Discrete Bessel functions and partial difference equations

Antonín Slavík
Charles University, Faculty of Mathematics and Physics,
Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail: slavik@karlin.mff.cuni.cz

Abstract

We introduce a new class of discrete Bessel functions and discrete modified Bessel functions of integer order. After obtaining some of their basic properties, we show that these functions lead to fundamental solutions of the discrete wave equation and discrete diffusion equation.

Keywords: Bessel function; modified Bessel function; Bessel difference equation; discrete wave equation; discrete diffusion equation; fundamental solution

MSC 2010 subject classification: 39A12, 39A14, 33C10, 33C05, 39A06, 39A10, 34A33

1 Introduction

In their recent paper [2], M. Bohner and T. Cuchta have proposed a new definition of the discrete Bessel function

$$J_n(t)=\frac{(-1)^n(-t)_n}{2^n n!}F\left(\frac{n-t}{2},\frac{n-t}{2}+\frac{1}{2};n+1;-1\right),\quad t\in\mathbb{N}_0, \tag{1.1}$$

where F is the hypergeometric function and $(x)_k$ denotes the Pochhammer symbol (also known as the rising factorial) given by

$$(x)_k=\begin{cases}x(x+1)\cdots(x+k-1) & \text{for } k\in\mathbb{N},\\ 1 & \text{for } k=0.\end{cases}$$

The discrete Bessel function given by (1.1) is different from the one studied in earlier papers [4, 5], and its advantage is that it shares many properties with the classical Bessel function. For example, it satisfies the difference equation

$$t(t-1)\Delta^2 y(t-2)+t\Delta y(t-1)+t(t-1)y(t-2)-n^2y(t)=0$$

(where $\Delta f(t)=f(t+1)-f(t)$ is the forward difference), which is a discrete analogue of the Bessel differential equation

$$t^2y''(t)+ty'(t)+(t^2-n^2)y(t)=0.$$

The goal of this paper is to introduce a new class of discrete Bessel functions denoted by J_n^c, where $n\in\mathbb{N}_0$ is the order and c is a parameter, and to show that these discrete Bessel functions provide fundamental solutions to the discrete wave equation

$$\Delta^2u(x,t)=c^2\big(u(x+1,t)-2u(x,t)+u(x-1,t)\big),\quad x\in\mathbb{Z},\quad t\in\mathbb{N}_0 \tag{1.2}$$

(with $\Delta^2u(x,t)$ being the second-order forward difference of u with respect to t). The discrete Bessel function J_n given by (1.1) is a special case of J_n^c corresponding to $c=1$.

We also introduce a new class of discrete modified Bessel functions denoted by I_n^c, which can be used to construct fundamental solutions of the discrete diffusion equation

$$\Delta u(x,t) = c\big(u(x+1,t) - 2u(x,t) + u(x-1,t)\big), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0 \tag{1.3}$$

(where $\Delta u(x,t)$ is the forward difference of u with respect to t).

Our motivation comes from the theory of lattice differential equations, i.e., equations with discrete space and continuous time. In this context, it is known that the fundamental solutions of the lattice wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\big(u(x+1,t) - 2u(x,t) + u(x-1,t)\big), \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}_0^+, \tag{1.4}$$

which is a semidiscrete analogue of (1.2), have the form $u_1(x,t) = \mathcal{J}_{2x}(2ct)$ and $u_2(x,t) = \int_0^t \mathcal{J}_{2x}(2cs)\,\mathrm{d}s$, where $\mathcal{J}_x$ is the classical Bessel function (see [8, Example 3.3]). Similarly, the fundamental solution of the lattice diffusion equation

$$\frac{\partial u}{\partial t}(x,t) = c\big(u(x+1,t) - 2u(x,t) + u(x-1,t)\big), \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}_0^+, \tag{1.5}$$

which is a semidiscrete analogue of (1.3), has the form $u(x,t) = e^{-2ct}\mathcal{I}_x(2ct)$, where $\mathcal{I}_x$ is the classical modified Bessel function (see [10, Example 3.1]).

The corresponding formulas for fundamental solutions of the partial difference equations (1.2) and (1.3) will be obtained in Sections 3 and 4, respectively. To achieve this goal, we need the discrete analogues of the functions $t \mapsto \mathcal{J}_n(ct)$ and $t \mapsto \mathcal{I}_n(ct)$ for an arbitrary $c > 0$, which are precisely the functions J_n^c and I_n^c mentioned earlier. While the functions $t \mapsto \mathcal{J}_n(ct)$ and $t \mapsto \mathcal{I}_n(ct)$ satisfy the differential equations

$$t^2 y''(t) + t y'(t) + (\pm c^2 t^2 - n^2) y(t) = 0,$$

we will show that their discrete counterparts J_n^c and I_n^c are solutions of the difference equations

$$t(t-1)\Delta^2 y(t-2) + t\Delta y(t-1) \pm c^2 t(t-1) y(t-2) - n^2 y(t) = 0. \tag{1.6}$$

By expanding the differences, we obtain the equivalent form

$$(t^2 - n^2) y(t) - t(2t-1) y(t-1) + (1 \pm c^2) t(t-1) y(t-2) = 0. \tag{1.7}$$

We remark that the fundamental solutions of the partial difference equations (1.2) and (1.3) are already available in the existing literature [8, 10], but they are expressed in a different form than we obtain in Sections 3 and 4. Expressing them in terms of the discrete Bessel functions can simplify the study of their properties. For example, following the method from [2], we prove that the function J_n^c is oscillatory. This fact implies that for each fixed x, the first fundamental solution to (1.2) is oscillatory as a function of t; this result is new and would be difficult to obtain by different methods.

2 Discrete Bessel functions

Both types of the Bessel functions, J_n^c and I_n^c, will be defined in terms of the hypergeometric series

$$F(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k.$$

Definition 2.1. For each $c \in \mathbb{C}$, we define the discrete Bessel function

$$J_n^c(t) = \frac{(-c/2)^n (-t)_n}{n!} F\left(\frac{n-t}{2}, \frac{n-t}{2} + \frac{1}{2}; n+1; -c^2\right), \quad t \in \mathbb{N}_0, \quad n \in \mathbb{N}_0, \tag{2.1}$$

and the discrete modified Bessel function

$$I_n^c(t) = \frac{(-c/2)^n(-t)_n}{n!} F\left(\frac{n-t}{2}, \frac{n-t}{2} + \frac{1}{2}; n+1; c^2\right), \quad t \in \mathbb{N}_0, \quad n \in \mathbb{N}_0. \tag{2.2}$$

Note that if $n > t$, then $(-t)_n = 0$ and therefore $J_n^c(t) = I_n^c(t) = 0$. Otherwise, if $n \le t$, then one of the fractions $\frac{n-t}{2}$ and $\frac{n-t}{2} + \frac{1}{2}$ is a nonpositive integer, which means that the hypergeometric series occurring in (2.1) and (2.2) have only finitely many nonzero terms. As in [2], the definition of $J_n^c(t)$ can be extended to all $t \in \mathbb{Z}$, but the same extension is not always possible for $I_n^c(t)$. Similarly, it would be possible to consider Bessel functions of non-integer orders n. However, for simplicity, we restrict ourselves to nonnegative integer values of t and n; this case is the most interesting one for applications to partial difference equations.

For $c = 1$, the function J_n^c coincides with the discrete Bessel function (1.1) introduced in [2]. For applications in partial difference equations, the most useful case is when c is a positive real number. One advantage of allowing c to be complex is the connection formula

$$I_n^c(t) = (-\mathrm{i})^n J_n^{\mathrm{i}c}(t),$$

which is a straightforward consequence of the definitions.

Our first goal is to prove that J_n^c and I_n^c satisfy the difference equations (1.6)–(1.7). The next result generalizes [2, Theorem 1].

Theorem 2.2. *If $c \in \mathbb{C}$ and $n \in \mathbb{N}_0$, then the function*

$$B_n(t) = \frac{(-c/2)^n(-t)_n}{n!} F\left(\frac{n-t}{2}, \frac{n-t}{2} + \frac{1}{2}; n+1; \pm c^2\right), \quad t \in \mathbb{N}_0,$$

satisfies the difference equation

$$t(t-1)\Delta^2 B_n(t-2) + t\Delta B_n(t-1) \mp c^2 t(t-1) B_n(t-2) - n^2 B_n(t) = 0, \quad t \ge 2,$$

or equivalently

$$(t^2 - n^2) B_n(t) - t(2t-1) B_n(t-1) + (1 \mp c^2) t(t-1) B_n(t-2) = 0, \quad t \ge 2.$$

Proof. We use the contiguous relation (see [7, formula 15.5.13])

$$(\gamma - \alpha - \beta) F(\alpha, \beta; \gamma; z) - (\gamma - \alpha) F(\alpha - 1, \beta; \gamma; z) + \beta(1-z) F(\alpha, \beta+1; \gamma; z) = 0$$

with $\alpha = \frac{n-t}{2} + 1$, $\beta = \frac{n-t}{2} + \frac{1}{2}$, $\gamma = n+1$, $z = \pm c^2$ to get

$$\left(t - \frac{1}{2}\right) F\left(\frac{n-t}{2} + 1, \frac{n-t}{2} + \frac{1}{2}; n+1; \pm c^2\right) - \left(\frac{n}{2} + \frac{t}{2}\right) F\left(\frac{n-t}{2}, \frac{n-t}{2} + \frac{1}{2}; n+1; \pm c^2\right)$$
$$+ \frac{n-t+1}{2}(1 \mp c^2) F\left(\frac{n-t}{2} + 1, \frac{n-t}{2} + \frac{3}{2}; n+1; \pm c^2\right) = 0.$$

By multiplying the equation with $2\frac{(-c/2)^n}{n!}(-t)_{n+1}$, using the definition of B_n and the symmetry of F in the first two arguments, we obtain

$$-t(2t-1) B_n(t-1) - (n+t)(-t+n) B_n(t) + (1 \mp c^2) t(t-1) B_n(t-2) = 0. \qquad \square$$

Corollary 2.3. *For each $c \in \mathbb{C}$ and $n \in \mathbb{N}_0$, the function J_n^c is a solution of the difference equation*

$$t(t-1)\Delta^2 y(t-2) + t\Delta y(t-1) + c^2 t(t-1) y(t-2) - n^2 y(t) = 0, \quad t \ge 2,$$

and the function I_n^c is a solution of the difference equation

$$t(t-1)\Delta^2 y(t-2) + t\Delta y(t-1) - c^2 t(t-1) y(t-2) - n^2 y(t) = 0, \quad t \ge 2.$$

The next task is to obtain expressions for differences of the discrete Bessel functions. The following result generalizes Theorems 5, 6 and Corollary 7 from [2]. Our proof is simpler than in [2] and relies on the contiguous relations for the hypergeometric function.

Theorem 2.4. *Assume that $c \in \mathbb{C}$. For each $n \in \mathbb{N}_0$, consider the function*

$$B_n(t) = \frac{(-c/2)^n(-t)_n}{n!} F\left(\frac{n-t}{2}, \frac{n-t}{2} + \frac{1}{2}; n+1; \pm c^2\right), \quad t \in \mathbb{N}_0.$$

Then we have the following identities:

$$t\Delta B_n(t-1) = nB_n(t) \pm ctB_{n+1}(t-1), \qquad n \geq 0, \quad t \geq 1, \tag{2.3}$$

$$t\Delta B_n(t-1) = -nB_n(t) + ctB_{n-1}(t-1), \qquad n \geq 1, \quad t \geq 1, \tag{2.4}$$

$$nB_n(t) = \frac{ct}{2}\big(B_{n-1}(t-1) \mp B_{n+1}(t-1)\big), \quad n \geq 1, \quad t \geq 1, \tag{2.5}$$

$$\Delta B_n(t) = \frac{c}{2}\big(B_{n-1}(t) \pm B_{n+1}(t)\big), \qquad n \geq 1, \quad t \geq 0. \tag{2.6}$$

Proof. To prove (2.3), we need to show that

$$0 = tB_n(t-1) + (n-t)B_n(t) \pm ctB_{n+1}(t-1).$$

Using the definition of B_n and dividing by $(-c/2)^n(-t)_{n+1}/(n+1)!$, we see it is enough to show that

$$\begin{aligned} 0 = &-(n+1)F\left(\frac{n-t}{2} + \frac{1}{2}, \frac{n-t}{2} + 1; n+1; \pm c^2\right) + (n+1)F\left(\frac{n-t}{2} + \frac{1}{2}, \frac{n-t}{2}; n+1; \pm c^2\right) \\ &\pm \frac{c^2}{2}(n+1-t)F\left(\frac{n-t}{2} + \frac{3}{2}, \frac{n-t}{2} + 1; n+2; \pm c^2\right). \end{aligned}$$

To prove this, we use the contiguous relations (see [7, formulas 15.5.13 and 15.5.16])

$$\begin{aligned} \alpha\gamma(1-z)F(\alpha+1, \beta+1; \gamma; z) &= \gamma\big((\gamma-\beta-1)F(\alpha, \beta; \gamma; z) - (\gamma-\alpha-\beta-1)F(\alpha, \beta+1; \gamma; z)\big), \\ \alpha\gamma(1-z)F(\alpha+1, \beta+1; \gamma; z) &= \alpha\gamma F(\alpha, \beta+1; \gamma; z) - \alpha(\gamma-\beta-1)zF(\alpha+1, \beta+1; \gamma+1; z). \end{aligned}$$

By equating the right-hand sides and dividing by $(\gamma - \beta - 1)$, we get

$$\gamma F(\alpha, \beta; \gamma; z) - \gamma F(\alpha, \beta+1; \gamma; z) = -\alpha z F(\alpha+1, \beta+1; \gamma+1; z).$$

The desired relation now follows by letting $\alpha = \frac{n-t}{2} + \frac{1}{2}$, $\beta = \frac{n-t}{2}$, $\gamma = n+1$, $z = \pm c^2$.

To prove (2.4), we have to show that

$$0 = tB_n(t-1) - (n+t)B_n(t) + ctB_{n-1}(t-1).$$

Using the definition of B_n and dividing by $(-c/2)^n(-t)_n/n!$, we see it is enough to show that

$$\begin{aligned} 0 = &-(-t+n)F\left(\frac{n-t}{2} + \frac{1}{2}, \frac{n-t}{2} + 1; n+1; \pm c^2\right) - (n+t)F\left(\frac{n-t}{2}, \frac{n-t}{2} + \frac{1}{2}; n+1; \pm c^2\right) \\ &+ 2nF\left(\frac{n-t}{2}, \frac{n-t}{2} + \frac{1}{2}; n; \pm c^2\right). \end{aligned}$$

To prove this, we use the contiguous relation (see [7, formula 15.5.15])

$$(\gamma - \alpha)F(\alpha, \beta; \gamma+1; z) + \alpha F(\alpha+1, \beta; \gamma+1; z) - \gamma F(\alpha, \beta; \gamma; z) = 0.$$

The desired relation now follows by letting $\alpha = \frac{n-t}{2}$, $\beta = \frac{n-t}{2} + \frac{1}{2}$, $\gamma = n$, $z = \pm c^2$.

Identity (2.5) is obtained by subtracting (2.3) from (2.4). To get the identity (2.6), add (2.3) and (2.4), divide by $2t$, and replace t by $t+1$. □

Corollary 2.5. *For each $c \in \mathbb{C}$, the following relations hold:*

$$
\begin{aligned}
t\Delta J_n^c(t-1) &= nJ_n^c(t) - ctJ_{n+1}^c(t-1), && n \geq 0, \quad t \geq 1, && (2.7)\\
t\Delta J_n^c(t-1) &= -nJ_n^c(t) + ctJ_{n-1}^c(t-1), && n \geq 1, \quad t \geq 1, && (2.8)\\
nJ_n^c(t) &= \frac{ct}{2}\big(J_{n-1}^c(t-1) + J_{n+1}^c(t-1)\big), && n \geq 1, \quad t \geq 1, && (2.9)\\
\Delta J_n^c(t) &= \frac{c}{2}\big(J_{n-1}^c(t) - J_{n+1}^c(t)\big), && n \geq 1, \quad t \geq 0, && (2.10)\\
t\Delta I_n^c(t-1) &= nI_n^c(t) + ctI_{n+1}^c(t-1), && n \geq 0, \quad t \geq 1, && (2.11)\\
t\Delta I_n^c(t-1) &= -nI_n^c(t) + ctI_{n-1}^c(t-1), && n \geq 1, \quad t \geq 1, && (2.12)\\
nI_n^c(t) &= \frac{ct}{2}\big(I_{n-1}^c(t-1) - I_{n+1}^c(t-1)\big), && n \geq 1, \quad t \geq 1, && (2.13)\\
\Delta I_n^c(t) &= \frac{c}{2}\big(I_{n-1}^c(t) + I_{n+1}^c(t)\big), && n \geq 1, \quad t \geq 0. && (2.14)
\end{aligned}
$$

The next theorem provides additional information about the values and differences of J_n^c and I_n^c.

Theorem 2.6. *For each $c \in \mathbb{C}$, the functions J_n^c and I_n^c have the following properties:*

- $J_0^c(0) = I_0^c(0) = 1$.
- $J_n^c(t) = I_n^c(t) = 0$ *for all $t \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that $n > t$.*
- $\Delta J_n^c(0) = \Delta I_n^c(0) = 0$ *for all $n \in \mathbb{N}_0 \setminus \{1\}$, and $\Delta J_1^c(0) = \Delta I_1^c(0) = c/2$.*

Proof. The first two statements follow from the definitions of J_n^c and I_n^c; note that $(-t)_n = 0$ for all $t \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that $n > t$.

Using (2.10) and (2.14), we get $\Delta J_n^c(0) = \frac{c}{2}\big(J_{n-1}^c(0) - J_{n+1}^c(0)\big)$ and $\Delta I_n^c(0) = \frac{c}{2}\big(I_{n-1}^c(0) + I_{n+1}^c(0)\big)$ for all $n \in \mathbb{N}$. Both expressions are equal to $c/2$ if $n = 1$, and zero for all $n \in \mathbb{N} \setminus \{1\}$. For $n = 0$, the relations (2.7) and (2.11) with $t = 1$ imply $\Delta J_0^c(0) = \Delta I_0^c(0) = 0$. □

The remaining results in this section are concerned with the sign of J_n^c and I_n^c if c is a real number. The first statement generalizes [2, Theorem 12].

Theorem 2.7. *For each $c \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}_0$, the function J_n^c is oscillatory (i.e., $J_n^c(t)$ changes sign or vanishes for infinitely many values of $t \in \mathbb{N}_0$).*

Proof. To simplify notation, we denote $y(t) = J_n^c(t)$. According to Corollary 2.3 with t replaced by $t+2$, we see that y satisfies the difference equation

$$(t+2)(t+1)\Delta^2 y(t) + (t+2)\Delta y(t+1) + c^2(t+2)(t+1)y(t) - n^2 y(t+2) = 0, \quad t \in \mathbb{N}_0.$$

Using the formulas $y(t) = y(t+1) - \Delta y(t+1) + \Delta^2 y(t)$ and $y(t+2) = \Delta y(t+1) + y(t+1)$, we obtain

$$\Delta^2 y(t)\big((t+2)(t+1)(1+c^2)\big) + \Delta y(t+1)\big(t+2-c^2(t+2)(t+1)-n^2\big) + y(t+1)\big(c^2(t+1)(t+2)-n^2\big) = 0,$$

and therefore

$$\Delta^2 y(t) = \frac{c^2(t+2)(t+1)+n^2-t-2}{(t+2)(t+1)(1+c^2)}\Delta y(t+1) - \frac{c^2(t+1)(t+2)-n^2}{(t+2)(t+1)(1+c^2)}y(t+1), \quad t \in \mathbb{N}_0.$$

Let

$$v(t) = \frac{\frac{2^t}{(1+c^2)^{t/2}}}{\binom{t}{\frac{t-n}{2}}} = \frac{2^t}{(1+c^2)^{t/2}} \frac{\Gamma(\frac{t-n}{2}+1)\Gamma(\frac{t+n}{2}+1)}{\Gamma(t+1)}, \quad t \in \{n, n+1, n+2, \ldots\}.$$

One can verify (using a computer system such as *Mathematica* or by a hand calculation similar to [2, Lemma 11]) that

$$\Delta v(t+1)+\Delta v(t)+v(t)\frac{c^2(t+1)(t+2)+n^2-t-2}{(t+2)(t+1)(1+c^2)}=0,\quad t\in\{n,n+1,n+2,\dots\},\tag{2.15}$$

$$\lim_{t\to\infty}\frac{v(t)}{v(t+1)}=\sqrt{1+c^2}.\tag{2.16}$$

Let

$$u(t)=v(t)y(t),\quad t\in\{n,n+1,n+2,\dots\}.$$

Using the product rule twice, we get

$$\begin{aligned}
\Delta u(t)&=y(t+1)\Delta v(t)+v(t)\Delta y(t),\\
\Delta^2 u(t)&=y(t+1)\Delta^2 v(t)+\big(\Delta v(t+1)+\Delta v(t)\big)\Delta y(t+1)+v(t)\Delta^2 y(t)\\
&=\frac{u(t+1)}{v(t+1)}\Delta^2 v(t)+\big(\Delta v(t+1)+\Delta v(t)\big)\Delta y(t+1)\\
&+v(t)\left(\frac{c^2(t+2)(t+1)+n^2-t-2}{(t+2)(t+1)(1+c^2)}\Delta y(t+1)-\frac{c^2(t+1)(t+2)-n^2}{(t+2)(t+1)(1+c^2)}\frac{u(t+1)}{v(t+1)}\right)\\
&=u(t+1)\left(\frac{\Delta^2 v(t)}{v(t+1)}-\frac{c^2(t+1)(t+2)-n^2}{(t+2)(t+1)(1+c^2)}\frac{v(t)}{v(t+1)}\right)\\
&+\Delta y(t+1)\left(\Delta v(t+1)+\Delta v(t)+v(t)\frac{c^2(t+2)(t+1)+n^2-t-2}{(t+2)(t+1)(1+c^2)}\right).
\end{aligned}$$

The last term vanishes thanks to (2.15), and therefore

$$\Delta^2 u(t)+u(t+1)\left(\frac{c^2(t+1)(t+2)-n^2}{(t+2)(t+1)(1+c^2)}\frac{v(t)}{v(t+1)}-\frac{\Delta^2 v(t)}{v(t+1)}\right)=0.$$

This is a second-order difference equation of the form $\Delta^2 u(t)+q(t)u(t+1)=0$, where

$$q(t)=\frac{c^2(t+1)(t+2)-n^2}{(t+2)(t+1)(1+c^2)}\frac{v(t)}{v(t+1)}-\frac{v(t+2)-2v(t+1)+v(t)}{v(t+1)}.$$

By Wintner's theorem (see [3, Theorem 4.45]), such equation is oscillatory if $\sum_{t=n}^{\infty}q(t)=\infty$. To verify this fact, it is enough to show that $\lim_{t\to\infty}q(t)>0$. Using (2.16), we calculate

$$\begin{aligned}
\lim_{t\to\infty}q(t)&=\frac{c^2}{c^2+1}\lim_{t\to\infty}\frac{v(t)}{v(t+1)}-\lim_{t\to\infty}\left(\frac{v(t+2)}{v(t+1)}-2+\frac{v(t)}{v(t+1)}\right)\\
&=\frac{c^2}{\sqrt{c^2+1}}-\frac{1}{\sqrt{c^2+1}}+2-\sqrt{1+c^2}=2-\frac{2}{\sqrt{c^2+1}}>0.
\end{aligned}$$

This shows that u is oscillatory. Since v is positive, y is oscillatory. □

Theorem 2.8. *For each $c\ge 0$ and $n\in\mathbb{N}_0$, the function I_n^c is nonnegative. For each $c<0$, the function I_n^c is nonnegative if n is even and nonpositive if n is odd.*

Proof. The first statement (where $c\ge 0$) is easily proved by induction with respect to t. For $t=0$, it follows from the first and second part of Theorem 2.6 that $I_n^c(0)\ge 0$ for all $n\in\mathbb{N}_0$.

Suppose that $I_n^c(t)\ge 0$ for all $n\in\mathbb{N}_0$. By the relation (2.14), we have $\Delta I_n^c(t)\ge 0$ for all $n\in\mathbb{N}$. If $n=0$, then the relation (2.11) with t replaced by $t+1$ implies $\Delta I_0^c(t)=cI_1^c(t)\ge 0$. Consequently, we have $I_n^c(t+1)=I_n^c(t)+\Delta I_n^c(t)\ge 0$ for all $n\in\mathbb{N}_0$.

The second statement (where $c<0$) is a consequence of the first part and the identity

$$I_n^c(t)=(-1)^n I_n^{|c|}(t),$$

which follows immediately from the definition. □

3 Discrete wave equation

In this section, we explore the relation between the discrete Bessel function J_n^c and the discrete wave equation

$$\Delta^2 u(x,t) = c^2\big(u(x+1,t) - 2u(x,t) + u(x-1,t)\big), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0$$

(the forward difference operator Δ always applies to the time variable t; differences with respect to the space variable x are never considered in this paper). Suppose that $u_1 : \mathbb{Z} \times \mathbb{N}_0 \to \mathbb{R}$ is the solution corresponding to the initial conditions

$$u_1(x,0) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases}$$
$$\Delta u_1(x,0) = 0, \quad x \in \mathbb{Z}.$$

Then it is not difficult to check that the function $u_2 : \mathbb{Z} \times \mathbb{N}_0 \to \mathbb{R}$ given by

$$u_2(x,t) = \sum_{s=0}^{t-1} u_1(x,s)$$

(where the sum is understood as empty if $t = 0$) is the solution of the discrete wave equation satisfying the conditions

$$u_2(x,0) = 0, \quad x \in \mathbb{Z},$$
$$\Delta u_2(x,0) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

In [8, Theorem 3.2], it is shown that for arbitrary bounded real sequences $\{u_x^0\}_{x\in\mathbb{Z}}$, $\{v_x^0\}_{x\in\mathbb{Z}}$, the function

$$u(x,t) = \sum_{k\in\mathbb{Z}} (u_k^0 \cdot u_1(x-k,t) + v_k^0 \cdot u_2(x-k,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0, \tag{3.1}$$

is the solution of the discrete wave equation satisfying

$$u(x,0) = u_x^0, \quad \Delta u(x,0) = v_x^0, \quad x \in \mathbb{Z}.$$

In fact, it is not difficult to see (use induction with respect to t) that we have $u_1(x,t) = u_2(x,t) = 0$ whenever $|x| > t$. Hence, on the right-hand side of the formula (3.1), the terms corresponding to $k \in \mathbb{Z}$ such that $|x-k| > t$ do not contribute to $u(x,t)$, and we can write

$$u(x,t) = \sum_{k=x-t}^{x+t} (u_k^0 \cdot u_1(x-k,t) + v_k^0 \cdot u_2(x-k,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0.$$

The solutions u_1, u_2 are referred to as the fundamental solutions of the discrete wave equation. The next theorem shows that u_1 (and consequently also u_2) can be expressed in terms of the discrete Bessel function J_n^c.

Theorem 3.1. *For each $c > 0$, the solution of the initial-value problem*

$$\Delta^2 u(x,t) = c^2\big(u(x+1,t) - 2u(x,t) + u(x-1,t)\big), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0, \tag{3.2}$$

$$u(x,0) = \begin{cases} 1 & \textit{if } x = 0, \\ 0 & \textit{if } x \neq 0, \end{cases} \tag{3.3}$$

$$\Delta u(x,0) = 0, \quad x \in \mathbb{Z}, \tag{3.4}$$

is given by

$$u(x,t) = J^{2c}_{2|x|}(t), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0. \tag{3.5}$$

Moreover, for each $x \in \mathbb{Z}$, the function $t \mapsto u(x,t)$ oscillatory.

Proof. Let u be defined by (3.5). The relations (3.3) and (3.4) follow from Theorem 2.6. If $x \geq 1$, we use the identity (2.10) to calculate

$$\Delta u(x,t) = \Delta J^{2c}_{2x}(t) = c\big(J^{2c}_{2x-1}(t) - J^{2c}_{2x+1}(t)\big),$$
$$\Delta^2 u(x,t) = c^2\big(J^{2c}_{2x-2}(t) - 2J^{2c}_{2x}(t) + J^{2c}_{2x+2}(t)\big) = c^2\big(u(x-1,t) - 2u(x,t) + u(x+1,t)\big).$$

Similarly, if $x \leq -1$, we obtain

$$\Delta u(x,t) = \Delta J^{2c}_{-2x}(t) = c\big(J^{2c}_{-2x-1}(t) - J^{2c}_{-2x+1}(t)\big),$$
$$\Delta^2 u(x,t) = c^2\big(J^{2c}_{-2x-2}(t) - 2J^{2c}_{-2x}(t) + J^{2c}_{-2x+2}(t)\big) = c^2\big(u(x+1,t) - 2u(x,t) + u(x-1,t)\big).$$

Finally, for $x = 0$, we use the identity (2.7) with $n = 0$ and t replaced by $t+1$ to get

$$\Delta u(0,t) = \Delta J^{2c}_0(t) = -2cJ^{2c}_1(t),$$

and consequently (by identity (2.10))

$$\Delta^2 u(0,t) = c^2\big(-2J^{2c}_0(t) + 2J^{2c}_2(t)\big) = c^2\big(J^{2c}_2(t) - 2J^{2c}_0(t) + J^{2c}_2(t)\big) = c^2\big(u(1,t) - 2u(0,t) + u(-1,t)\big).$$

Thus, the relation (3.2) holds for all $x \in \mathbb{Z}$, $t \in \mathbb{N}_0$.

The fact that $t \mapsto u(x,t)$ is oscillatory follows from Theorem 2.7. □

Remark 3.2. The first fundamental solution of the discrete wave equation can be alternatively expressed using the multinomial coefficients as follows (see [8, Example 3.5]):

$$u(x,t) = \sum_{j=0}^{t} \binom{t}{j,\, t-2j-2x,\, j+2x} (-1)^j c^{2j+2x}$$

4 Discrete diffusion equation

We now turn our attention to the discrete diffusion equation

$$\Delta w(x,t) = d\big(w(x+1,t) - 2w(x,t) + w(x-1,t)\big), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0.$$

The solution $w : \mathbb{Z} \times \mathbb{N}_0 \to \mathbb{R}$ corresponding to the initial conditions

$$w(x,0) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases}$$

is called the fundamental solution. In [9, Corollary 3.8], it is shown that for an arbitrary bounded real sequence $\{u^0_x\}_{x\in\mathbb{Z}}$, the function

$$u(x,t) = \sum_{k\in\mathbb{Z}} u^0_k \cdot w(x-k,t), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0, \tag{4.1}$$

is the solution of the discrete diffusion equation satisfying

$$u(x,0) = u^0_x, \quad x \in \mathbb{Z}.$$

Observing that $w(x,t) = 0$ whenever $|x| > t$ (use induction with respect to t), we can simplify the formula (4.1) to

$$u(x,t) = \sum_{k=x-t}^{x+t} u_k^0 \cdot w(x-k,t), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0.$$

For $d \neq 1/2$, the next theorem shows that the fundamental solution w can be constructed using the discrete modified Bessel function I_n^c.

Theorem 4.1. *For each $d \neq 1/2$, the solution of the initial-value problem*

$$\Delta w(x,t) = d\big(w(x+1,t) - 2w(x,t) + w(x-1,t)\big), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0, \tag{4.2}$$

$$w(x,0) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases} \tag{4.3}$$

is given by

$$w(x,t) = (1-2d)^t I_{|x|}^{2d/(1-2d)}(t), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0. \tag{4.4}$$

Moreover, for each $x \in \mathbb{Z}$, the function $t \mapsto w(x,t)$ is nonnegative if $d \in (0,1/2)$, and oscillatory if $d > 1/2$.

Proof. Let w be defined by (4.4). To simplify notation, let $c = d/(1-2d)$. The relation (4.3) follows from Theorem 2.6. Let

$$z(x,t) = I_{|x|}^{2c}(t), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0.$$

If $x \geq 1$, we use the identity (2.14) to calculate

$$\Delta z(x,t) = \Delta I_x^{2c}(t) = c\big(I_{x+1}^{2c}(t) + I_{x-1}^{2c}(t)\big) = c\big(z(x+1,t) + z(x-1,t)\big).$$

Similarly, if $x \leq -1$, we obtain

$$\Delta z(x,t) = \Delta I_{-x}^{2c}(t) = c\big(I_{-x+1}^{2c}(t) + I_{-x-1}^{2c}(t)\big) = c\big(z(x-1,t) + z(x+1,t)\big).$$

Finally, if $x = 0$, identity (2.11) with $n = 0$ and t replaced by $t+1$ implies that

$$\Delta z(0,t) = \Delta I_0^{2c}(t) = 2cI_1^{2c}(t) = c\big(z(1,t) + z(-1,t)\big).$$

Note that

$$\Delta(1-2d)^t = (1-2d)^{t+1} - (1-2d)^t = -2d(1-2d)^t.$$

Hence, by the product rule, we have

$$\begin{aligned} \Delta w(x,t) &= \Delta\left(z(x,t)(1-2d)^t\right) = \Delta z(x,t)(1-2d)^{t+1} + z(x,t)\Delta(1-2d)^t \\ &= \frac{d}{1-2d}\big(z(x+1,t) + z(x-1,t)\big)(1-2d)^{t+1} - z(x,t)2d(1-2d)^t \\ &= d\big(z(x+1,t) - 2z(x,t) + z(x-1,t)\big)(1-2d)^t = d\big(w(x+1,t) - 2w(x,t) + w(x-1,t)\big). \end{aligned}$$

The fact that $t \mapsto w(x,t)$ is nonnegative if $d \in (0,1/2)$ and oscillatory if $d > 1/2$ follows from the definition of w and Theorem 2.8. □

Remark 4.2. An alternative form of the fundamental solution to the discrete diffusion equation is (see [10, Example 3.3])

$$w(x,t) = \sum_{j=0}^{t} \binom{t}{j,\, t-2j-x,\, j+x} d^{2j+x}(1-2d)^{t-2j-x}.$$

This formula is valid also for $d = 1/2$, when it reduces to

$$w(x,t) = \begin{cases} \binom{t}{\frac{t+x}{2}} \left(\frac{1}{2}\right)^t & \text{if } t+x \text{ is even}, \\ 0 & \text{if } t+x \text{ is odd}. \end{cases}$$

5 Conclusion

We conclude the paper by pointing out two possible directions for further research:

- The classical Bessel functions have their multivariable counterparts [1], which found applications in various areas of physics (see, e.g., [6] and the references there). Is there a reasonable extension of the discrete Bessel functions to several variables? If yes, is it related to the higher-dimensional discrete diffusion/wave equations? (Note that an explicit formula for the fundamental solution of the n-dimensional discrete diffusion equation, which does not rely on Bessel functions, can be found in [10]).
- In Sections 3 and 4, we were dealing with the discrete diffusion/wave equations whose left-hand sides involve forward differences of first and second order with respect to time. In some situations, it might be more appropriate to consider the backward first-order difference for the diffusion equation, and the central or backward second-order difference for the wave equation. Is it possible to express their solutions with the help of some Bessel-type functions?

Acknowledgement

This paper was supported by the Czech Science Foundation, grant no. GA15-07690S. I am grateful to the anonymous referees for their suggestions, which helped to improve the exposition of the paper.

References

[1] P. Appell, *Sur l'inversion approchée de certaines intégrales réelles et sur l'extension de l'équation de Kepler et des fonctions de Bessel*, C. R. Acad. Sci. 160 (1915), 419–423.

[2] M. Bohner, T. Cuchta, *The Bessel difference equation*, Proc. Amer. Math. Soc. 145 (2017), 1567–1580.

[3] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[4] R. H. Boyer, *Discrete Bessel functions*, J. Math. Anal. Appl. 2 (1961), 509–524.

[5] J. J. Gergen, *Bessel difference systems of zero order*, J. Math. Anal. Appl. 13 (1966), 102–117.

[6] H. J. Korsch, A. Klumpp, D. Witthaut, *On two-dimensional Bessel functions*, J. Phys. A, Math. Gen. 39 (2006), 14947–14964.

[7] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, eds., *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010. Online version at `http://dlmf.nist.gov/`.

[8] A. Slavík, *Discrete-space systems of partial dynamic equations and discrete-space wave equation*, Qual. Theory Dyn. Syst. 16 (2017), 299–315.

[9] A. Slavík, P. Stehlík, *Dynamic diffusion-type equations on discrete-space domains*, J. Math. Anal. Appl. 427 (2015), 525–545.

[10] A. Slavík, P. Stehlík, *Explicit solutions to dynamic diffusion-type equations and their time integrals.* Appl. Math. Comput. 234 (2014), 486–505.