2019

In the first year significant progress was made in three of the announced projects of the grant:

A: The existence theory for 3-dimensional elastic deformations in a three dimensional fluid was developed *via the methods of the calculus of variations*. Both the embedding of this theory into the frame-work of fluid-structure interaction, as well as the existence result itself are without prefiguration (see WIP 1).

B: The existence theory for compressible heat-conducting fluids was developed being the first result for weak solutions in that setting (see WIP 2).

C: The weak-strong uniqueness could shown to be satisfied for Koiter plates interacting with incompressible fluids (see WIP 5). This is the first weak-strong uniqueness result in 3 dimensions for FSI involving elastic deformable structures.

Further the preprint on non-linear elastic plate deformations in collaboration with B. Muha has been sent for publication (see Preprint 4).

Kontakt: S. Schwarzacher: schwarz@karlin.mff.cuni.cz or B. Benešová: benesova@karlin.mff.cuni.cz