Písemka 10. 3. 2020

(1) $a_n = \frac{1}{\pi}\int_{-\pi}^{0} \cos x \cos nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx \, dx =$

$= \frac{1}{2\pi}\left(\int_{-\pi}^{0} (\cos(n+1)x + \cos(n-1)x)\,dx + \int_0^\pi (\sin(n+1)x - \sin(n-1)x)\,dx\right) =$

$n \neq 1: = \frac{1}{2\pi}\left( \underbrace{\left[\frac{\sin(n+1)x}{n+1}\right]_{-\pi}^{0}}_{=0} + \underbrace{\left[\frac{\sin(n-1)x}{n-1}\right]_{-\pi}^{0}}_{=0} - \left[\frac{\cos(n+1)x}{n+1}\right]_0^\pi + \left[\frac{\cos(n-1)x}{n-1}\right]_0^\pi \right)$

$= \frac{1}{2\pi}\left( \frac{1-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}-1}{n-1} \right) = \frac{1}{2\pi}\, \frac{n-1-(-1)^{n+1}(n-1)+(-1)^{n-1}(n+1)-n-1}{n^2-1}$

$= \frac{1}{2\pi}\, \frac{-2 \quad -2\,(-1)^n}{n^2-1} = -\frac{1}{\pi}\, \frac{1+(-1)^n}{n^2-1}$

$a_1 = \frac{1}{2\pi}\left(\int_{-\pi}^{0} (\cos 2x + 1)\,dx + \int_0^\pi \sin 2x \, dx\right)$

$= \frac{1}{2\pi}\left( \underbrace{\left[\frac{\sin 2x}{2}\right]_{-\pi}^{0}}_{=0} + \pi - \underbrace{\left[\frac{\cos 2x}{2}\right]_0^\pi}_{=0} \right) = \frac{1}{2}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{0} \cos x \sin nx \, dx + \frac{1}{\pi}\int_0^\pi \sin x \sin nx \, dx =$

$= \frac{1}{2\pi}\left(\int_{-\pi}^{0} (\sin(n+1)x + \sin(n-1)x)\,dx + \int_0^\pi (\cos(n-1)x - \cos(n+1)x)\,dx\right)$

$n \neq 1: = \frac{1}{2\pi}\left( -\left[\frac{\cos(n+1)x}{n+1}\right]_{-\pi}^{0} - \left[\frac{\cos(n-1)x}{n-1}\right]_{-\pi}^{0} + \underbrace{\left[\frac{\sin(n-1)x}{n-1}\right]_0^\pi}_{=0} - \underbrace{\left[\frac{\sin(n+1)x}{n+1}\right]_0^\pi}_{=0} \right)$

$= \frac{1}{2\pi}\left( \frac{(-1)^{n+1}-1}{n+1} - \frac{1-(-1)^{n-1}}{n-1} \right) = \frac{1}{2\pi}\, \frac{(-1)^{n+1}(n-1)-n+1-n-1+(-1)^{n-1}(n+1)}{n^2-1}$

$= \frac{1}{2\pi}\, \frac{-2n \quad +(-1)^{n+1}2n}{n^2-1} = -\frac{1}{\pi}\, \frac{n(1+(-1)^n)}{n^2-1}$

$b_1 = \frac{1}{2\pi}\left(\int_{-\pi}^{0} \sin 2x \, dx + \int_0^\pi (1-\cos 2x)\,dx\right) =$

$= \frac{1}{2\pi}\left( -\underbrace{\left[\frac{\cos 2x}{2}\right]_{-\pi}^{0}}_{=0} + \pi - \underbrace{\left[\frac{\sin 2x}{2}\right]_0^\pi}_{=0} \right) = \frac{1}{2}$

$F(x) = \frac{1}{\pi} + \frac{1}{2}\cos x + \frac{1}{2}\sin x + \sum_{n=2}^{\infty}\left( -\frac{1}{\pi}\frac{1+(-1)^n}{n^2-1}\cos nx - \frac{1}{\pi}\frac{n(1+(-1)^n)}{n^2-1}\sin nx \right)$

$= \frac{1}{\pi} + \frac{1}{2}\cos x + \frac{1}{2}\sin x + \sum_{k=1}^{\infty}\left( -\frac{1}{\pi}\frac{2}{4k^2-1}\cos 2kx - \frac{1}{\pi}\frac{4k}{4k^2-1}\sin 2kx \right)$

$F(x) = \begin{cases} f(x), & x \in (-\pi,0)\cup(0,\pi) \\ 1/2, & x = 0 \\ -1/2, & x = \pm\pi \end{cases}$

$x = 0: \quad \frac{1}{2} = \frac{1}{\pi} + \frac{1}{2} + \sum_{k=1}^{\infty}\left(-\frac{1}{\pi}\frac{2}{4k^2-1}\right) \Rightarrow \sum_{k=1}^{\infty}\frac{1}{4k^2-1} = -\frac{1}{\pi}\left(-\frac{\pi}{2}\right) = \frac{1}{2}$

$x = \frac{\pi}{2}: \quad 1 = \frac{1}{\pi} + \frac{1}{2} + \sum_{k=1}^{\infty}\left(-\frac{1}{\pi}\frac{2}{4k^2-1}(-1)^k\right)$

$\underbrace{\left(\frac{1}{\pi} - \frac{1}{2}\right)\frac{\pi}{2}}_{\frac{1}{2}-\frac{\pi}{4}} = \sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1}$

② $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \cos x \, e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos x \, e^{(1-in)x}\,dx =$$

$$= \frac{1}{2\pi}\Big( \underbrace{\big[\sin x \, e^{(1-in)x}\big]_{-\pi}^{\pi}}_{=0} - \int_{-\pi}^{\pi} \sin x \,(1-in)\, e^{(1-in)x}\,dx \Big) =$$

$$= \frac{-(1-in)}{2\pi}\Big( \underbrace{\big[-\cos x \, e^{(1-in)x}\big]_{-\pi}^{\pi}}_{-e^{(1-in)(-\pi)} + e^{(1-in)\pi}} + (1-in)\underbrace{\int_{-\pi}^{\pi} \cos x \, e^{(1-in)x}\,dx}_{2\pi c_n} \Big)$$

$$c_n = \frac{-(1-in)}{2\pi}\left(-e^{-\pi}e^{in\pi} + e^{\pi}e^{-in\pi}\right) - (1-in)^2 c_n$$

$$c_n(1+1-2in-n^2) = \frac{-(1-in)}{2\pi}\left(-e^{-\pi}+e^{\pi}\right)(-1)^n$$

$$c_n = \frac{(1-in)\left(e^{-\pi}-e^{\pi}\right)(-1)^n}{2\pi(2-2in-n^2)}$$

$$F(x) = \sum_{n\in\mathbb{Z}} \frac{(1-in)\left(e^{-\pi}-e^{\pi}\right)(-1)^n}{2\pi(2-2in-n^2)}\, e^{inx}$$

$$F(x) = \begin{cases} f(x), & x \in (-\pi,\pi) \\ \dfrac{-e^{\pi}-e^{-\pi}}{2}, & x = \pm\pi \end{cases}$$