Discrete Calculus and Weighted Fibonacci and Tribonacci Sums

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Abstract. We show the power of discrete calculus, especially the summation by parts formula, in evaluating Fibonacci and Tribonacci sums with polynomial and exponential weights. We provide new, short, and elementary proofs of several known results, and derive new identities involving Fibonacci, Lucas, and Tribonacci numbers.

1. INTRODUCTION. Everyone is familiar with integration by parts as a basic method for evaluating definite integrals; surprisingly, it seems that its discrete version, summation by parts, which is useful for calculating the values of finite sums, is much less known. The technique goes back to N. H. Abel's 1826 paper [**1**] on the binomial series, and forms the core of modern proofs of Abel's criterion for the convergence of an infinite series. One goal of the present paper is to show that summation by parts is not merely an ad-hoc trick serving to prove Abel's test, and to convince the reader that it should be in every mathematician's toolbox. A second goal is to demonstrate the power of this technique by providing short and elementary proofs of some old and new identities involving the Fibonacci, Lucas, and Tribonacci numbers.

We will focus on weighted Fibonacci sums of the form $\sum_{i=0}^{n} F_i w_i$, where F_i are the Fibonacci numbers, and w_i are prescribed weights. We are mainly interested in polynomial weights, which are discussed in the classical book [**12**], as well as exponential weights, which are treated in [**5**]. The identities that we will obtain apply not only to the Fibonacci sequence, but to any sequence $\{G_i\}_{i=0}^{\infty}$ satisfying the recurrence relation $G_{i+2} = G_{i+1} + G_i$ and having arbitrary initial values G_0 and G_1. Following the book [**4**], such sequences are known as the Gibonacci numbers (shorthand for generalized Fibonacci numbers). One advantage is that we do not need to specify whether we deal with the Fibonacci sequence whose initial values are $F_0 = 0$ and $F_1 = 1$, or with the sequence whose initial values are $F_0 = F_1 = 1$; both choices appear frequently in literature. More importantly, our identities apply also to the Lucas sequence, which is defined by the recurrence relation $L_{i+2} = L_{i+1} + L_i$ and the initial values $L_0 = 2$, $L_1 = 1$.

For example, using the summation by parts formula, we can easily derive identities such as

$$\sum_{i=0}^{n} G_i c^i = \frac{G_n c^{n+2} + G_{n+1} c^{n+1} + (c-1)G_0 - G_1 c}{c^2 + c - 1},$$

which holds for any Gibonacci sequence (see Section 4).

To demonstrate that the methods of discrete calculus are not limited to weighted Gibonacci sums, we will also consider weighted sums $\sum_{i=0}^{n} T_i w_i$ involving the Tribonacci numbers. These numbers satisfy the relation $T_{i+3} = T_{i+2} + T_{i+1} + T_i$, and their history goes back to C. Darwin's *Origin of Species* as well as the 1914 article [**3**] by the Russian mathematician N. A. Agronomof. More details can be found in the enlightening recent paper [**20**].

The outline of the paper is as follows: We begin with a brief introduction to discrete calculus—a discrete version of differential and integral calculus, whose basic operations are differences and antidifferences (for more information on discrete calculus, see [**10, 16, 18**]). We include a short derivation of the summation by parts formula, and then show how to find antidifferences for the Gibonacci and Tribonacci numbers. Using these results, we will successively evaluate several types of weighted Gibonacci and Tribonacci sums with exponential and polynomial weights. We will provide simple proofs of several known results, and obtain new identities involving the Gibonacci and Tribonacci numbers. For example, the recurrence formulas for Gibonacci and Tribonacci sums with polynomial weights presented in Sections 5 and 6 are new. We will conclude the exposition with a brief look at sums containing squares of Gibonacci and Tribonacci numbers. Throughout the whole paper, we include several exercises for the reader.

There are numerous sources dealing with weighted Fibonacci and Tribonacci sums, see e.g., [**2, 7, 11**] and the references therein, but as far as we are aware, none of them uses summation by parts. The reader is invited to compare our approach with Gauthier's differential method described in [**8**] and [**12**, Chapter 25], which is more laborious and uses Binet's explicit formula for the Fibonacci numbers, and with the derivations in [**5**], which are based on generating functions.

2. A SHORT INTRODUCTION TO DISCRETE CALCULUS. Given a real or complex sequence $\{a_i\}_{i\in\mathbb{Z}}$, the basic operation in discrete calculus is the difference

$$\Delta a_i = a_{i+1} - a_i, \quad i \in \mathbb{Z}.$$

Conversely, an antidifference (or indefinite sum) of $\{a_i\}_{i\in\mathbb{Z}}$ is an arbitrary sequence $\{b_i\}_{i\in\mathbb{Z}}$ such that $\Delta b_i = a_i$ for all $i \in \mathbb{Z}$; we write

$$\sum a_i = b_i.$$

Antidifferences are similar to antiderivatives; for example, they are unique up to a constant. The expression $\sum a_i$ is only a formal symbol, but it suggests than antidifferences are useful for calculating finite sums (similarly, antiderivatives are helpful in calculating definite integrals). Indeed, if $\{b_i\}_{i\in\mathbb{Z}}$ is an antidifference of $\{a_i\}_{i\in\mathbb{Z}}$, then

$$\sum_{i=p}^{q} a_i = \sum_{i=p}^{q}(b_{i+1} - b_i) = [b_i]_{i=p}^{q+1} = b_{q+1} - b_p \tag{1}$$

whenever $p, q \in \mathbb{Z}$ and $p \le q$; this is the fundamental theorem of discrete calculus.

Finding antidifferences in closed form is not always easy; some basic results and techniques are described in [**10, 16**]. In short, formulas for differences often give rise to formulas for antidifferences. Here we restrict ourselves only to a single example that will be needed throughout the rest of the paper. The difference of the geometric progression $\{c^i\}_{i\in\mathbb{Z}}$ is

$$\Delta c^i = c^{i+1} - c^i = c^i(c-1). \tag{2}$$

If $c \ne 1$, we can divide by $c - 1$ and obtain

$$\Delta\left(\frac{c^i}{c-1}\right) = c^i.$$

Hence, the antidifference of $\{c_i\}_{i\in\mathbb{Z}}$ is

$$\sum c^i = \frac{c^i}{c-1}.$$

Using this result and the fundamental identity (1), we obtain the well-known formula for the sum of a finite geometric series:

$$\sum_{i=0}^{n} c^i = \left[\frac{c^i}{c-1}\right]_{i=0}^{n+1} = \frac{c^{n+1}-1}{c-1}.$$

The product rule for differences is

$$\begin{aligned}\Delta(a_i b_i) = a_{i+1}b_{i+1} - a_i b_i &= a_{i+1}b_{i+1} - a_{i+1}b_i + a_{i+1}b_i - a_i b_i \\ &= a_{i+1}\Delta b_i + (\Delta a_i) b_i.\end{aligned}$$

Rearranging gives

$$(\Delta a_i) b_i = \Delta(a_i b_i) - a_{i+1}\Delta b_i,$$

and summing over $i = p, \ldots, q$ yields the summation by parts formula

$$\sum_{i=p}^{q} (\Delta a_i) b_i = [a_i b_i]_{i=p}^{q+1} - \sum_{i=p}^{q} a_{i+1}\Delta b_i. \tag{3}$$

This is the discrete version of integration by parts, and will be our main tool. We have already mentioned that the identity goes back to Abel's paper [**1**], where it appeared in the proof of "Lehrsatz III" (this part of the paper and its English translation is also available in [**19**, subsection 16.1.3]) in the slightly different but equivalent form

$$\begin{aligned}&\varepsilon_0 p_0 + \varepsilon_1(p_1 - p_0) + \varepsilon_2(p_2 - p_1) + \cdots + \varepsilon_m(p_m - p_{m-1}) \\ &= p_0(\varepsilon_0 - \varepsilon_1) + p_1(\varepsilon_1 - \varepsilon_2) + \cdots + p_{m-1}(\varepsilon_{m-1} - \varepsilon_m) + p_m\varepsilon_m.\end{aligned}$$

This equality is obvious by inspection, and Abel felt no need to explain it. On the other hand, the above-mentioned form (3) seems more useful for calculations, and exhibits the similarity to integration by parts.

To provide an example, we evaluate the sum $\sum_{i=0}^{n} 2^i i$. Noting that $2^i = \Delta 2^i$ (cf. (2) with $c = 2$) and $\Delta i = (i+1) - i = 1$, we use the formula (3) with $a_i = 2^i$, $b_i = i$ to calculate

$$\begin{aligned}\sum_{i=0}^{n} 2^i i = \sum_{i=0}^{n} (\Delta 2^i) i &= [2^i i]_{i=0}^{n+1} - \sum_{i=0}^{n} 2^{i+1} \\ &= 2^{n+1}(n+1) - 2(2^{n+1}-1) = (n-1)2^{n+1} + 2.\end{aligned}$$

3. THE ART OF FINDING ANTIDIFFERENCES. To be able to evaluate weighted Gibonacci sums, we need the differences and antidifferences for the Gibonacci numbers. The recurrence relation $G_{i+2} = G_{i+1} + G_i$ makes it possible to extend the definition of G_i to all $i \in \mathbb{Z}$. Consequently, for all $i \in \mathbb{Z}$ we have

$$\Delta G_i = G_{i+1} - G_i = G_{i-1},$$

and therefore

$$\sum G_i = G_{i+1}.$$

With this result, it is easy to recover the classical result $\sum_{i=1}^{n} G_i = G_{n+2} - G_2$; see [**4**, p. 24] for a nice combinatorial proof, and [**21**, Chapter 1] for a proof in the spirit of discrete calculus. Indeed, the fundamental equation (1) yields

$$\sum_{i=1}^{n} G_i = \sum_{i=1}^{n} \Delta G_{i+1} = [G_{i+1}]_{i=1}^{n+1} = G_{n+2} - G_2. \tag{4}$$

If we restrict ourselves to Gibonacci numbers with even or odd indices, we get

$$\Delta G_{2i} = G_{2i+2} - G_{2i} = G_{2i+1},$$
$$\Delta G_{2i-1} = G_{2i+1} - G_{2i-1} = G_{2i},$$

and therefore

$$\sum G_{2i+1} = G_{2i}, \quad \sum G_{2i} = G_{2i-1}.$$

From these relations and the fundamental equation (1), we obtain the classical identities

$$\sum_{i=1}^{n} G_{2i} = \sum_{i=1}^{n} \Delta G_{2i-1} = [G_{2i-1}]_{i=1}^{n+1} = G_{2n+1} - G_1, \tag{5}$$

$$\sum_{i=1}^{n} G_{2i-1} = \sum_{i=1}^{n} \Delta G_{2i-2} = [G_{2i-2}]_{i=1}^{n+1} = G_{2n} - G_0 \tag{6}$$

(see e.g., [**4**, p. 32], or [**21**, Chapter 1] for the special case $G_i = F_i$).

We also need an antidifference for the Tribonacci numbers, which satisfy the recurrence relation

$$T_{i+3} = T_{i+2} + T_{i+1} + T_i. \tag{7}$$

Again, we leave the initial values T_0, T_1, T_2 unspecified to make our formulas more general. For example, the papers [**5, 20**] take $T_0 = T_1 = 0$ and $T_2 = 1$, while [**3, 7**] use $T_0 = 0$ and $T_1 = T_2 = 1$. Another natural choice is $T_0 = T_1 = 1$ and $T_2 = 2$; in this setting, T_n corresponds to the number of tilings of a $1 \times n$ rectangle with monominoes, dominoes, and trominoes.[1]

[1]This combinatorial interpretation of the Tribonacci numbers immediately leads to Agronomof's identity discussed in [**20**], namely

$$T_{n+p} = T_p T_n + T_{p-1} T_{n-1} + T_{p-2} T_{n-1} + T_{p-1} T_{n-2}$$

(the subscripts in [**20**] are shifted because of the initial conditions). The first term $T_p T_n$ on the right-hand side gives the number of tilings of a $1 \times (n+p)$ rectangle such that no tile covers cells p and $p+1$ at the same time, the second term $T_{p-1} T_{n-1}$ is the number of tilings such that cells p and $p+1$ are covered by a single domino, and the last two terms $T_{p-2} T_{n-1}$ and $T_{p-1} T_{n-2}$ count the tilings where cells p and $p+1$ are covered by a single tromino (there are two ways of doing this).

The recurrence relation (7) makes it possible to consider T_i for all $i \in \mathbb{Z}$ (in particular, our later calculations will involve T_{-1}, which is just $T_2 - T_1 - T_0$). Then, for all $i \in \mathbb{Z}$, we have

$$\Delta T_{i+1} = T_{i+2} - T_{i+1} = T_i + T_{i-1},$$
$$\Delta T_{i-1} = T_i - T_{i-1} = T_{i-2} + T_{i-3}.$$

Therefore, $\Delta(T_{i+1} + T_{i-1}) = 2T_i$, which means that

$$\sum T_i = \frac{1}{2}(T_{i+1} + T_{i-1}). \tag{8}$$

An immediate application of the previous equality and the fundamental formula (1) is the identity

$$\sum_{i=1}^{n} T_i = \left[\frac{1}{2}(T_{i+1} + T_{i-1})\right]_{i=1}^{n+1} = \frac{1}{2}(T_{n+2} + T_n - T_2 - T_0). \tag{9}$$

This result includes as a special case formula (7) from [**20**], where $T_0 = T_1 = 0$ and $T_2 = 1$.

We close this section with a pair of exercises:

- Verify that $\sum T_{2i} = \frac{1}{2}(T_{2i-1} + T_{2i-2})$ and $\sum T_{2i+1} = \frac{1}{2}(T_{2i-1} + T_{2i})$. Use the fundamental identity (1) to determine $\sum_{i=0}^{n} T_{2i}$ and $\sum_{i=0}^{n} T_{2i+1}$.
- Verify that $\sum T_{3i} = \frac{1}{2}(T_{3i-1} - T_{3i-3})$. Try to guess $\sum T_{3i+1}$ and $\sum T_{3i+2}$. Calculate $\sum_{i=0}^{n} T_{3i}$, $\sum_{i=0}^{n} T_{3i+1}$ and $\sum_{i=0}^{n} T_{3i+2}$.

The sums $\sum_{i=0}^{n} T_{2i}$ and $\sum_{i=0}^{n} T_{3i}$ were evaluated in [**7**] in the special case when $T_0 = 0$ and $T_1 = T_2 = 1$.

4. GIBONACCI SUMS WITH EXPONENTIAL WEIGHTS. We are finally ready to proceed to the next goal of this paper and demonstrate that summation by parts is an excellent tool for the evalutation of weighted Fibonacci and Tribonacci sums.

Let us begin with Gibonacci sums of the form $E(n) = \sum_{i=0}^{n} G_i c^i$, i.e., sums with exponential weights. We will encounter a phenomenon that is familiar from integration by parts: Summation by parts does not lead directly to the result, but instead yields an equation for the unknown value $E(n)$. Assume that $c \neq 0$, recall that $G_i = \Delta G_{i+1}$, and apply the formula (3) with $a_i = G_{i+1}$ and $b_i = c^i$ to get

$$\begin{aligned} E(n) &= \sum_{i=0}^{n} G_i c^i = \sum_{i=0}^{n} \Delta(G_{i+1}) c^i = \left[G_{i+1} c^i\right]_{i=0}^{n+1} - \sum_{i=0}^{n} G_{i+2}(c^{i+1} - c^i) \\ &= G_{n+2} c^{n+1} - G_1 - \frac{c-1}{c^2} \sum_{i=2}^{n+2} G_i c^i \\ &= G_{n+2} c^{n+1} - G_1 - \frac{c-1}{c^2} \big(E(n) + G_{n+1} c^{n+1} + G_{n+2} c^{n+2} - G_0 - G_1 c\big). \end{aligned}$$

Solving this equation for $E(n)$ and simplifying the result using $G_{n+2} = G_n + G_{n-1}$, we obtain the identity

$$\sum_{i=0}^{n} G_i c^i = \frac{G_n c^{n+2} + G_{n+1} c^{n+1} + (c-1)G_0 - G_1 c}{c^2 + c - 1}, \tag{10}$$

which holds for all $c \in \mathbb{C}$ such that $c^2 + c - 1 \neq 0$ (note that the result is obviously true for $c = 0$).[2] This identity generalizes formula (11) from [**5**], which deals with the case $G_i = F_i$.

In a similar way, we can deal with weighted sums involving only Gibonacci numbers with either even or odd indices, which we denote by $A(n) = \sum_{i=0}^{n} G_{2i}c^i$ and $B(n) = \sum_{i=0}^{n} G_{2i+1}c^i$. Recalling that $G_{2i} = \Delta G_{2i-1}$ and $G_{2i+1} = \Delta G_{2i}$, we apply the summation by parts formula (3) with $b_i = c^i$, and $a_i = G_{2i-1}$ or $a_i = G_{2i}$, respectively. If $c \neq 0$, we get

$$A(n) = \left[G_{2i-1}c^i\right]_{i=0}^{n+1} - (c-1)\sum_{i=0}^{n} G_{2i+1}c^i = G_{2n+1}c^{n+1} - G_{-1} - (c-1)B(n),$$

$$B(n) = \left[G_{2i}c^i\right]_{i=0}^{n+1} - (c-1)\sum_{i=0}^{n} G_{2i+2}c^i = G_{2n+2}c^{n+1} - G_0 - \frac{c-1}{c}\sum_{i=1}^{n+1} G_{2i}c^i$$

$$= G_{2n+2}c^{n+1} - G_0 - \frac{c-1}{c}(G_{2n+2}c^{n+1} + A(n) - G_0).$$

Solving this system of two linear equations for $A(n)$ and $B(n)$ yields

$$A(n) = \frac{(c-1)c^{n+1}G_{2n+2} - c^{n+2}G_{2n+1} + cG_{-1} + (1-c)G_0}{c^2 - 3c + 1},$$

$$B(n) = \frac{(c-1)c^{n+1}G_{2n+1} - c^{n+1}G_{2n+2} + (1-c)G_{-1} + G_0}{c^2 - 3c + 1}.$$

Finally, applying the identities $G_{2n+2} = G_{2n} + G_{2n+1}$ and $G_{-1} = G_1 - G_0$, we get

$$\sum_{i=0}^{n} G_{2i}c^i = \frac{(c-1)c^{n+1}G_{2n} - c^{n+1}G_{2n+1} + cG_1 + (1-2c)G_0}{c^2 - 3c + 1},$$

$$\sum_{i=0}^{n} G_{2i+1}c^i = \frac{(c-2)c^{n+1}G_{2n+1} - c^{n+1}G_{2n} + (1-c)G_1 + cG_0}{c^2 - 3c + 1}$$

for each $c \in \mathbb{C}$ such that $c^2 - 3c + 1 \neq 0$. (Again, the remaining cases can be dealt with using L'Hôpital's rule.) These formulas might be known, but we were unable to find them in the literature even in the special case when $G_i = F_i$.

5. GIBONACCI SUMS WITH POLYNOMIAL WEIGHTS. We now proceed to Gibonacci sums of the form $R_k(n) = \sum_{i=1}^{n} G_i i^k$, i.e., sums with polynomial weights. This section will be somewhat formula-heavy, but all calculations are elementary, relying only on the binomial theorem and summation by parts. However, they demonstrate an interesting technique: Instead of giving the explicit value of $R_k(n)$, summation by parts leads to a new recurrence relation that expresses $R_k(n)$ in terms of $R_0(n), \ldots, R_{k-1}(n)$.

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[2]What happens if $c^2 + c - 1 = 0$? This equation has roots $c_{1,2} = (-1 \pm \sqrt{5})/2$. The left-hand side of (10) is a continuous of function of c; hence, we can calculate its value at c_j as the limit of the right-hand side of (10) for $c \to c_j$. Using L'Hôpital's rule, we obtain

$$\sum_{i=0}^{n} G_i c_j^i = \frac{G_n(n+2)c_j^{n+1} + G_{n+1}(n+1)c_j^n + G_0 - G_1}{2c_j + 1}, \quad j \in \{1, 2\}.$$

For $k \geq 1$, the binomial theorem yields

$$\Delta i^k = (i+1)^k - i^k = \sum_{j=0}^{k-1} \binom{k}{j} i^j. \tag{11}$$

Therefore, the summation by parts formula (3) with $a_i = G_{i+1}$ and $b_i = i^k$ gives

$$\begin{aligned} R_k(n) &= \left[G_{i+1} i^k\right]_{i=1}^{n+1} - \sum_{i=1}^{n} G_{i+2} \sum_{j=0}^{k-1} \binom{k}{j} i^j \\ &= \left[G_{i+1} i^k\right]_{i=1}^{n+1} - \sum_{j=0}^{k-1} \binom{k}{j} \sum_{i=1}^{n} G_{i+2} (i+2-2)^j \\ &= G_{n+2}(n+1)^k - G_2 - \sum_{j=0}^{k-1} \binom{k}{j} \sum_{l=0}^{j} \binom{j}{l} (-2)^{j-l} \sum_{i=1}^{n} G_{i+2}(i+2)^l. \end{aligned}$$

Here we have interchanged the order of sums, and then invoked the binomial theorem. Since $\sum_{i=1}^{n} G_{i+2}(i+2)^l = R_l(n+2) - R_l(2)$, we get the promised recurrence relation

$$\begin{aligned} R_k(n) &= G_{n+2}(n+1)^k - G_2 \\ &\quad - \sum_{j=0}^{k-1} \binom{k}{j} \sum_{l=0}^{j} \binom{j}{l} (-2)^{j-l} \left(R_l(n+2) - G_1 - 2^l G_2\right), \quad k \in \mathbb{N}. \end{aligned}$$

Using the value of $R_0(n)$ from (4), we can now calculate (it is best to use computer software such as *Wolfram Mathematica*)

$$\begin{aligned} \sum_{i=1}^{n} G_i i &= (n+1)G_{n+2} - G_{n+4} + G_1 + G_2, \\ \sum_{i=1}^{n} G_i i^2 &= (n+1)^2 G_{n+2} - (2n+3)G_{n+4} + 2G_{n+6} - 3G_1 - 5G_2, \\ \sum_{i=1}^{n} G_i i^3 &= (n+1)^3 G_{n+2} - (3n^2+9n+7)G_{n+4} + 6(n+2)G_{n+6} \\ &\qquad - 6G_{n+8} + 19G_1 + 31G_2, \end{aligned}$$

etc. In the special cases when $G_i = F_i$ or $G_i = L_i$, the formulas up to $k = 4$ are given in [**12**, Section 25.2]. Also, the first formula is equivalent to Glaister's result from [**9**].

Essentially the same method works for sums with polynomial weights and Gibonacci numbers with even or odd indices; denote them by $E_k(n) = \sum_{i=1}^{n} G_{2i} i^k$ and $O_k(n) = \sum_{i=1}^{n} G_{2i-1} i^k$. The new idea here is that summation by parts will lead to a system of recurrence relations for E_k and O_k. Indeed, for $k \geq 1$, we use (11) and summation by parts to obtain

$$E_k(n) = \left[G_{2i-1} i^k\right]_{i=1}^{n+1} - \sum_{i=1}^{n} G_{2i+1} \sum_{j=0}^{k-1} \binom{k}{j} i^j$$

$$= \left[G_{2i-1}i^k\right]_{i=1}^{n+1} - \sum_{j=0}^{k-1}\binom{k}{j}\sum_{i=1}^{n} G_{2i+1}(i+1-1)^j$$

$$= G_{2n+1}(n+1)^k - G_1 - \sum_{j=0}^{k-1}\binom{k}{j}\sum_{l=0}^{j}\binom{j}{l}(-1)^{j-l}\sum_{i=1}^{n} G_{2i+1}(i+1)^l$$

$$= G_{2n+1}(n+1)^k - G_1 - \sum_{j=0}^{k-1}\binom{k}{j}\sum_{l=0}^{j}\binom{j}{l}(-1)^{j-l}(O_l(n+1) - G_1)$$

for all $k \in \mathbb{N}$. Similarly,

$$O_k(n) = \left[G_{2i-2}i^k\right]_{i=1}^{n+1} - \sum_{i=1}^{n} G_{2i}\sum_{j=0}^{k-1}\binom{k}{j} i^j$$

$$= \left[G_{2i-2}i^k\right]_{i=1}^{n+1} - \sum_{j=0}^{k-1}\binom{k}{j}\sum_{i=1}^{n} G_{2i}i^j$$

$$= G_{2n}(n+1)^k - G_0 - \sum_{j=0}^{k-1}\binom{k}{j} E_j(n)$$

for all $k \in \mathbb{N}$. Using these recurrence formulas for $E_k(n)$, $O_k(n)$ and the values of $E_0(n)$, $O_0(n)$ from (5) and (6), one can calculate

$$\sum_{i=1}^{n} G_{2i}i = (n+1)G_{2n+1} - G_{2n+2} + G_0, \tag{12}$$

$$\sum_{i=1}^{n} G_{2i-1}i = (n+1)G_{2n} - G_{2n+1} - G_0 + G_1, \tag{13}$$

$$\sum_{i=1}^{n} G_{2i}i^2 = (n+1)^2 G_{2n+1} - (2n+3)G_{2n+2} + 2G_{2n+3} + G_0 - 2G_1, \tag{14}$$

$$\sum_{i=1}^{n} G_{2i-1}i^2 = (n+1)^2 G_{2n} - (2n+3)G_{2n+1} + 2G_{2n+2} - 3G_0 + G_1, \tag{15}$$

etc. Special cases of the formula (14) corresponding to $G_i = F_i$ and $G_i = L_i$ are in agreement with identities (11) and (14) in [**13**].

With the help of (12), it is straightforward to calculate

$$\sum_{j=1}^{n} G_{2n-2j}j = -\sum_{j=1}^{n} G_{2(n-j)}(n-j) + n\sum_{j=1}^{n} G_{2(n-j)} = G_{2n} - nG_1 + (n-1)G_0.$$

This result generalizes Theorems 4.1 and 4.2 from [**6**], which deal with the Fibonacci and Lucas numbers. In a similar way, using (12) again, we obtain

$$\sum_{j=1}^{n-1} G_{2n-2j}j = -\sum_{j=1}^{n-1} G_{2(n-j)}(n-j) + n\sum_{j=1}^{n-1} G_{2(n-j)} = G_{2n} - nG_1 - G_0.$$

A result of this type appeared as identity 3.12 in the recent paper [**14**], where the coefficients of G_1 and G_0 are swapped; a numerical calculation indicates that our version is correct.

6. WEIGHTED TRIBONACCI SUMS. We now switch to weighted sums involving the Tribonacci numbers. This section will be a short one because the methods of Sections 4 and 5 are still applicable without any significant changes. We just need to recall that, according to (8), we have $\sum T_i = \frac{1}{2}(T_{i+1} + T_{i-1})$.

First, we evaluate the weighted Tribonacci sum $H(n) = \sum_{i=0}^{n} T_i c^i$ with exponential weights. For $c \neq 0$, the summation by parts formula (3) with $a_i = \frac{1}{2}(T_{i+1} + T_{i-1})$ and $b_i = c^i$ gives

$$H(n) = \sum_{i=0}^{n} \Delta\left(\frac{1}{2}(T_{i+1} + T_{i-1})\right) c^i = \frac{1}{2}\left[(T_{i+1} + T_{i-1})c^i\right]_{i=0}^{n+1}$$
$$-\frac{c-1}{2}\sum_{i=0}^{n}(T_{i+2} + T_i)c^i = \frac{T_{n+2} + T_n}{2}c^{n+1} - \frac{T_1 + T_{-1}}{2}$$
$$-\frac{c-1}{2}\left(\frac{1}{c^2}\left(T_{n+2}c^{n+2} + T_{n+1}c^{n+1} + H(n) - T_0 - T_1 c\right) + H(n)\right).$$

Solving this equation for $H(n)$ leads to the identity

$$\sum_{i=0}^{n} T_i c^i = \frac{c^{n+1}(1-c)T_{n+1} + c^{n+2}T_{n+2} + c^{n+3}T_n + c^2 T_{-1} + (c-1)T_0 - cT_1}{c^3 + c^2 + c - 1},$$

which holds for all $c \in \mathbb{C}$ such that $c^3 + c^2 + c - 1 \neq 0$. This result generalizes formula (14) from [**5**], which deals with the case $T_0 = T_1 = 0$ and $T_2 = 1$.

Second, consider the weighted sum $U_k(n) = \sum_{i=1}^{n} T_i i^k$ with polynomial weights. For $k \geq 1$, summation by parts with $a_i = \frac{1}{2}(T_{i+1} + T_{i-1})$ and $b_i = i^k$ gives

$$U_k(n) = \sum_{i=1}^{n} \Delta\left(\frac{1}{2}(T_{i+1} + T_{i-1})\right) i^k = \left[\frac{1}{2}(T_{i+1} + T_{i-1})i^k\right]_{i=1}^{n+1}$$
$$-\sum_{i=1}^{n}\frac{1}{2}(T_{i+2} + T_i)((i+1)^k - i^k) = \frac{1}{2}(T_{n+2} + T_n)(n+1)^k - \frac{1}{2}(T_2 + T_0)$$
$$-\frac{1}{2}\sum_{j=0}^{k-1}\binom{k}{j}\left(\sum_{i=1}^{n} T_i i^j + \sum_{i=1}^{n} T_{i+2}(i+2-2)^j\right).$$

Applying the binomial theorem as in Section 5, we get the recurrence relation

$$U_k(n) = \frac{1}{2}(T_{n+2} + T_n)(n+1)^k - \frac{1}{2}(T_2 + T_0)$$
$$-\sum_{j=0}^{k-1}\binom{k}{j}\left(U_j(n) + \sum_{l=0}^{j}\binom{j}{l}(-2)^{j-l}(U_l(n+2) - T_1 - 2^l T_2)\right),$$

which expresses $U_k(n)$, $k \in \mathbb{N}$, in terms of $U_{k-1}(n), \ldots, U_0(n)$. For example, letting $k = 1$ and using the value of $U_0(n)$ from (9), a straightforward calculation gives

$$\sum_{i=1}^{n} T_i i = \frac{1}{4}\left((2n+1)T_n + 2nT_{n+2} - T_{n+4} + 2T_1 + 2T_2\right).$$

For $T_0 = T_1 = 0$ and $T_2 = 1$, this identity agrees with the formulas in [**5**, p. 515] and [**2**, Eq. (2.17)], where they are written in a slightly different but equivalent form.

An alternative method of evaluating Tribonacci sums with polynomial weights is described in [**2**, Section 2].

Following what we did for the Gibonacci numbers, we could continue investigating sums involving only Tribonacci numbers with even or odd indices. Instead, we leave this task as an exercise for the reader: Using summation by parts and the antidifferences mentioned in the end of Section 4, find the values of the sums $\sum_{i=0}^{n} T_{2i}c^i$ and $\sum_{i=0}^{n} T_{2i+1}c^i$. For $c = -1$, your results should agree with the formulas for $\sum_{i=0}^{n}(-1)^i T_{2i}$ and $\sum_{i=0}^{n}(-1)^i T_{2i+1}$ presented in [**5, 7**].

7. GIBONACCI AND TRIBONACCI SQUARED. We hope the reader is now eager to discover further identities using discrete calculus. Why not explore sums with squares of Gibonacci and Tribonacci numbers? Of course, we need antidifferences for the squared sequences. For the Gibonacci numbers, we have

$$G_i^2 = G_i(G_{i+1} - G_{i-1}) = G_iG_{i+1} - G_iG_{i-1} = \Delta(G_{i-1}G_i),$$

and therefore

$$\sum G_i^2 = G_{i-1}G_i. \tag{16}$$

This result immediately leads to the formula for the sum of squared Gibonacci numbers

$$\sum_{i=1}^{n} G_i^2 = [G_{i-1}G_i]_{i=1}^{n+1} = G_nG_{n+1} - G_0G_1$$

(see e.g., [**4**, identity 67]).

Sometimes we also need a second-order antidifference of G_i^2, i.e., a first-order antidifference of $G_{i-1}G_i$. Clearly, it suffices to find an antidifference for the shifted sequence G_iG_{i+1}, which is done as follows. We calculate

$$G_iG_{i+1} = (G_{i+1} - G_{i-1})G_{i+1} = G_{i+1}^2 - G_{i+1}G_{i-1},$$

and apply the Cassini-type identity $G_{i+1}G_{i-1} = G_i^2 + (-1)^i(G_1^2 - G_0G_2)$ (see [**4**, identity 46]) to get

$$\begin{aligned} G_iG_{i+1} &= G_{i+1}^2 - G_i^2 + (-1)^{i+1}(G_1^2 - G_0G_2) \\ &= \Delta G_i^2 + (G_1^2 - G_0G_2)\frac{(-1)^{i+1} - (-1)^i}{2} = \Delta\left(G_i^2 + (G_1^2 - G_0G_2)\frac{(-1)^i}{2}\right). \end{aligned}$$

Thus, we see that

$$\sum G_iG_{i+1} = G_i^2 + (G_1^2 - G_0G_2)\frac{(-1)^i}{2}. \tag{17}$$

The relations (16) and (17) allow us to calculate antidifferences of G_i^2 or arbitrarily high order. Armed with this information, the reader will have no trouble solving the following exercises:

- Use summation by parts to calculate $\sum_{i=1}^n G_i^2 i$. (For $G_i = F_i$ and $G_i = L_i$, the result should agree with the identities in [**13**].)
- Use repeated summation by parts to calculate $\sum_{i=1}^n G_i^2 c^i$.

And what about squared Tribonacci numbers? The antidifference

$$\sum T_i^2 = T_{i-1}T_i - \frac{1}{4}(T_{i-1} + T_{i-3})^2 \tag{18}$$

is not easy to discover, but the verification is routine, and we leave this task to the reader.[3] Once we have the formula, it is obvious that

$$\sum_{i=1}^n T_i^2 = T_n T_{n+1} - \frac{1}{4}(T_n + T_{n-2})^2 - T_0 T_1 + \frac{1}{4}(T_0 + T_{-2})^2.$$

A slightly different but equivalent formula was presented in [**15**], whose author merely remarked it can be proved by induction. For an alternative derivation based on Agronomof's identity, see [**20**]; note, however, this derivation deals only with the case $T_0 = T_1 = 0$ and $T_2 = 1$.

8. CONCLUSION. This is the end of our journey into the world of discrete calculus and weighted sums. We hope the readers will enjoy discovering and proving additional identities using the tools described in this article. Of course, there is no need to restrict oneself to Gibonacci and Tribonacci sums. A possible project is to consider sums involving the Jacobsthal numbers, which are given by

$$J_0 = 0, \quad J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2} \quad \text{for } n \geq 2.$$

The simplest combinatorial interpretation is that J_n corresponds to the number of tilings of a $2 \times (n-1)$ rectangle with dominoes and 2×2 squares. Additional problems related to the Jacobsthal numbers are listed in [**17**]. From the viewpoint of discrete calculus, an attractive feature of these numbers is that their difference and antidifference are easy to express in terms of J_n. A few basic identities involving the Jacobsthal numbers, which can serve as an inspiration, are available in [**22**]. The readers will surely find further ideas to explore.

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