# Intrinsically Defined Curves and Special Functions

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The curvature of a plane curve measures the rate of rotation of the unit tangent vector as we move along the curve. In particular, the curvature of a unit speed curve at a given point equals the instantaneous angular velocity of the tangent vector; positive curvature corresponds to a counterclockwise rotation, and negative to a clockwise rotation. Conversely, the curvature function contains enough information to recover the original curve up to translation and rotation. In this case, the curve is said to be defined intrinsically.

In the present article, we show that even for the simplest curvature functions (power functions and the sine function), the problem of reconstructing the curve leads to integrals that cannot be evaluated in terms of elementary functions. Still, it is possible to study both qualitative and quantitative properties of the corresponding curves.

## Preliminaries

We start by recalling some basic facts from differential geometry. (See, for example, Pressley's text [**9**].) A plane curve is a differentiable mapping $\boldsymbol{c} : I \to \mathbb{R}^2$, where $I$ is an interval on the real line. We restrict our attention to unit speed curves, for which $\|\boldsymbol{c}'(s)\| = 1$ for every $s \in I$. The unit tangent vector is defined as $\boldsymbol{T}(s) = \boldsymbol{c}'(s)$. Let $\alpha : I \to \mathbb{R}$ be a continuous function such that $\boldsymbol{T}(s) = (\cos\alpha(s), \sin\alpha(s))$ for every $s \in I$ (in other words, $\alpha(s)$ measures the oriented angle between $\boldsymbol{T}(s)$ and the horizontal axis). Then the curvature $\kappa$ of $\boldsymbol{c}$ satisfies $\kappa(s) = \alpha'(s)$ for every $s \in I$. (This formula can even serve as the definition of curvature.)

Given a continuous function $\kappa : I \to \mathbb{R}$, how do we find a unit speed curve $\boldsymbol{c} : I \to \mathbb{R}^2$ whose curvature at $\boldsymbol{c}(s)$ is $\kappa(s)$? Let $\boldsymbol{c}(s) = (x(s), y(s))$, so that $x(s)$, $y(s)$ are the components of the unknown function $\boldsymbol{c}$. Combining the relations $\alpha'(s) = \kappa(s)$ and $\boldsymbol{T}(s) = (\cos\alpha(s), \sin\alpha(s))$, we obtain the following system of differential equations:

$$\alpha'(s) = \kappa(s), \quad x'(s) = \cos\alpha(s), \quad y'(s) = \sin\alpha(s).$$

The solution

$$\alpha(s) = \int \kappa(s)\,\mathrm{d}s, \quad x(s) = \int \cos\alpha(s)\,\mathrm{d}s, \quad y(s) = \int \sin\alpha(s)\,\mathrm{d}s,$$

depends on the choice of integration constants, which correspond to translation and rotation of the curve. Throughout this paper we will assume the initial conditions $x(0) = y(0) = \alpha(0) = 0$. With these conditions,

$$\alpha(s) = \int_0^s \kappa(u)\,\mathrm{d}u, \quad x(s) = \int_0^s \cos\alpha(u)\,\mathrm{d}u, \quad y(s) = \int_0^s \sin\alpha(u)\,\mathrm{d}u.$$

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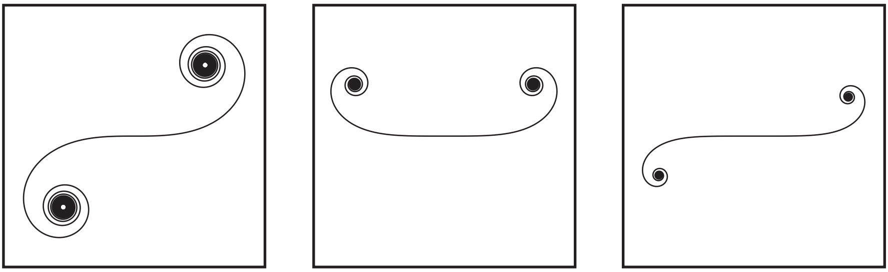

**Figure 1** Curves with curvature $\kappa(s) = s^n$, $n \in \{1, 2, 3\}$

## Power functions

With the formulas for recovering a curve from its curvature at hand, we can start experimenting with various choices of the function $\kappa$. An obvious first choice to try is $\kappa(s) = s^n$, where $n$ is a nonnegative integer. It is not too surprising that $n = 0$ gives $\alpha(s) = s$, $x(s) = \sin s$, $y(s) = -\cos s$, i.e., a circle of unit radius. For $n \geq 1$, we obtain $\alpha(s) = s^{n+1}/(n+1)$, but

$$x(s) = \int_0^s \cos \frac{u^{n+1}}{n+1}\, \mathrm{d}u, \quad y(s) = \int_0^s \sin \frac{u^{n+1}}{n+1}\, \mathrm{d}u$$

cannot be integrated in terms of elementary functions.

FIGURE 1 shows computer-generated plots of the curves corresponding to $n = 1, 2, 3$. Our plots were created in *Mathematica*, but any other software capable of numerical integration can serve the same purpose. Wolfram Research hosts a freely available interactive demonstration that shows plots like these for some other curvature funtions [**4**].

For $n = 1$, the curve is known as the Euler spiral, Cornu spiral, or clothoid. It has a long history, which can be traced back to Jacob Bernoulli's 1694 treatise *Curvatura Laminae Elasticae*. Besides the theory of elasticity, clothoids are useful in the construction of railroads and freeways (since they enable a smooth transition between straight and circular sections), and also show up in diffraction theory. The history and applications are described by Levien [**7**].

Is there anything else we can say about these spirals? One obvious fact is their symmetry: Each spiral consists of two parts, which are centrally symmetric with respect to the origin for odd values of $n$; for even values of $n$, we have two parts that are symmetric with respect to the vertical axis.

When $s \to \infty$, each spiral seems to approach a certain point; let us call it the center of the spiral (the other center corresponds to $s \to -\infty$). The coordinates of the center are

$$x_\infty = \lim_{s\to\infty} x(s) = \int_0^\infty \cos \frac{u^{n+1}}{n+1}\, \mathrm{d}u,$$
$$y_\infty = \lim_{s\to\infty} y(s) = \int_0^\infty \sin \frac{u^{n+1}}{n+1}\, \mathrm{d}u.$$

When we ask *Mathematica* to calculate these two integrals, we learn that

$$x_\infty = \Gamma\left(1 + \frac{1}{n+1}\right) \sqrt[n+1]{n+1}\, \cos \frac{\pi}{2(n+1)}, \tag{1}$$

$$y_\infty = \Gamma\left(1 + \frac{1}{n+1}\right) \sqrt[n+1]{n+1}\, \sin \frac{\pi}{2(n+1)}, \tag{2}$$

where $\Gamma$ is the gamma function, which is defined by $\Gamma(w) = \int_0^\infty t^{w-1}e^{-t}\,\mathrm{d}t$ and satisfies the recurrence $\Gamma(w+1) = w\,\Gamma(w)$ when $w > 0$. In particular, when $n = 1$, we have $x_\infty = y_\infty = \Gamma(3/2) = 1/2\,\Gamma(1/2) = \sqrt{\pi}/2$. Where did the values in (1) and (2) come from and how can we verify their correctness?

Most complex analysis textbooks show how to derive the formulas

$$\int_0^\infty \cos t^2\,\mathrm{d}t = \int_0^\infty \sin t^2\,\mathrm{d}t = \sqrt{\frac{\pi}{8}}$$

by means of Cauchy's integral theorem. These integrals are known as the Fresnel integrals [**11**]. A linear change of variables then gives

$$\int_0^\infty \cos\frac{u^2}{2}\,\mathrm{d}u = \int_0^\infty \sin\frac{u^2}{2}\,\mathrm{d}u = \frac{\sqrt{\pi}}{2},$$

which agrees with the above-mentioned values of $x_\infty$ and $y_\infty$. The coordinates of the clothoid's center were first obtained by Euler in 1781. (See [**5**]; according to Levien [**7**], it took Euler 38 years to discover a method of calculating these two improper integrals.)

The situation is slightly more complicated when $n$ is a general positive integer. By applying the change of variables $t = u^{n+1}/(n+1)$, or $u = \sqrt[n+1]{n+1}\,t^{\frac{1}{n+1}}$, we get

$$x_\infty = \int_0^\infty \cos\frac{u^{n+1}}{n+1}\,\mathrm{d}u = \frac{\sqrt[n+1]{n+1}}{n+1}\int_0^\infty t^{\frac{1}{n+1}-1}\cos t\,\mathrm{d}t, \tag{3}$$

$$y_\infty = \int_0^\infty \sin\frac{u^{n+1}}{n+1}\,\mathrm{d}u = \frac{\sqrt[n+1]{n+1}}{n+1}\int_0^\infty t^{\frac{1}{n+1}-1}\sin t\,\mathrm{d}t. \tag{4}$$

These improper integrals represent real and imaginary parts of $\int_0^\infty t^{w-1}e^{-it}\,\mathrm{d}t$, where $w = \frac{1}{n+1}$, and it is known that

$$\int_0^\infty t^{w-1}e^{-it}\,\mathrm{d}t = \Gamma(w)\,e^{-\pi i w/2}, \quad w \in (0,1). \tag{5}$$

Let us pause for a moment to explain why the last formula is true. Complex analysis comes to the rescue again: We integrate the function $g(z) = z^{w-1}e^{-z}$ along the two paths shown in FIGURE 2; by Cauchy's integral theorem, we have

$$\int_{c_1+c_2} g(z)\,\mathrm{d}z = \int_{c_3+c_4} g(z)\,\mathrm{d}z.$$

What happens if $r \to 0$ and $R \to \infty$? We see that

$$\lim_{r\to 0,\,R\to\infty}\int_{c_1} g(z)\,\mathrm{d}z = \int_0^\infty t^{w-1}e^{-t}\,\mathrm{d}t = \Gamma(w),$$

$$\lim_{r\to 0,\,R\to\infty}\int_{c_4} g(z)\,\mathrm{d}z = \int_0^\infty (it)^{w-1}e^{-it}i\,\mathrm{d}t = i^w\int_0^\infty t^{w-1}e^{-it}\,\mathrm{d}t.$$

Further, it can be shown that both $\int_{c_3} g(z)\,\mathrm{d}z$ and $\int_{c_2} g(z)\,\mathrm{d}z$ approach zero. (In the first case, note that $|g(re^{i\varphi})| < r^{w-1}$, $\varphi \in [0, \pi/2]$. In the second case, use the estimates $|g(Re^{i\varphi})| < R^{w-1}e^{-R\cos\varphi}$ and $\cos\varphi \ge 1 - 2\varphi/\pi$, $\varphi \in [0, \pi/2]$; for more details, see [**10**, p. 52]). Equation (5) then follows by writing $i^w$ as $(e^{i\pi/2})^w$.

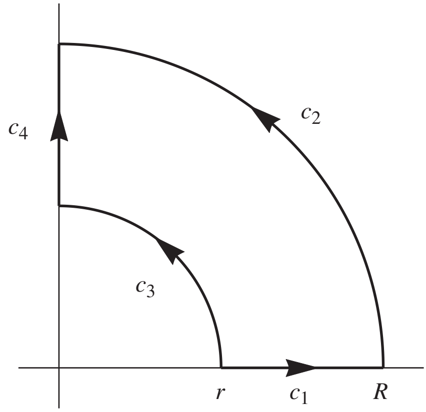


**Figure 2** Quarter of an annulus in the complex plane

Now, back to our original problem. Decomposition of (5) into real and imaginary parts leads to

$$\int_0^\infty t^{w-1} \cos t \, \mathrm{d}t = \Gamma(w) \cos \frac{\pi w}{2},$$

$$\int_0^\infty t^{w-1} \sin t \, \mathrm{d}t = \Gamma(w) \sin \frac{\pi w}{2}.$$

We now substitute $w = \frac{1}{n+1}$ and use (3) and (4) to obtain the final result:

$$\begin{aligned} x_\infty &= \Gamma\left(\frac{1}{n+1}\right) \frac{\sqrt[n+1]{n+1}}{n+1} \cos \frac{\pi}{2(n+1)} \\ &= \Gamma\left(1 + \frac{1}{n+1}\right) \sqrt[n+1]{n+1} \cos \frac{\pi}{2(n+1)}, \\ y_\infty &= \Gamma\left(\frac{1}{n+1}\right) \frac{\sqrt[n+1]{n+1}}{n+1} \sin \frac{\pi}{2(n+1)} \\ &= \Gamma\left(1 + \frac{1}{n+1}\right) \sqrt[n+1]{n+1} \sin \frac{\pi}{2(n+1)}. \end{aligned}$$

## Sine function

After $\kappa(s) = s^n$, it is natural to try other elementary functions. A particularly nice family of curves is obtained by letting $\kappa(s) = a \sin s$, where $a > 0$ is a parameter. Then

$$\alpha(s) = \int_0^s a \sin u \, \mathrm{d}u = a - a \cos s,$$

which leads to the parametrization

$$x(s) = \int_0^s \cos(a - a \cos u) \, \mathrm{d}u,$$

$$y(s) = \int_0^s \sin(a - a \cos u) \, \mathrm{d}u.$$

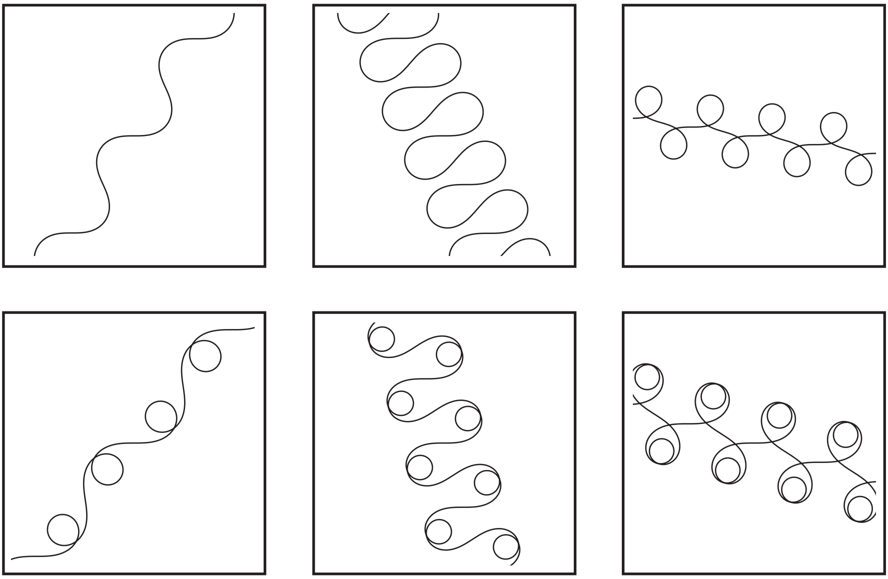

**Figure 3** Curves with curvature $\kappa(s) = a \sin s$, $a \in \{1, 2, 3, 4, 5, 6\}$

We are unable to express these integrals in terms of elementary functions, and have to resort to numerical integration again.

FIGURE 3 shows the curves corresponding to $\kappa(s) = a \sin s$, where $a$ is an integer. (The plots do not have the same scale.) More experimentation with various choices of $a$ (perhaps with the interactive demonstration [**4**]) reveals that a value near to 2.4 leads to a closed figure-eight shaped curve (the first plot in FIGURE 4). Other values of $a$ which give closed curves can be found near 5.52, 8.65 (second and third plots in FIGURE 4), 11.79, and 14.93.

This raises some natural questions:

- Can we prove rigorously that these curves are indeed closed?
- What are the magical values of $a$ that produce these closed curves, and are there infinitely many of them? If yes, how fast do they grow?

For $s \in [0, \pi]$, the curvature is nonnegative and the tangent vector turns counterclockwise. This corresponds to the loops on the right sides of the curves in FIGURE 4. The left loops must be symmetric because $\kappa(s + \pi) = -\kappa(s)$. Consequently, the curve $\boldsymbol{c}$ is closed if and only if $\boldsymbol{c}(\pi) = \boldsymbol{c}(0) = 0$, and our task is to find the values of $a$ such

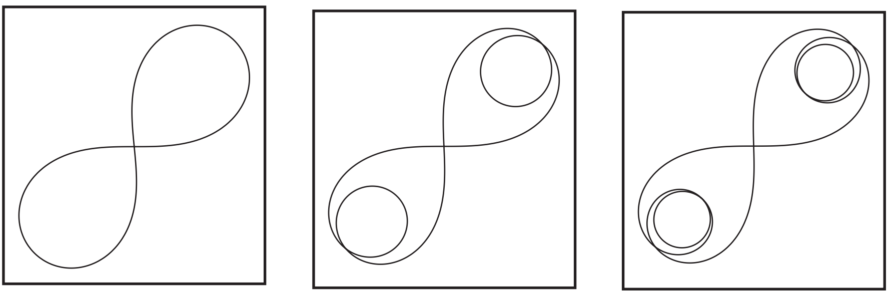

**Figure 4** Closed curves with $\kappa(s) = a \sin s$, for $a$ close to 2.40, 5.52, and 8.65

that

$$0 = x(\pi) = \int_0^\pi \cos(a - a\cos u)\,\mathrm{d}u,$$

$$0 = y(\pi) = \int_0^\pi \sin(a - a\cos u)\,\mathrm{d}u.$$

Applying the difference identities for the sine and cosine, we see that we need

$$0 = \cos a \int_0^\pi \cos(a\cos u)\,\mathrm{d}u + \sin a \int_0^\pi \sin(a\cos u)\,\mathrm{d}u,$$

$$0 = \sin a \int_0^\pi \cos(a\cos u)\,\mathrm{d}u - \cos a \int_0^\pi \sin(a\cos u)\,\mathrm{d}u.$$

However, by making the change of variables $u = \pi - v$, we obtain

$$\int_{\pi/2}^\pi \sin(a\cos u)\,\mathrm{d}u = \int_0^{\pi/2} \sin(a\cos(\pi - v))\,\mathrm{d}v = -\int_0^{\pi/2} \sin(a\cos v)\,\mathrm{d}v,$$

so that $\int_0^\pi \sin(a\cos u)\,\mathrm{d}u = 0$ for all $a \in \mathbb{R}$. Consequently,

$$x(\pi) = \cos a \int_0^\pi \cos(a\cos u)\,\mathrm{d}u,$$

$$y(\pi) = \sin a \int_0^\pi \cos(a\cos u)\,\mathrm{d}u.$$

Thus, the function

$$F(a) = \int_0^\pi \cos(a\cos u)\,\mathrm{d}u$$

plays a key role in the solution of our problem: the curve is periodic if and only if $F(a) = 0$.

When we ask *Mathematica* to calculate $F(a)$, we learn that

$$F(a) = \pi J_0(a),$$

where $J_0$ is the zeroth-order Bessel function. The history of Bessel functions goes back to Daniel Bernoulli [**13**]; before Bessel, these functions were usually defined as the sums of infinite series. The definition used by Bessel in his 1824 paper [**3**] is

$$J_n(a) = \frac{1}{\pi} \int_0^\pi \cos(nu - a\sin u)\,\mathrm{d}u, \tag{6}$$

where $J_n$ is the $n$th-order Bessel function. Although modern textbooks usually define Bessel functions in a different way (we will return to this topic shortly), the formula (6) can be found in various compendia devoted to special functions [**1, 6, 8, 13**]. For $n = 0$, we have

$$J_0(a) = \frac{1}{\pi} \int_0^\pi \cos(a\sin u)\,\mathrm{d}u,$$

which looks slightly different from $\frac{1}{\pi}F(a)$. However,

$$F(a) = \int_0^\pi \cos(a\cos u)\,\mathrm{d}u = \int_0^{\pi/2} \cos(a\cos u)\,\mathrm{d}u + \int_{\pi/2}^\pi \cos(a\cos u)\,\mathrm{d}u$$

$$= \int_{\pi/2}^{\pi} \cos(a\cos(v - \pi/2))\,\mathrm{d}v + \int_{0}^{\pi/2} \cos(a\cos(v + \pi/2))\,\mathrm{d}v$$

$$= \int_{\pi/2}^{\pi} \cos(a\sin v)\,\mathrm{d}v + \int_{0}^{\pi/2} \cos(a\sin v)\,\mathrm{d}v = \int_{0}^{\pi} \cos(a\sin v)\,\mathrm{d}v = \pi J_0(a),$$

which proves that $F(a) = \pi J_0(a)$. Indeed, the graph of $J_0$ (FIGURE 5) confirms that the zeros of this function are exactly those values of $a$ that we found experimentally. The calculation of these zeros is a delicate problem; the basic idea is to isolate each zero in a suitable interval and then use a numerical root-finding algorithm. Fortunately, modern software packages such as *Mathematica* have built-in commands to calculate the zeros to any prescribed degree of accuracy.

How can we be sure that the zeroth-order Bessel function has an infinite number of zeros? This fact was already conjectured by Daniel Bernoulli and Fourier [**13**], but the proof had to wait for Bessel. He showed [**3**] that $J_0$ is positive in the interval $[m\pi, (m + 1/2)\pi]$ when $m$ is even, and negative when $m$ is odd. Watson's treatise has a proof that is accessible to the modern reader [**13**, pp. 478–9]. Since $J_0$ is continuous, it must have an odd number of zeros in each of the intervals $[(m + 1/2)\pi, (m + 1)\pi]$, $m \in \mathbb{N}_0$.

Nowadays, the proof of the fact that $J_0$ has infinitely many zeros is usually based on the oscillation theory of differential equations. We need to recall that the $n$th-order Bessel function is a solution of the second-order linear differential equation

$$a^2 J_n''(a) + a J_n(a) + (a^2 - n^2) J_n(a) = 0.$$

In fact, this property (together with a suitable initial condition) often serves as the definition of Bessel functions; the differential equation arises from various problems in physics [**12**]. For $n = 0$, this implies

$$aF''(a) + F'(a) + aF(a) = 0. \tag{7}$$

(As an exercise, the reader can try to verify this fact by using the definition of $F$ and differentiating under the integral sign.) Performing the change of variables $F(a) = G(a)a^{-1/2}$, the differential equation (7) is transformed to

$$G''(a) + \left(1 + \frac{1}{4a^2}\right) G(a) = 0, \quad a > 0. \tag{8}$$

Note that for $a \in (0, \infty)$, the zeros of $F$ and $G$ coincide. It is now sufficient to apply the following theorem: If $q : [t_0, \infty) \to \mathbb{R}$ is a nonnegative function such that

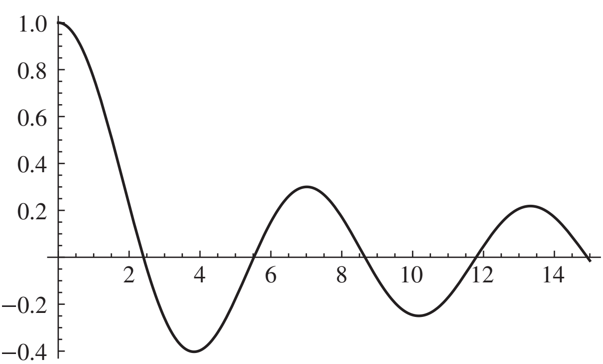


**Figure 5** Graph of the zeroth-order Bessel function $J_0$

$\int_{t_0}^{\infty} q(t)\,\mathrm{d}t = \infty$, then every nontrivial solution of $x''(t) + q(t)x(t) = 0$ on $[t_0, \infty)$ is oscillatory, i.e., has an infinite number of zeros (see [**12**, p. 120] for a short elementary proof).

Finally, let us note that if $a_n$ is the $n$th zero of $F$ in $[0, \infty)$, then

$$\lim_{n\to\infty} (a_{n+1} - a_n) = \pi. \tag{9}$$

Intuitively, this follows from the fact that when $a \to \infty$, the term $(1 + 1/4a^2)$ in the differential equation (8) approaches 1, and all solutions of $G''(a) + G(a) = 0$ have distance $\pi$ between their successive zeros. A rigorous proof of (9) can be based on the use of Sturm's comparison theorem [**2**, Theorem 31.1 and Problems 31.3, 31.4].

## Conclusion

One aim of this paper has been to show that problems in differential geometry can serve as a motivation for the study of more advanced mathematics. We started by experimenting with intrinsically defined curves, and this simple geometrical problem has led us to a number of interesting topics in advanced analysis, such as the gamma and Bessel functions, evaluation of integrals by complex analytic methods, or the oscillation theory of second-order differential equations. The computer played a significant role in our initial investigation of intrinsically defined curves, and also in performing the subsequent calculations. Computer experiments often lead to the discovery of surprising phenomena, which are then proved by formal arguments.

An interested reader might enjoy learning more about the two families of curves studied in this paper, as well as exploring other types of intrinsically defined curves. Finally, let us mention the article by Zahn and Roskies [**14**], where the problem of closed curves is discussed from a different perspective. The authors establish a relation between simple closed plane curves and their Fourier descriptors, roughly corresponding to the Fourier series coefficients of the primitive function to $\kappa$. Conversely, they characterize closed curves as those where one of the Fourier descriptors is a zero of the first-order Bessel function.

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**Summary** We study two classes of plane curves with prescribed curvature. First, we investigate spirals whose curvature is a power function, and express coordinates of the spirals' centers in terms of the gamma function. For curves in the second family, the curvature is a multiple of the sine function. We show that this family contains infinitely many closed curves and provide their characterization in terms of the zeroth-order Bessel function.

## Algebra Made Difficult

Anonymous graffiti found on a blackboard one late afternoon in April 2012 at the Mathematical Sciences Research Institute in Berkeley, California:

"Problem: Solve $x = ax + b$ for $x$.

Solution:

$$\begin{aligned} x &= a(ax+b)+b \ = \ a^2x+ab+b \\ &= a(a(ax+b)+b)+b \ = a^3x+a^2b+ab+b \\ &\ \vdots \\ &= (\text{assuming } |a|<1) \lim_{n\to\infty} a^n x + b\sum_{i=0}^{\infty} a^i \\ &= 0 + b/(1-a). \end{aligned}$$

This also holds by analytic continuation for all $a \neq 1$."

(submitted by James Propp, University of Massachusetts Lowell)