Duelling Idiots and Abel Sums

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Abstract. We investigate a puzzle involving the winning probabilities in a duel of two players. The problem of calculating limiting probabilities leads to the summation of a divergent infinite series. The solution admits a generalization that applies to a wide class of duels.

There is a wealth of literature dealing with mathematical duels and their three-person versions called truels, see for example [**4, 5, 6**] and the references therein. The basic form of a duel is the simple sequential duel, where two players fight against each other. They share a single gun, which they exchange after each unsuccessful shot, until one player finally eliminates the opponent. Another version is the random duel, where the player who is going to shoot is chosen at random in each round. In any case, the problem is to calculate the winning probabilities for both players. Truels are even more interesting, since each player can either choose the target at random, or adopt some deterministic strategy. For example, if all three players have different shooting skills, it is natural for each of them to aim at the stronger of the two opponents. In this case, it is known that the worst player is most likely to win the truel.

In the present paper, we begin by investigating a specific sequential duel proposed by Paul Nahin. The problem of calculating limiting winning probabilities leads to the evaluation of a certain Abel sum, and we will accomplish this task by finding the Cesàro sum of a divergent series. Thus, a problem in recreational mathematics leads naturally to some basic concepts of summability theory. Finally, we will show that the solution admits a generalization to a wide class of sequential duels.

One of Paul Nahin's enjoyable books, *Duelling Idiots and Other Probability Puzzlers*, contains the following amusing exercise [**8**, p. 20]: Players A and B have one gun – a revolver with six chambers – and one bullet. Inserting it into the gun's cylinder, A will spin the cylinder and shoot at B (who is impossible to miss). If the gun doesn't fire then A will give the gun to B, who will spin the cylinder and then shoot at A. If the gun doesn't fire, B spins the cylinder again and gets a second try. If the gun still doesn't fire, B gives the gun to A, who gets a maximum of three trigger pulls (with a spin of the cylinder between pulls). The duel continues in a similar way, and each player gets an extra trigger pull for each turn, until the gun eventually fires. What is the probability that A wins the duel?

We generalize the problem by considering a revolver with n chambers. The probability that any player's shot will be successful is $1/n$. Note that A can win after shot 1, or after shots 4, 5, 6, then after shots 11, 12, 13, 14, 15, etc. Thus, the winning probability of A is

$$\frac{1}{n}+\left(\left(1-\frac{1}{n}\right)^3+\left(1-\frac{1}{n}\right)^4+\left(1-\frac{1}{n}\right)^5\right)\frac{1}{n}$$
$$+\left(\left(1-\frac{1}{n}\right)^{10}+\cdots+\left(1-\frac{1}{n}\right)^{14}\right)\frac{1}{n}+\cdots.$$

We factor $1/n$ out of the whole sum, and observe that the infinite series is divided into groups containing powers of $1-1/n$. The numbers of terms in these groups are 1, 3, 5, etc.; in general, the m-th group has $2m-1$ terms for each $m \in \mathbb{N}$. The exponents of the first terms in each group are 0, $1+2$, $1+2+3+4$, etc.; the first exponent in the m-th group is $\sum_{i=1}^{2m-2} i = (2m-1)(m-1)$. In this way, we rewrite the winning probability of A in the form

$$\frac{1}{n}\sum_{m=1}^{\infty}\left(1-\frac{1}{n}\right)^{(2m-1)(m-1)}\sum_{j=0}^{2m-2}\left(1-\frac{1}{n}\right)^{j}$$
$$=\sum_{m=1}^{\infty}\left(1-\frac{1}{n}\right)^{(2m-1)(m-1)}\left(1-\left(1-\frac{1}{n}\right)^{2m-1}\right).$$

Nahin considers only the case $n=6$ and evaluates the sum numerically to a high precision (25 digits), saying it is “a number I feel safe in saying has never appeared in print before until now” [**8**, p. 85].

Indeed, it seems difficult to find the value of the sum analytically, so let us stick with numerical calculations. Table 1 shows that for small integers n, the probability that A wins the duel is greater than $1/2$. We also see that it decreases with n, and seems to approach $1/2$ for $n \to \infty$. An intuitive explanation might be that for large n, the probability that A wins in the first round will be negligible, and there is no major advantage in being the first player. Our goal is to supply a rigorous calculation of the limiting probability.

n	P
2	0.610322
3	0.557085
4	0.538937
5	0.529620
6	0.523919
7	0.520065
8	0.517283
9	0.515180
10	0.513533

Table 1. Winning probabilities for player A and revolver with n chambers.

Among mathematicians, the person who is most famous for being seriously involved in duels is Évariste Galois, but our problem is closer to the work of his contemporary Niels Henrik Abel, and boils down to an interesting exercise in the summability of divergent infinite series.

We denote $q = 1 - 1/n$ to simplify the formula for the winning probability of A:

$$P(q)=\sum_{m=1}^{\infty} q^{(2m-1)(m-1)}\left(1-q^{2m-1}\right). \tag{1}$$

In fact, if we forget about the revolver having n chambers, we can assume that q is the probability that a player misses the opponent (for whatever reason), and this version of the problem makes sense not just for q of the form $1-1/n$, but for any $q \in (0,1)$.

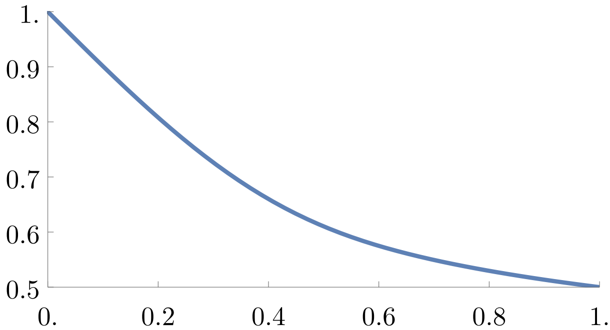

Figure 1: The probability $P(q)$ as a function of q.

In his review [**1**] of Nahin's book, Michael Fox pointed out that one can obtain the result (1) in a simpler way: $q^{(2m-1)(m-1)}$ is the probability that the gun does not fire during the first $(2m-1)(m-1)$ shots, and $1 - q^{2m-1}$ is the probability that it fires during the subsequent $2m-1$ shots.

The first few terms of the infinite series (1) are

$$q^0 - q^1 + q^3 - q^6 + q^{10} - q^{15} + \cdots ,$$

which suggests that the sum can be written in the alternative form

$$P(q) = \sum_{k=0}^{\infty} (-1)^k q^{\frac{1}{2}k(k+1)}. \tag{2}$$

Indeed, this power series has radius of convergence 1, and the sum of two adjacent terms corresponding to $k = 2m-2$ and $k = 2m-1$, $m \in \mathbb{N}$, is

$$(-1)^{2m-2} q^{\frac{1}{2}(2m-2)(2m-1)} + (-1)^{2m-1} q^{\frac{1}{2}(2m-1)2m} = q^{(m-1)(2m-1)} - q^{(2m-1)m},$$

which agrees with the terms of the series (1).

Plotting the values of $P(q)$ given by the power series (2) for $q \in [0,1)$, we obtain the graph in Figure 1, which provides further evidence that $P(q)$ tends to $1/2$ for $q \to 1-$ (note that this corresponds to $n \to \infty$ in our earlier calculation). Thus, our goal is to show that

$$\lim_{q \to 1-} \left(\sum_{k=0}^{\infty} (-1)^k q^{\frac{1}{2}k(k+1)} \right) = \frac{1}{2}.$$

The infinite series (2) is divergent for $q = 1$, which makes it impossible to calculate the limit using Abel's theorem. However, given a power series of the form $\sum_{k=0}^{\infty} a_k x^k$ with radius of convergence 1, the limit

$$\lim_{x \to 1-} \left(\sum_{k=0}^{\infty} a_k x^k \right)$$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a_k	1	-1	0	1	0	0	-1	0	0	0	1	0	0	0	0
s_k	1	0	0	1	1	1	0	0	0	0	1	1	1	1	1

Table 2. The sequence $\{a_k\}_{k=0}^{\infty}$ and its partial sums $\{s_k\}_{k=0}^{\infty}$.

(provided it exists) is called the Abel sum of the series $\sum_{k=0}^{\infty} a_k$. Thus, we are trying to calculate the Abel sum of the series whose terms are

$$a_k = \begin{cases} (-1)^j & \text{if } k = \frac{j(j+1)}{2} \text{ for some } j \in \mathbb{N} \cup \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

One possible way of calculating an Abel sum is to recall an old result by Georg Frobenius, which says that if an infinite series has a Cesàro sum, then its Abel sum exists and has the same value (see [**2**] or [**3**, Section 5.12]). The Cesàro sum is defined as

$$\lim_{N\to\infty} \frac{\sum_{k=0}^{N} s_k}{N+1},$$

where $s_k = \sum_{j=0}^{k} a_j$, $k \in \mathbb{N} \cup \{0\}$, are the partial sums of $\sum_{k=0}^{\infty} a_k$.

The first few partial sums of our series are given in Table 2. Since the nonzero terms of $\sum_{k=0}^{\infty} a_k$ alternate between 1 and -1, the partial sums alternate between 1 and 0. The changes occur at positions $k = \frac{j(j+1)}{2}$, $j \in \mathbb{N} \cup \{0\}$. If j is even, then a block of ones starts at position k; if j is odd, there is a block of zeros beginning at position k. The lengths of the blocks are 1, 2, 3, 4, 5, etc. At the beginning of each block containing ones, the average value

$$\sigma_N = \frac{\sum_{k=0}^{N} s_k}{N+1},$$

of the partial sums begins to increase with N, and reaches a local maximum at the end of that block. Similarly, at the beginning of each block of zeros, σ_N begins to decrease with N, and reaches a local minimum at the end of the block. Let us calculate the values of these maxima and minima.

The end of the l-th block of ones, counting from $l = 0$, occurs at position

$$N = \frac{(2l+1)(2l+2)}{2} - 1 = (l+1)(2l+1) - 1,$$

and the total number of ones until that point is $1 + 3 + \cdots + (2l+1) = l(l+1)$. Hence, at this point, we have

$$\sigma_N = \frac{l(l+1)}{(l+1)(2l+1)} = \frac{l}{2l+1}.$$

The end of the l-th block of zeros occurs at position

$$N = \frac{(2l+2)(2l+3)}{2} - 1 = (l+1)(2l+3) - 1,$$

and the total number of ones until that point is still $l(l+1)$. Hence, at this point, we have

$$\sigma_N = \frac{l(l+1)}{(l+1)(2l+3)} = \frac{l}{2l+3}.$$

For $l \to \infty$, the values of these local maxima and minima tend to $1/2$. This proves that the Cesàro sum is

$$\lim_{N\to\infty} \sigma_N = \frac{1}{2},$$

and consequently $\lim_{q\to 1-} P(q) = 1/2$, as we wished to prove.

Observe that if we denote

$$A = \{k \in \mathbb{N} \cup \{0\} : s_k = 1\},$$

then $\sum_{k=0}^{N} s_k = |A \cap \{0, \dots, N\}|$, and the Cesàro sum

$$\lim_{N\to\infty} \sigma_N = \lim_{N\to\infty} \frac{|A \cap \{0, \dots, N\}|}{N+1}$$

is simply the density of the set A in $\mathbb{N} \cup \{0\}$. Thus, the key point of the previous calculation was to verify that the density of the set $\{0, 3, 4, 5, 10, \dots\}$, containing the numbers of all rounds in which A is the active player (counting the rounds from 0), is $1/2$.

Let us consider a more general version of the original problem, namely a duel in which the numbers of shots for the two players are no longer given by the sequence 1, 2, 3, 4, …, but by a sequence of positive integers

$$a_1, b_1, a_2, b_2, \dots. \tag{3}$$

The winning probability for player A is now given by

$$\begin{aligned} P(q) &= (1-q) \sum_{m=0}^{\infty} q^{\sum_{i=1}^{m}(a_i+b_i)} (1 + q + \cdots + q^{a_{m+1}-1}) \\ &= \sum_{m=0}^{\infty} q^{\sum_{i=1}^{m}(a_i+b_i)} (1 - q^{a_{m+1}}) \\ &= q^0 - q^{a_1} + q^{a_1+b_1} - q^{a_1+b_1+a_2} + q^{a_1+b_1+a_2+b_2} - \cdots . \end{aligned} \tag{4}$$

The existence as well as the value of $\lim_{q\to 1-} P(q)$ now depend on the choice of the sequence (3). For example, it is known that

$$\lim_{q\to 1-} \left(\sum_{n=0}^{\infty} (-1)^n q^{2^n} \right)$$

does not exist [**3**, Section 4.10]; hence, if we choose (3) in such a way that $a_1 = 2$, $a_1 + b_1 = 2^2$, $a_1 + b_1 + a_2 = 2^3$, etc., then

$$P(q) = 1 - \sum_{n=0}^{\infty} (-1)^n q^{2^n},$$

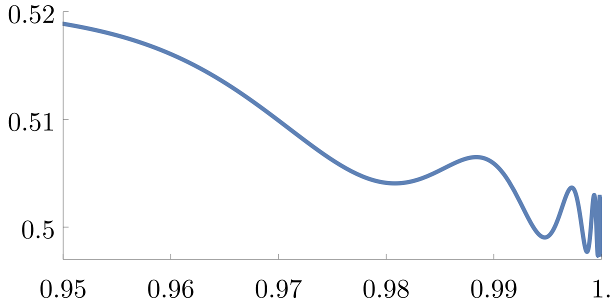

Figure 2: A duel where $\lim_{q\to 1-} P(q)$ does not exist.

and $\lim_{q\to 1-} P(q)$ does not exist, see Figure 2. This example also shows that P need not be a monotonic function of q.

However, using the relation between the Abel and Cesàro sums as before, we arrive at the following result.

Theorem 1. *In a duel where the numbers of shots for the two players are given by the sequence* (3)*, the winning probability of player A satisfies* $\lim_{q\to 1-} P(q) = \alpha$*, provided that the set*

$$\{0, \dots, a_1 - 1, a_1 + b_1, \dots, a_1 + b_1 + a_2 - 1, a_1 + b_1 + a_2 + b_2, \dots\} \tag{5}$$

has density α *in* $\mathbb{N} \cup \{0\}$.

A simple example, also considered by Nahin in [**8**, Chapter 2], is the duel where the players always alternate after one shot, i.e., $a_n = b_n = 1$ for all $n \in \mathbb{N}$; this is the basic sequential dual. Then $\lim_{q\to 1-} P(q) = 1/2$, because the set $\{0, 2, 4, \dots\}$ has density $1/2$. Alternatively, one can observe that $P(q) = \sum_{m=0}^{\infty} (-1)^m q^m$, and $\lim_{q\to 1-} P(q)$ equals the Cesàro sum of Grandi's series $\sum_{m=0}^{\infty} (-1)^m$, which is $1/2$.

In general, the density of the set (5) might be difficult to calculate. Therefore, we propose yet another method of finding $\lim_{q\to 1-} P(q)$. It is based on the next result, which follows from a corollary given in [**7**].

Proposition 2. *If* $\{\lambda_n\}_{n=0}^{\infty}$ *is a strictly increasing sequence of nonnegative integers such that*

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \frac{\lambda_{2n+1} - \lambda_{2n}}{\lambda_{2n+2} - \lambda_{2n}} = \alpha, \tag{6}$$

then

$$\lim_{s\to 0+} \left(\sum_{n=0}^{\infty} (-1)^n e^{-\lambda_n s} \right) = \alpha. \tag{7}$$

Although the result in [**7**] assumes that λ_n are positive, it remains true even if $\lambda_0 = 0$, because neither the assumptions in (6) nor the limit in (7) depend on λ_0 (the limit of the zeroth term of the series, i.e., $\lim_{s\to 0+} e^{-\lambda_0 s}$, is always 1). The basic idea of the proof presented in [**7**] is to write

$$\sum_{n=0}^{\infty} (-1)^n e^{-\lambda_n s} = \sum_{n=0}^{\infty} \int_{\lambda_{2n}}^{\lambda_{2n+1}} s e^{-sx} \, \mathrm{d}x = \int_0^{\infty} \left(s e^{-sx} \sum_{n=0}^{\infty} \mathbf{1}_{[\lambda_{2n}, \lambda_{2n+1}]}(x) \right) \mathrm{d}x,$$

integrate by parts, and calculate the limit for $s \to 0+$ using the dominated convergence theorem (the assumptions from (6) are needed to determine the limit of the integrand).

To see why Proposition 2 is useful in connection with duels, substitute $s = -\ln q$, where $q \in (0, 1)$. This transforms the relation (7) into

$$\lim_{q \to 1-} \left(\sum_{n=0}^{\infty} (-1)^n q^{\lambda_n} \right) = \alpha,$$

which is exactly the type of a limit we are interested in. Returning to the probability $P(q)$ given by (4), we see that we need to take

$$\begin{aligned} \lambda_0 &= 0, \\ \lambda_1 &= a_1, \\ \lambda_2 &= a_1 + b_1, \\ \lambda_3 &= a_1 + b_1 + a_2, \\ \lambda_4 &= a_1 + b_1 + a_2 + b_2, \end{aligned}$$

etc., which is a strictly increasing sequence of nonnegative integers as required by Proposition 2. Next, we calculate

$$\frac{\lambda_{n+1}}{\lambda_n} = \begin{cases} 1 + \frac{a_{k+1}}{a_1+b_1+\cdots+a_k+b_k} & \text{if } n = 2k, \\ 1 + \frac{b_{k+1}}{a_1+b_1+\cdots+a_k+b_k+a_{k+1}} & \text{if } n = 2k+1. \end{cases}$$

Consequently, the first assumption in (6) will be satisfied if

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_1 + b_1 + \cdots + a_k + b_k} = 0 \tag{8}$$

and

$$\lim_{k \to \infty} \frac{b_{k+1}}{a_1 + b_1 + \cdots + a_k + b_k + a_{k+1}} = 0. \tag{9}$$

As for the second assumption, we have

$$\frac{\lambda_{2n+1} - \lambda_{2n}}{\lambda_{2n+2} - \lambda_{2n}} = \frac{a_{n+1}}{a_{n+1} + b_{n+1}},$$

and therefore we need

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_{n+1} + b_{n+1}} = \alpha. \tag{10}$$

To sum up, we have proved the following result.

Theorem 3. *In a duel where the numbers of shots for the two players are given by the sequence* (3)*, the winning probability of player A satisfies* $\lim_{q \to 1-} P(q) = \alpha$*, provided that* (8)*,* (9) *and* (10) *hold.*

As a quick check, observe that the original version of the duel corresponds to $a_n = 2n - 1$ and $b_n = 2n$ for all $n \in \mathbb{N}$ (or $\lambda_k = \frac{k(k+1)}{2}$ for each $k \in \mathbb{N} \cup \{0\}$). The two conditions (8) and (9) are satisfied, and the relation (10) holds with $\alpha = 1/2$. This confirms our earlier result $\lim_{q \to 1-} P(q) = 1/2$.

The simple sequential duel with $a_n = b_n = 1$ for all $n \in \mathbb{N}$ is even more trivial, since (8), (9), and (10) are satisfied with $\alpha = 1/2$.

On the other hand, one can easily find sequences (3) for which the limiting probability is no longer $1/2$. For example, if we choose any two numbers $p, q \in \mathbb{N}$ and let $a_n = p$ and $b_n = q$ for all $n \in \mathbb{N}$, then conditions (8), (9), and (10) are satisfied with $\alpha = \frac{p}{p+q}$. In this way, we can achieve the limiting probability to be any rational number from $(0, 1)$.

We leave it up to the reader to find sequences (3) for which the limiting probabilities are irrational. Another exercise is to check that conditions (8), (9), and (10) hold if a_n and b_n are given by polynomials in n with nonnegative coefficients, and to determine the corresponding limit α.

Once we have Theorem 3, we may combine it with Theorem 1 to calculate densities of various sets having the form (5). A set of nonnegative integers can be written in this form if i) it contains zero, ii) it is infinite, and iii) its complement in $\mathbb{N} \cup \{0\}$ is also infinite. The numbers $a_1, a_2, \ldots$ simply correspond to the lengths of successive blocks of integers that are included in the set, while $b_1, b_2, \ldots$ are the lengths of gaps. In view of this, condition (10) is quite natural, and if it holds together with conditions (8) and (9) (e.g., if a_n and b_n are polynomials in n with nonnegative coefficients), then the density of the set (5) is α. The previously mentioned conditions i)–iii) are not restrictive: We can always include zero in a set without changing its density, a finite set has density 0, and a set whose complement is finite has density 1.

The results obtained in this paper apply not only to duels in Nahin's sense, but also to all kinds of sequential two-player knock-out games where both players have the same winning probability $1 - q$ in each round. We leave it as an exercise to extend Theorem 1 to the case of multiple-player sequential games, such as sequential truels in which three players alternate after prescribed numbers of shots. In this situation, the limit of each player's winning probability for $q \to 1-$ equals the density of the set containing numbers of all rounds in which he/she is the active player.

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REFERENCES

1. Fox, M. (2002). Review of Duelling Idiots and Other Probability Puzzlers, by P. J. Nahin. *Math. Gaz.* 86(505): 167–169.
2. Frobenius, G. (1880). Ueber die Leibnitzsche Reihe. *J. Reine Angew. Math.* 89: 262–264.
3. Hardy, G. H. (1949). *Divergent Series*. Oxford: Clarendon Press.
4. Kilgour, D. M., Brams, S. J. (1997). The truel. *Math. Mag.* 70(5): 315–326.
5. Knuth, D. E. (1973). The triel: a new solution. *J. Recreat. Math.* 6(1): 1–7.
6. Lanphier, D. (2017). Duels, truels, gruels, and survival of the unfittest. In: Beineke, J., Rosenhouse, J., eds. *The Mathematics of Various Entertaining Subjects*, Vol. 2. Princeton, NJ: Princeton University Press, pp. 169–194.
7. Lee, S. (2018). Proving $\lim_{x \to 0+} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^x} = \frac{1}{2}$. `https://math.stackexchange.com/q/2866902`.
8. Nahin, P. (2000). *Duelling Idiots and Other Probability Puzzlers*. Princeton, NJ: Princeton University Press.

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