A general form of Gronwall inequality with Stieltjes integrals

Claudio A. Gallegos[1], Ignacio Márquez Albés[2], Antonín Slavík[3]

Abstract

We present a new Gronwall inequality for Stieltjes integrals, which improves numerous existing results, and has a simple proof based on the quotient rule for Stieltjes integrals. As an application, we obtain uniqueness theorems for measure differential equations and nabla dynamic equations. Finally, we revisit the topic from the perspective of Stieltjes derivatives.

Keywords: Gronwall inequality, Stieltjes integral, Stieltjes derivative, uniqueness of solutions

MSC 2020 subject classification: 26D10, 34A06, 34A12, 34N05

1 Introduction

The Gronwall inequality is a fundamental tool in the theory of differential and integral equations. Its classical statement is as follows (see, for instance, [12, Corollary 1.9.1]): If $u, K, L\colon [t_0, t_0+T] \to [0,\infty)$ are continuous functions satisfying the integral inequality

$$u(t) \le K(t) + \int_{t_0}^{t} L(s)u(s)\,\mathrm{d}s, \quad t \in [t_0, t_0+T], \tag{1.1}$$

then

$$u(t) \le K(t) + \int_{t_0}^{t} K(s)L(s)\exp\left(\int_{s}^{t} L(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [t_0, t_0+T]. \tag{1.2}$$

Special cases of this result were obtained by T. H. Gronwall (see [6]) and later by R. Bellman (see [2, p. 35]), who called it the *fundamental lemma*. For this reason, the result is sometimes referred to as the Gronwall–Bellman inequality. Note that if equality holds in (1.1), then it also holds in (1.2). Thus, in fact, the statement provides a comparison between the solutions of an integral inequality and the corresponding integral equation, respectively. In the special case when K is a constant function, we get the simpler estimate

$$u(t) \le K\exp\left(\int_{t_0}^{t} L(\tau)\,\mathrm{d}\tau\right), \tag{1.3}$$

whose right-hand side is the solution of the integral equation $u(t) = K + \int_{t_0}^{t} L(s)u(s)\,\mathrm{d}s$.

In the present paper, we are interested in Gronwall-type results with ordinary integrals replaced by Stieltjes integrals. A natural generalization of (1.1) is

$$u(t) \le K(t) + \int_{t_0}^{t} L(s)u(s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0+T],$$

[1]Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Casilla 653, Las Palmeras 3425, Santiago, Chile. E-mail: `claudio.gallegos.castro@gmail.com`

[2]Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic. E-mail: `marquez@math.cas.cz`, ORCID iD: 0000-0002-0754-9544.

[3]Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8, Czech Republic. E-mail: `slavik@karlin.mff.cuni.cz`, ORCID iD: 0000-0003-3941-7375.

where g is a nondecreasing function. However, using the substitution theorem (see Theorem 2.2), the integral on the right-hand side can be rewritten as $\int_{t_0}^t u(s)\,\mathrm{d}P(s)$, where $P(s) = \int_{t_0}^s L(\tau)\,\mathrm{d}g(\tau)$. Hence, it suffices to study the simpler integral inequality

$$u(t) \leq K(t) + \int_{t_0}^t u(s)\,\mathrm{d}P(s), \quad t \in [t_0, t_0 + T],$$

where P is a nondecreasing function.

Our goal is to obtain a priori estimates for u similar to those in (1.2) and (1.3). Several results of this type, which are useful in the study of measure differential equations and Stieltjes differential equations, are available in the literature, see e.g. [5, 10, 11, 14, 20, 23]. However, their assumptions are not completely satisfactory. In short, it is usually assumed that K is a constant function, or that P is a left-continuous or right-continuous function. We will show that these assumptions are not necessary, and provide a concise proof of a new general version of Gronwall's lemma.

The article is organized as follows. In Section 2, we recall the substitution theorem and quotient rule for Stieltjes integrals, as well as the notion of the generalized exponential function. These preliminaries are crucial for Section 3, where we establish a new version of the Gronwall inequality along with a detailed comparison with the results available in the literature. Section 4 contains two applications – uniqueness theorems for measure differential equations and for nabla dynamic equations on time scales. Finally, in Section 5, we revisit the inequalities obtained before in terms of Stieltjes derivatives, and present a differential counterpart of the earlier results.

Unless otherwise specified, all Stieltjes integrals in Sections 2, 3, 4 will be understood as gauge integrals, i.e., in the Kurzweil–Stieltjes sense (or, equivalently, in the Perron–Stieltjes sense). The definition of the Kurzweil–Stieltjes integral $\int_a^b f\,\mathrm{d}g$ is based on sums of the form $\sum_{i=1}^m f(\xi_i)(g(t_i) - g(t_{i-1}))$, where $a = t_0 < \cdots < t_m = b$ is a partition of $[a, b]$, and $\xi_i \in [t_{i-1}, t_i]$ for each $i \in \{1, \ldots, m\}$. The precise definition can be found e.g. in [20, Section 6.2] or in [17, Section 7.1], but it is not too important here, since we use only some basic properties together with the results presented in Section 2. Later, in Section 5, we focus on Lebesgue-Stieltjes integrals. Readers interested in Gronwall inequalities for other types of Stieltjes integrals might consult the paper [22], which deals with modified Stieltjes integrals and Dushnik integrals. The articles [9] and [18] provide an abstract approach to Gronwall-type inequalities independent on the choice of the integral. Additional interesting results for the abstract Lebesgue integral are available in [7, 8]. However, none of these articles fully covers the results obtained in the present paper.

2 Preliminaries

To prove our main result, we need the following quotient rule for the Kurzweil-Stieltjes integral, which was recently proved in [16, Theorem 6.6].

Theorem 2.1. *If $f, g : [a, b] \to \mathbb{R}$ have bounded variation and for each $t \in [a, b]$, we have $g(t) \neq 0$, $g(t-) \neq 0$, and $g(t+) \neq 0$, then*

$$\frac{f(b)}{g(b)} - \frac{f(a)}{g(a)} = \int_a^b \frac{\mathrm{d}f(t)}{g(t+)} - \int_a^b \frac{f(t-)\,\mathrm{d}g(t)}{g(t-)g(t+)},$$

with the convention that $g(a-) = g(a)$ and $g(b+) = g(b)$.

The following substitution theorem is well known in the case when f is bounded (see e.g. [20, Theorem 6.6.1]). However, this assumption is not necessary, as shown in [17, Chapter 7, Exercise 2]. Because the book need not be easily accessible and the final part of the proof provided there is slightly confusing, we include the proof here.

Theorem 2.2. *Assume that $g,h\colon [a,b]\to\mathbb{R}$ are such that $\int_a^b g\,\mathrm{d}h$ exists. Then for each function $f\colon [a,b]\to\mathbb{R}$, we have*
$$\int_a^b f(x)\,\mathrm{d}\left(\int_a^x g(z)\,\mathrm{d}h(z)\right)=\int_a^b f(x)g(x)\,\mathrm{d}h(x),$$
whenever either side of the equation exists.

Proof. Denote $w(x)=\int_a^x g\,\mathrm{d}h$, $x\in[a,b]$. It suffices to check that for each $\varepsilon>0$, there exists a gauge $\delta\colon [a,b]\to(0,\infty)$ such that if $(\xi_i,[\alpha_{i-1},\alpha_i])_{i=1}^m$ is a δ-fine partition of $[a,b]$, then
$$\left|\sum_{i=1}^m f(\xi_i)(w(\alpha_i)-w(\alpha_{i-1}))-\sum_{i=1}^m f(\xi_i)g(\xi_i)(h(\alpha_i)-h(\alpha_{i-1}))\right|\le\varepsilon.$$

The definition of $\int_a^b g\,\mathrm{d}h$ implies that for each $n\in\mathbb{N}$, there exists a gauge $\delta_n\colon [a,b]\to(0,\infty)$ such that if $(\xi_i,[\alpha_{i-1},\alpha_i])_{i=1}^m$ is a δ_n-fine partition of $[a,b]$, then
$$\left|\int_a^b g\,\mathrm{d}h-\sum_{i=1}^m g(\xi_i)(h(\alpha_i)-h(\alpha_{i-1}))\right|<\frac{\varepsilon}{n\cdot 2^{n+1}}.$$
Now, let
$$E_n=\{x\in[a,b]: n-1\le|f(x)|<n\},\quad n\in\mathbb{N},$$
and define $\delta\colon [a,b]\to(0,\infty)$ by
$$\delta(x)=\delta_n(x)\quad\text{whenever } x\in E_n.$$
If $(\xi_i,[\alpha_{i-1},\alpha_i])_{i=1}^m$ is a δ-fine partition of $[a,b]$, then for each $n\in\mathbb{N}$, the Saks-Henstock lemma (see [20, Corollary 6.5.2]) implies
$$\sum_{i:\xi_i\in E_n}\left|\int_{\alpha_{i-1}}^{\alpha_i} g\,\mathrm{d}h-g(\xi_i)(h(\alpha_i)-h(\alpha_{i-1}))\right|\le\frac{\varepsilon}{n\cdot 2^n}.$$
Therefore,
$$\begin{aligned}&\left|\sum_{i=1}^m f(\xi_i)(w(\alpha_i)-w(\alpha_{i-1}))-\sum_{i=1}^m f(\xi_i)g(\xi_i)(h(\alpha_i)-h(\alpha_{i-1}))\right|\\
&=\left|\sum_{i=1}^m f(\xi_i)\left(\int_{\alpha_{i-1}}^{\alpha_i} g\,\mathrm{d}h-g(\xi_i)(h(\alpha_i)-h(\alpha_{i-1}))\right)\right|\\
&=\left|\sum_{n=1}^\infty\sum_{i:\xi_i\in E_n} f(\xi_i)\left(\int_{\alpha_{i-1}}^{\alpha_i} g\,\mathrm{d}h-g(\xi_i)(h(\alpha_i)-h(\alpha_{i-1}))\right)\right|\\
&\le\sum_{n=1}^\infty n\sum_{i:\xi_i\in E_n}\left|\int_{\alpha_{i-1}}^{\alpha_i} g\,\mathrm{d}h-g(\xi_i)(h(\alpha_i)-h(\alpha_{i-1}))\right|\le\sum_{n=1}^\infty\frac{\varepsilon}{2^n}=\varepsilon,\end{aligned}$$
and the proof is complete. □

Finally, we need the definition and basic properties of the generalized exponential function. If $P\colon [a,b]\to\mathbb{R}$ is a regulated function (i.e., it has one-sided limits at each point), we denote $\Delta^+P(t)=P(t+)-P(t)$, $\Delta^-P(t)=P(t)-P(t-)$, and $\Delta P(t)=P(t+)-P(t-)$ for all $t\in[a,b]$, with the convention that $P(a-) = P(a)$ and $P(b+) = P(b)$. Recall that if $P\colon [a,b] \to \mathbb{R}$ has bounded variation and satisfies $1 + \Delta^+ P(t) \neq 0$ for all $t \in [a, t_0)$ and $1 - \Delta^- P(t) \neq 0$ for all $t \in (t_0, b]$, then the linear integral equation

$$x(t) = 1 + \int_{t_0}^{t} x(s)\,\mathrm{d}P(s), \quad t \in [a,b], \tag{2.1}$$

has a unique solution, which is known as the generalized exponential function, and is denoted by $t \mapsto e_{\mathrm{d}P}(t, t_0)$. It has the following basic properties (see [19] and [20, Section 8.5]):

- The function $t \mapsto e_{\mathrm{d}P}(t, t_0)$ is regulated on $[a, b]$ and satisfies

$$e_{\mathrm{d}P}(t+, t_0) = (1 + \Delta^+ P(t))\, e_{\mathrm{d}P}(t, t_0), \quad t \in [a, b), \tag{2.2}$$

$$e_{\mathrm{d}P}(t-, t_0) = (1 - \Delta^- P(t))\, e_{\mathrm{d}P}(t, t_0), \quad t \in (a, b]. \tag{2.3}$$

- $e_{\mathrm{d}P}(t, s)\, e_{\mathrm{d}P}(s, r) = e_{\mathrm{d}P}(t, r)$ for every $t, s, r \in [a, b]$.
- $e_{\mathrm{d}P}(t, s) = e_{\mathrm{d}P}(s, t)^{-1}$ for every t, $s \in [a, b]$.
- We have the explicit formula

$$e_{\mathrm{d}P}(t, t_0) = \begin{cases} 1, & t = t_0, \\ \dfrac{e^{P(t-)-P(t_0+)}}{e^{\sum_{s\in(t_0,t)} \Delta P(s)}} \; \dfrac{\prod\limits_{s\in[t_0,t)} (1 + \Delta^+ P(s))}{\prod\limits_{s\in(t_0,t]} (1 - \Delta^- P(s))}, & t > t_0, \\ \dfrac{e^{\sum_{s\in(t,t_0)} \Delta P(s)}}{e^{P(t_0-)-P(t+)}} \; \dfrac{\prod\limits_{s\in(t,t_0]} (1 - \Delta^- P(s))}{\prod\limits_{s\in[t,t_0)} (1 + \Delta^+ P(s))}, & t < t_0. \end{cases} \tag{2.4}$$

- If $1 + \Delta^+ P(t) > 0$ for all $t \in [a, b)$ and $1 - \Delta^- P(t) > 0$ for all $t \in (a, b]$, then $e_{\mathrm{d}P}(t, t_0) > 0$ for all $t \in [a, b]$.

3 A general form of Gronwall inequality

We now present our main result, a general version of the Gronwall inequality with Stieltjes integrals. The proof is inspired by the proof of [14, Proposition 4.3], which is based on Stieltjes derivatives.

Theorem 3.1. *Let $P\colon [t_0, t_0 + T] \to \mathbb{R}$ be a nondecreasing function such that $1 - \Delta^- P(s) > 0$ for all $s \in (t_0, t_0 + T]$. If $K\colon [t_0, t_0 + T] \to [0, \infty)$ is such that $\int_{t_0}^{t_0+T} K(s)\,\mathrm{d}P(s)$ exists and $u\colon [t_0, t_0 + T] \to \mathbb{R}$ satisfies*

$$u(t) \le K(t) + \int_{t_0}^{t} u(s)\,\mathrm{d}P(s), \qquad t \in [t_0, t_0 + T], \tag{3.1}$$

then

$$u(t) \le K(t) + \int_{t_0}^{t} \frac{K(s) e_{\mathrm{d}P}(t, s)}{(1 + \Delta^+ P(s))(1 - \Delta^- P(s))}\,\mathrm{d}P(s), \qquad t \in [t_0, t_0 + T], \tag{3.2}$$

with the convention that $\Delta^- P(t_0) = 0$ and $\Delta^+ P(s) = 0$ if $s = t$. Moreover, if equality holds in (3.1)*, then it also holds in* (3.2)*.*

In addition, if K is bounded on $[t_0, t] \subset [t_0, t_0 + T]$, then

$$u(t) \le \left(\sup_{\xi\in[t_0,t]} K(\xi) \right) e_{\mathrm{d}P}(t, t_0). \tag{3.3}$$

Proof. Let $U\colon [t_0, t_0+T] \to \mathbb{R}$ be given by

$$U(t) = \int_{t_0}^{t} u(s)\,\mathrm{d}P(s), \quad t \in [t_0, t_0+T].$$

The assumptions on P imply that $e_{\mathrm{d}P}(s,t_0) > 0$, $e_{\mathrm{d}P}(s+,t_0) > 0$ and $e_{\mathrm{d}P}(s-,t_0) > 0$ for all $s \in [t_0, t_0+T]$. Using Theorem 2.1 with $f(t) = U(t)$ and $g(t) = e_{\mathrm{d}P}(t,t_0)$ and then Theorem 2.2 together with (2.1), (2.2), (2.3), we obtain

$$\begin{aligned}
\frac{U(t)}{e_{\mathrm{d}P}(t,t_0)} &= \int_{t_0}^{t} \frac{\mathrm{d}U(s)}{e_{\mathrm{d}P}(s+,t_0)} - \int_{t_0}^{t} \frac{U(s-)}{e_{\mathrm{d}P}(s-,t_0)e_{\mathrm{d}P}(s+,t_0)}\,\mathrm{d}(e_{\mathrm{d}P}(s,t_0)) \\
&= \int_{t_0}^{t} \frac{u(s)\,\mathrm{d}P(s)}{(1+\Delta^+P(s))e_{\mathrm{d}P}(s,t_0)} - \int_{t_0}^{t} \frac{U(s-)\,\mathrm{d}P(s)}{(1-\Delta^-P(s))(1+\Delta^+P(s))e_{\mathrm{d}P}(s,t_0)}
\end{aligned}$$

for all $t \in [t_0, t_0+T]$, with the convention that $U(t_0-) = U(t_0)$, $\Delta^-P(t_0) = 0$ and $\Delta^+P(s) = 0$ if $s = t$.

Since $U(s-) = U(s) - u(s)\Delta^-P(s)$ (see [20, Corollary 6.5.5]), we deduce

$$\begin{aligned}
\frac{U(t)}{e_{\mathrm{d}P}(t,t_0)} &= \int_{t_0}^{t} \frac{u(s)\,\mathrm{d}P(s)}{(1+\Delta^+P(s))e_{\mathrm{d}P}(s,t_0)} + \int_{t_0}^{t} \frac{(-U(s)+u(s)\Delta^-P(s))\,\mathrm{d}P(s)}{(1-\Delta^-P(s))(1+\Delta^+P(s))e_{\mathrm{d}P}(s,t_0)} \\
&= \int_{t_0}^{t} \frac{1}{(1+\Delta^+P(s))e_{\mathrm{d}P}(s,t_0)} \left(u(s) + \frac{-U(s)+u(s)\Delta^-P(s)}{1-\Delta^-P(s)} \right) \mathrm{d}P(s) \\
&= \int_{t_0}^{t} \frac{u(s)-U(s)}{(1+\Delta^+P(s))(1-\Delta^-P(s))e_{\mathrm{d}P}(s,t_0)}\,\mathrm{d}P(s).
\end{aligned}$$

Now, since (3.1) holds and $1 - \Delta^-P(s) > 0$ for all $s \in (t_0, t_0+T]$, we obtain the estimate

$$\frac{U(t)}{e_{\mathrm{d}P}(t,t_0)} \le \int_{t_0}^{t} \frac{K(s)}{(1+\Delta^+P(s))(1-\Delta^-P(s))e_{\mathrm{d}P}(s,t_0)}\mathrm{d}P(s), \qquad t \in [t_0, t_0+T]. \tag{3.4}$$

Note that the integral on the right-hand side exists: Indeed, the Kurzweil-Stieltjes integral $\int_{t_0}^{t} K(s)\,\mathrm{d}P(s)$ exists by assumption. Since K is nonnegative, the corresponding Lebesgue-Stieltjes integral exists as well (see [20, Theorem 6.12.7]) and is finite. Consider the function

$$\varphi(s) = \frac{1}{(1+\Delta^+P(s))(1-\Delta^-P(s))e_{\mathrm{d}P}(s,t_0)}.$$

We have $\lim_{\xi\to s} \Delta^+P(\xi) = 0$ for each $s \in [t_0, t_0+T)$, and $\lim_{\xi\to s} \Delta^-P(\xi) = 0$ for each $s \in (t_0, t_0+T]$ (see [20, Corollary 4.1.7]). Therefore,

$$\begin{aligned}
\lim_{\xi\to s+} e_{\mathrm{d}P}(\xi,t_0)(1+\Delta^+P(\xi))(1-\Delta^-P(\xi)) &= e_{\mathrm{d}P}(s+,t_0) \neq 0, \\
\lim_{\xi\to s-} e_{\mathrm{d}P}(\xi,t_0)(1+\Delta^+P(\xi))(1-\Delta^-P(\xi)) &= e_{\mathrm{d}P}(s-,t_0) \neq 0.
\end{aligned}$$

It follows that φ is regulated, and therefore Borel measurable and bounded. Hence, the Lebesgue-Stieltjes integral of $K \cdot \varphi$ with respect to P exists and is finite, and therefore the Kurzweil-Stieltjes integral $\int_{t_0}^{t} K(s)\varphi(s)\,\mathrm{d}P(s)$ exists (see [20, Theorem 6.12.3]).

Multiplying inequality (3.4) by $e_{\mathrm{d}P}(t,t_0)$ and noting that $e_{\mathrm{d}P}(t,t_0)/e_{\mathrm{d}P}(s,t_0) = e_{\mathrm{d}P}(t,s)$, (3.1) yields

$$u(t) \le K(t) + U(t) \le K(t) + \int_{t_0}^{t} \frac{e_{\mathrm{d}P}(t,s)K(s)}{(1+\Delta^+P(s))(1-\Delta^-P(s))}\,\mathrm{d}P(s), \qquad t \in [t_0, t_0+T].$$

Hence, inequality (3.2) holds. An inspection of the proof reveals that if equality holds in (3.1), then it holds in (3.4) as well, and consequently also in (3.2).

If we in addition assume that K is bounded on $[t_0, t] \subset [t_0, t_0 + T]$, then

$$u(t) \le \left(\sup_{\xi \in [t_0, t]} K(\xi) \right) \left(1 + \int_{t_0}^{t} \frac{e_{\mathrm{d}P}(t, s)}{(1 + \Delta^+ P(s))(1 - \Delta^- P(s))} \,\mathrm{d}P(s) \right).$$

To finish the proof, it suffices to show that the last term equals $e_{\mathrm{d}P}(t, t_0)$. To see this, we use Theorem 2.1 with $f(t) = 1$ and $g(t) = e_{\mathrm{d}P}(t, t_0)$ to get

$$\begin{aligned} \frac{1}{e_{\mathrm{d}P}(t, t_0)} - \frac{1}{e_{\mathrm{d}P}(t_0, t_0)} &= -\int_{t_0}^{t} \frac{1}{e_{\mathrm{d}P}(s-, t_0) e_{\mathrm{d}P}(s+, t_0)} \mathrm{d}(e_{\mathrm{d}P}(s, t_0)) \\ &= -\int_{t_0}^{t} \frac{\mathrm{d}P(s)}{(1 + \Delta^+ P(s))(1 - \Delta^- P(s)) e_{\mathrm{d}P}(s, t_0)}, \qquad t \in [t_0, t_0 + T], \end{aligned}$$

where the last equality follows from the formulas (2.1), (2.2), (2.3) and Theorem 2.2. Multiplying by $e_{\mathrm{d}P}(t, t_0)$ and recalling that $e_{\mathrm{d}P}(t, t_0)/e_{\mathrm{d}P}(s, t_0) = e_{\mathrm{d}P}(t, s)$, we get

$$1 - e_{\mathrm{d}P}(t, t_0) = -\int_{t_0}^{t} \frac{e_{\mathrm{d}P}(t, s) \,\mathrm{d}P(s)}{(1 + \Delta^+ P(s))(1 - \Delta^- P(s))},$$

and the proof is complete. □

Remark 3.2. Instead of assuming that the function K is nonnegative and $\int_{t_0}^{t_0+T} K(s) \,\mathrm{d}P(s)$ exists, it is possible to assume that K is Borel measurable and $\int_{t_0}^{t_0+T} |K(s)| \,\mathrm{d}P(s)$ exists. This guarantees that the Lebesgue-Stieltjes integral of K with respect to P exists and is finite, and the rest of the proof remains unchanged.

Let us compare our Theorem 3.1 with similar results available in the literature:

- [23, Corollary 1.43] has stronger assumptions than our result. It assumes that K is a constant and P is left-continuous, and claims that $u(t) \le K e^{P(t) - P(t_0)}$; this follows from our estimate (3.3) and the fact that if P is left-continuous and such that $1 + \Delta^+ P(t) > 0$ for $t \ge t_0$, then $e_{\mathrm{d}P}(t, t_0) \le e^{P(t) - P(t_0)}$ for all $t \ge t_0$ (see [20, Corollary 8.5.5]). The last inequality is sharp if P is not right-continuous (cf. the explicit formula (2.4)), and therefore our estimate is better.
- The same remarks apply to [20, Theorem 7.5.3]. Our Theorem 3.1 is not only stronger, but the proof is simpler than in [20].
- [11, Theorem 22.3] has stronger assumptions on K and P than our Theorem 3.1: It assumes that K is a constant and P is nondecreasing with $1 - \Delta^- P(t) > 0$ for $t \in (t_0, t_0 + T]$, and $1 - \Delta^+ P(t) > 0$ for $t \in [t_0, t_0 + T)$; these assumptions are given on p. 145 of [11]. The conclusion is that $u(t) \le K e_{\mathrm{d}P}(t, t_0)$ for $t \in [t_0, t_0 + T]$; note that [11] does not use the notation $e_{\mathrm{d}P}$, but instead refers to the solution of Eq. (2.1).
- [14, Proposition 4.3] is formulated in the context of Stieltjes derivatives, and therefore assumes that P is left-continuous. Since the Kurzweil-Stieltjes integral of a nonnegative function with respect to a left-continuous nondecreasing function over $[a, b]$ coincides with the Lebesgue-Stieltjes integral over $[a, b)$, the first part of [14, Proposition 4.3] can be obtained as a consequence of our Theorem 3.1. However, to get an analogue of our inequality (3.3), [14, Proposition 4.3] assumes that $t \mapsto K(t)(1 + \Delta^+ P(t))$ is nondecreasing, which is too restrictive (for example, if K is constant, then P needs to be continuous). In Section 5, we will show that this condition can be avoided even when proving the result by means of Stieltjes derivatives.

- Section 2 of [5] contains a Gronwall-type result that is similar to ours, but assumes that K is a constant function and u has bounded variation. The assumptions on P are the same as in our Theorem 3.1. However, the author says that the proof is tedious, and is based on the approximation of P by functions with finitely many discontinuities. He omits the proof, and only later gives a proof for the case when u and P are right-continuous.

- [10, Lemma 7.11] is a special case of our Theorem 3.1 where K is constant, P is right-continuous and such that $\frac{1}{2} - \Delta^- P(s) \geq 0$ for all $s \in (t_0, t_0 + T]$.

- There is a time scale version of the Gronwall inequality in [3, Theorem 6.4]: If $\mathbb{T} \subset \mathbb{R}$ is a time scale, and u, K, L are rd-continuous functions with $L \geq 0$, then $u(t) \leq K(t) + \int_{t_0}^t L(s)u(s)\,\Delta s$ implies

$$u(t) \leq K(t) + \int_{t_0}^{t} e_L(t, \sigma(s))K(s)L(s)\,\Delta s, \tag{3.5}$$

  where σ is the forward jump operator given by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and e_L is the time scale exponential function. This result can be also derived from our Theorem 3.1 and Remark 3.2 by taking into account the relation between Δ-integrals and Kurzweil-Stieltjes integrals, see e.g. [20, Section 8.6]. In particular, we let $g(t) = \inf\{s \in \mathbb{T} : s \geq t\}$ and $P(t) = \int_{t_0}^t L(s)\,\mathrm{d}g(s)$. Then g and P are left-continuous, and the integral in (3.2) becomes

$$\begin{aligned}\int_{t_0}^{t} \frac{K(s)e_{\mathrm{d}P}(t,s)}{(1+\Delta^+P(s))(1-\Delta^-P(s))}\,\mathrm{d}P(s) &= \int_{t_0}^{t} \frac{K(s)}{(1+\Delta^+P(s))e_{\mathrm{d}P}(s,t)}\,\mathrm{d}P(s)\\ = \int_{t_0}^{t} \frac{K(s)L(s)}{e_{\mathrm{d}P}(s+,t)}\,\mathrm{d}g(s) &= \int_{t_0}^{t} K(s)L(s)e_L(t,\sigma(s))\,\Delta s,\end{aligned}$$

  which coincides with the integral on the right-hand side of (3.5).

  In a similar way, one can deal with ∇-integrals: Suppose that $u(t) \leq K(t) + \int_{t_0}^t L(s)u(s)\,\nabla s$. Recall that ∇-integrals are special cases of Kurzweil-Stieltjes integrals with respect to the nondecreasing right-continuous function $h(t) = \sup\{s \in \mathbb{T} : s \leq t\}$. Letting $P(t) = \int_{t_0}^t L(s)\,\mathrm{d}h(s)$ and using (3.2), we get

$$\begin{aligned}u(t) \leq K(t) + \int_{t_0}^{t} \frac{K(s)e_{\mathrm{d}P}(t,s)}{(1+\Delta^+P(s))(1-\Delta^-P(s))}\,\mathrm{d}P(s) &= K(t) + \int_{t_0}^{t} \frac{K(s)}{(1-\Delta^-P(s))e_{\mathrm{d}P}(s,t)}\,\mathrm{d}P(s)\\ = K(t) + \int_{t_0}^{t} \frac{K(s)L(s)}{e_{\mathrm{d}P}(s-,t)}\,\mathrm{d}h(s) &= K(t) + \int_{t_0}^{t} K(s)L(s)\hat{e}_L(t,\rho(s))\,\nabla s,\end{aligned}$$

  where ρ is the backward jump operator given by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, and $\hat{e}_L$ is the nabla exponential function. As far as we are aware, this nabla version of Gronwall's inequality is not available in the literature.

We finalize this section with a result similar to Theorem 3.1, but the integrals on the right-hand sides of the inequalities will now be functions of their lower limits instead of upper limits. Although it would be possible to adapt the proof of Theorem 3.1 by considering the function $U(t) = \int_t^{t_0+T} u(s)\,\mathrm{d}P(s)$, $t \in [t_0, t_0 + T]$, we prefer to give a different proof and deduce the result from Theorem 3.1 by a "time-reversal" argument, which is inspired by the proof of [23, Corollary 1.42].

Theorem 3.3. *Let $P\colon [t_0, t_0 + T] \to \mathbb{R}$ be a nondecreasing function such that $1 - \Delta^+P(s) > 0$ for all $s \in [t_0, t_0 + T)$. If $K\colon [t_0, t_0 + T] \to [0, \infty)$ is such that $\int_{t_0}^{t_0+T} K(s)\,\mathrm{d}P(s)$ exists and $u\colon [t_0, t_0 + T] \to \mathbb{R}$ satisfies*

$$u(t) \leq K(t) + \int_{t}^{t_0+T} u(s)\,\mathrm{d}P(s), \qquad t \in [t_0, t_0 + T], \tag{3.6}$$

then

$$u(t) \le K(t) + \int_t^{t_0+T} \frac{K(s)e_{\mathrm{d}(-P)}(t,s)}{(1-\Delta^+P(s))(1+\Delta^-P(s))}\,\mathrm{d}P(s), \qquad t \in [t_0, t_0+T], \tag{3.7}$$

with the convention that $\Delta^+P(t_0+T) = 0$ *and* $\Delta^-P(s) = 0$ *if* $s = t$*. Moreover, if equality holds in* (3.6)*, then it also holds in* (3.7)*.*

In addition, if K *is bounded on* $[t, t_0+T] \subset [t_0, t_0+T]$*, then*

$$u(t) \le \left(\sup_{\xi \in [t, t_0+T]} K(\xi)\right) e_{\mathrm{d}(-P)}(t, t_0+T). \tag{3.8}$$

Proof. Let $\widetilde{P}\colon [-t_0-T, -t_0] \to \mathbb{R}$ be given by

$$\widetilde{P}(\sigma) = -P(-\sigma), \quad \sigma \in [-t_0-T, -t_0].$$

This function is nondecreasing, and we have the identities

$$\widetilde{P}(\sigma+) = -P(-\sigma-), \quad \sigma \in [-t_0-T, -t_0), \tag{3.9}$$

$$\widetilde{P}(\sigma-) = -P(-\sigma+), \quad \sigma \in (-t_0-T, -t_0]. \tag{3.10}$$

Consequently,

$$\Delta^+\widetilde{P}(\sigma) = \Delta^-P(-\sigma), \quad \sigma \in [-t_0-T, -t_0), \tag{3.11}$$

$$\Delta^-\widetilde{P}(\sigma) = \Delta^+P(-\sigma), \quad \sigma \in (-t_0-T, -t_0], \tag{3.12}$$

$$1-\Delta^-\widetilde{P}(\sigma) = 1-\Delta^+P(-\sigma) > 0, \quad \sigma \in (-t_0-T, -t_0]. \tag{3.13}$$

Using the change of variables theorem $\int_{\phi(c)}^{\phi(d)} f(s)\,\mathrm{d}g(s) = \int_c^d f(\phi(t))\,\mathrm{d}g(\phi(t))$ (see [20, Theorem 6.6.5 and Exercise 6.6.6]) with $\phi(x) = -x$, we find that for every $t \in [t_0, t_0+T]$, the integral on the right-hand side of (3.6) equals

$$\int_t^{t_0+T} u(s)\,\mathrm{d}P(s) = \int_{-t}^{-t_0-T} u(-s)\,\mathrm{d}P(-s) = -\int_{-t_0-T}^{-t} u(-s)\,\mathrm{d}P(-s) = \int_{-t_0-T}^{-t} u(-s)\,\mathrm{d}\widetilde{P}(s). \tag{3.14}$$

Therefore, the inequality (3.6) can be rewritten as

$$u(t) \le K(t) + \int_{-t_0-T}^{-t} u(-s)\,\mathrm{d}\widetilde{P}(s), \qquad t \in [t_0, t_0+T],$$

i.e.,

$$\widetilde{u}(t) \le \widetilde{K}(t) + \int_{-t_0-T}^{t} \widetilde{u}(s)\,\mathrm{d}\widetilde{P}(s), \qquad t \in [-t_0-T, -t_0], \tag{3.15}$$

where $\widetilde{u}(\sigma) = u(-\sigma)$ and $\widetilde{K}(\sigma) = K(-\sigma)$ for all $\sigma \in [-t_0-T, -t_0]$. Hence, the functions $\widetilde{u}$, $\widetilde{K}$, $\widetilde{P}\colon [-t_0-T, -t_0] \to \mathbb{R}$ satisfy the assumptions of Theorem 3.1, and we obtain

$$\tilde{u}(t) \le \widetilde{K}(t) + \int_{-t_0-T}^{t} \frac{\widetilde{K}(s)e_{\mathrm{d}\widetilde{P}}(t,s)}{(1+\Delta^+\widetilde{P}(s))(1-\Delta^-\widetilde{P}(s))}\mathrm{d}\widetilde{P}(s), \qquad t \in [-t_0-T, -t_0],$$

i.e.,

$$u(t) \le K(t) + \int_{-t_0-T}^{-t} \frac{K(-s)e_{\mathrm{d}\widetilde{P}}(-t,s)}{(1+\Delta^+\widetilde{P}(s))(1-\Delta^-\widetilde{P}(s))}\mathrm{d}\widetilde{P}(s), \qquad t \in [t_0, t_0+T], \tag{3.16}$$

with the convention that $\Delta^-\widetilde{P}(-t_0-T)=0$ and $\Delta^+\widetilde{P}(s)=0$ if $s=t$. To proceed, we need to show that

$$e_{\mathrm{d}\widetilde{P}}(-t,s)=e_{\mathrm{d}(-P)}(t,-s),\quad \text{for all } t\in[t_0,t_0+T],\ s\in[-t_0-T,-t_0],\ s<-t.$$

This assertion can be obtained by using the explicit formula (2.4), along with Eq. (3.9)–(3.12). Indeed, given $t\in[t_0,t_0+T]$, $s\in[-t_0-T,-t_0]$ such that $s<-t$, we have

$$\begin{aligned}
e_{\mathrm{d}\widetilde{P}}(-t,s)&=\frac{e^{\widetilde{P}(-t-)-\widetilde{P}(s+)}}{e^{\sum_{\sigma\in(s,-t)}\Delta\widetilde{P}(\sigma)}}\cdot\frac{\prod\limits_{\sigma\in[s,-t)}(1+\Delta^+\widetilde{P}(\sigma))}{\prod\limits_{\sigma\in(s,-t]}(1-\Delta^-\widetilde{P}(\sigma))}\\
&=\frac{e^{P(-s-)-P(t+)}}{e^{\sum_{\sigma\in(s,-t)}\Delta P(-\sigma)}}\cdot\frac{\prod\limits_{\sigma\in[s,-t)}(1+\Delta^-P(-\sigma))}{\prod\limits_{\sigma\in(s,-t]}(1-\Delta^+P(-\sigma))}\\
&=\frac{e^{\sum_{\sigma\in(s,-t)}\Delta(-P)(-\sigma)}}{e^{(-P)(-s-)-(-P)(t+)}}\cdot\frac{\prod\limits_{\sigma\in[s,-t)}(1-\Delta^-(-P)(-\sigma))}{\prod\limits_{\sigma\in(s,-t]}(1+\Delta^+(-P)(-\sigma))}\\
&=\frac{e^{\sum_{\sigma\in(t,-s)}\Delta(-P)(\sigma)}}{e^{(-P)(-s-)-(-P)(t+)}}\cdot\frac{\prod\limits_{\sigma\in(t,-s]}(1-\Delta^-(-P)(\sigma))}{\prod\limits_{\sigma\in[t,-s)}(1+\Delta^+(-P)(\sigma))}=e_{\mathrm{d}(-P)}(t,-s).
\end{aligned}$$

Applying the previous identity, the relations (3.11) and (3.12), and the previously mentioned change of variables theorem with $\phi(x)=-x$, we deduce that the integral on the right-hand side of (3.16) equals

$$\begin{aligned}
&\int_{-t_0-T}^{-t}\frac{K(-s)e_{\mathrm{d}\widetilde{P}}(-t,s)}{(1+\Delta^+\widetilde{P}(s))(1-\Delta^-\widetilde{P}(s))}\,\mathrm{d}\widetilde{P}(s)=-\int_{-t_0-T}^{-t}\frac{K(-s)e_{\mathrm{d}\widetilde{P}}(-t,s)}{(1+\Delta^-P(-s))(1-\Delta^+P(-s))}\,\mathrm{d}P(-s)\\
&=\int_{-t}^{-t_0-T}\frac{K(-s)e_{\mathrm{d}(-P)}(t,-s)}{(1+\Delta^-P(-s))(1-\Delta^+P(-s))}\,\mathrm{d}P(-s)=\int_{t}^{t_0+T}\frac{K(s)e_{\mathrm{d}(-P)}(t,s)}{(1+\Delta^-P(s))(1-\Delta^+P(s))}\,\mathrm{d}P(s),
\end{aligned}$$

with the convention that $\Delta^+P(t_0+T)=0$ and $\Delta^-P(s)=0$ if $s=t$.

Therefore, from (3.16) we conclude that the relation (3.7) holds.

If K is bounded from above on $[t,t_0+T]\subset[t_0,t_0+T]$, then $\widetilde{K}$ is bounded by the same constant on $[-t_0-T,t]$. Therefore, by the inequality (3.15) and Theorem 3.1, we have

$$\widetilde{u}(t)\le\left(\sup_{\xi\in[-t_0-T,t]}\widetilde{K}(\xi)\right)e_{\mathrm{d}\widetilde{P}}(t,-t_0-T),\qquad t\in[-t_0-T,-t_0],$$

i.e.,

$$u(t)\le\left(\sup_{\xi\in[t,t_0+T]}K(\xi)\right)e_{\mathrm{d}(-P)}(t,t_0+T),\qquad t\in[t_0,t_0+T].$$

□

Remark 3.4. Suppose that the assumptions of Theorem 3.3 hold and, moreover, K is constant and P is right-continuous. Then, combining (3.8) and (2.4), we obtain the estimate

$$u(t)\le Ke_{\mathrm{d}(-P)}(t,t_0+T)=K\frac{e^{\sum_{s\in(t,t_0+T)}\Delta^-(-P)(s)}}{e^{(-P)(t_0+T-)-(-P)(t)}}\prod_{s\in(t,t_0+T]}(1-\Delta^-(-P)(s))$$

$$= Ke^{P(t_0+T)-P(t)} \frac{\prod_{s\in(t,t_0+T]}(1+\Delta^- P(s))}{e^{\sum_{s\in(t,t_0+T]}\Delta^- P(s)}} \le Ke^{P(t_0+T)-P(t)},$$

where the last inequality holds because $(1+\Delta^- P(s))/e^{\Delta^- P(s)} \le 1$ for each s. This result agrees with [20, Exercise 7.5.4] and [23, Corollary 1.44], which are special cases of our Theorem 3.3.

4 Uniqueness results

A classical application of Gronwall-type inequalities is in proving uniqueness of solutions to various types of equations. Having the general Gronwall inequality presented in Theorem 3.1, we can study uniqueness of solutions for the measure differential equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, \mathrm{d}g(s), \qquad t \in [t_0, t_0 + T], \tag{4.1}$$

where $f\colon [t_0, t_0+T]\times\mathbb{R}^n \to \mathbb{R}^n$, and $g\colon [t_0, t_0+T] \to \mathbb{R}$ has bounded variation. The integral on the right-hand side is considered in the sense of Kurzweil-Stieltjes. Clearly, if $x\colon [t_0, t_0+T] \to \mathbb{R}^n$ is a solution of (4.1), then it is regulated on $[t_0, t_0+T]$. In the next result, we present sufficient conditions for uniqueness of solutions to (4.1). We use the symbol $\mathrm{var}_c^d g$ to denote the variation of g over an interval $[c, d]$.

Theorem 4.1. *Consider functions $f\colon [t_0, t_0+T]\times\mathbb{R}^n \to \mathbb{R}^n$ and $g\colon [t_0, t_0+T] \to \mathbb{R}$, where g has bounded variation. Suppose there exists a function $L\colon [t_0, t_0+T] \to [0,\infty)$ such that $1 - L(s)|\Delta^- g(s)| > 0$ for all $s \in (t_0, t_0+T]$, and*

$$\left\| \int_c^d [f(s, x(s)) - f(s, y(s))]\, \mathrm{d}g(s) \right\| \le \int_c^d L(s) \|x(s) - y(s)\|\, \mathrm{d}(\mathrm{var}_{t_0}^s g) \tag{4.2}$$

for all $[c, d] \subseteq [t_0, t_0+T]$ and all regulated functions $x, y\colon [t_0, t_0+T] \to \mathbb{R}^n$. Then Eq. (4.1) has at most one solution on $[t_0, t_0+T]$.

Proof. Assume that $x, y : [t_0, t_0+T] \to \mathbb{R}^n$ are solutions of Eq. (4.1), and let $u(t) = \|x(t) - y(t)\|$ for all $t \in [t_0, t_0+T]$. Applying (4.2), we obtain the estimate

$$u(t) \le \int_{t_0}^{t} L(s) u(s)\, \mathrm{d}(\mathrm{var}_{t_0}^s g) = \int_{t_0}^{t} u(s)\, \mathrm{d}P(s), \qquad t \in [t_0, t_0+T],$$

where $P(s) = \int_{t_0}^{s} L(\xi)\, \mathrm{d}(\mathrm{var}_{t_0}^{\xi} g)$ for all $s \in [t_0, t_0+T]$. Note that P is a nondecreasing function which satisfies $1 - \Delta^- P(s) > 0$ for all $s \in (t_0, t_0+T]$. Indeed, since $\Delta^-(\mathrm{var}_{t_0}^s g) = |\Delta^- g(s)|$ (see [20, Lemma 2.3.3]), we have

$$1 - \Delta^- P(s) = 1 - L(s)\Delta^-(\mathrm{var}_{t_0}^s g) = 1 - L(s)|\Delta^- g(s)| > 0, \qquad s \in (t_0, t_0+T].$$

Therefore, we can apply the Gronwall lemma (Theorem 3.1) with $K = 0$, and conclude that $u = 0$ on $[t_0, t_0+T]$, which in turn implies that $x = y$ on $[t_0, t_0+T]$. □

A similar result was presented in [5, Section 3], where the author assumed the Lipschitz condition

$$\|f(s, x) - f(s, y)\| \le L\|x - y\|,$$

with $L > 0$ such that $1 - L|\Delta^- g(s)|$ is bounded away from zero for $s \in [t_0, t_0+T]$. These hypotheses clearly imply that (4.2) hold. However, the integral version of the Lipschitz condition in (4.2) is more general: L need not be a constant, and (4.2) imposes no condition on f on intervals where g is constant.

An interesting corollary of Theorem 4.1 is a uniqueness theorem for nabla dynamic equations on time scales. Recall that a time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. We are interested in nabla dynamic equations of the form

$$x^{\nabla}(t) = f(t, x(t)), \quad t \in [t_0, t_0 + T]_{\mathbb{T}}, \tag{4.3}$$

where $f\colon [t_0, t_0 + T]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function, and we use the notation $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$.

If ν is the so-called backward graininess given by $\nu(t) = t - \rho(t)$, where $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ is the usual backward jump operator, then the nabla derivative $x^{\nabla}(t)$ is

$$x^{\nabla}(t) = \begin{cases} \frac{x(t)-x(\rho(t))}{\nu(t)} & \text{if } \nu(t) > 0, \\ x'(t) & \text{if } \nu(t) = 0. \end{cases}$$

Nabla dynamic equations are implicit in the sense that if $\nu(t) > 0$ and if we know the value $x(\rho(t))$, then finding the value $x(t)$ requires solving the equation

$$x(t) = x(\rho(t)) + f(t, x(t))\nu(t).$$

Numerical analysts are well aware that implicit difference equations are uniquely solvable if f is a "well-behaved" function and the step size $\nu(t)$ is sufficiently small.

A basic reference for nabla dynamic equations on general time scales is [1]. However, we were unable to find a source dealing with uniqueness of solutions for nabla dynamic equations. A result of this type can be easily derived from Theorem 4.1. For this purpose, we switch from Eq. (4.3) to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, \nabla s, \qquad t \in [t_0, t_0 + T]_{\mathbb{T}}, \tag{4.4}$$

where the integral on the right-hand side is the nabla integral. For our purposes, it is convenient to interpret it as a Henstock-Kurzweil nabla integral (cf. [20, Section 8.6]).

Theorem 4.2. *Consider a function* $f : [t_0, t_0 + T]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$. *Suppose there exists a function* $L\colon [t_0, t_0 + T]_{\mathbb{T}} \to [0, \infty)$ *such that* $1 - L(s)\nu(s) > 0$ *for all* $s \in (t_0, t_0 + T]_{\mathbb{T}}$, *and*

$$\left\| \int_{c}^{d} [f(s, x(s)) - f(s, y(s))]\, \nabla s \right\| \le \int_{c}^{d} L(s)\|x(s) - y(s)\|\, \nabla s$$

for all $[c, d] \subseteq [t_0, t_0 + T]_{\mathbb{T}}$ *and all regulated functions* $x, y\colon [t_0, t_0 + T]_{\mathbb{T}} \to \mathbb{R}^n$. *Then Eq.* (4.4) *has at most one solution on* $[t_0, t_0 + T]_{\mathbb{T}}$.

Proof. It suffices to recall that nabla integrals are special cases of Kurzweil-Stieltjes integrals with respect to the function $g(t) = \sup\{s \in [t_0, t_0 + T]_{\mathbb{T}} : s \le t\}$, which is nondecreasing and right-continuous with $\Delta^- g(s) = \nu(s)$ if $s \in \mathbb{T}$, and $\Delta^- g(s) = 0$ otherwise. The rest follows from Theorem 4.1. $\square$

5 Gronwall inequalities via Stieltjes derivatives

In what follows, we consider $g\colon \mathbb{R} \to \mathbb{R}$ to be a nondecreasing and left-continuous function. The aim of this section is to prove Theorems 3.1 and 3.3, as well as their differential counterpart, by means of Stieltjes derivatives.

All integrals in the present section will be understood as Lebesgue-Stieltjes integrals. In particular, we denote by μ_g the Lebesgue-Stieltjes measure on $\mathbb{R}$ corresponding to g, and by $\mathcal{L}^1_{\mu_g}(I)$ the class of all μ_g-measurable functions $f : I \to \mathbb{R}$ such that $\int_I |f| \,\mathrm{d}\mu_g$ is finite.

By $f'_g(t)$ we denote the Stieltjes derivative of a function f with respect to g at a point t. The definition of this derivative along with its properties can be found in [4, 13, 14, 15], and we do not repeat it here. We also need the concept of g-absolutely continuous functions, which generalizes the classical notion of absolute continuity. In fact, it suffices to keep in mind the following two results:

- Given $f \in \mathcal{L}^1_{\mu_g}([t_0, t_0+T))$, the map $F(t) = \int_{[t_0,t)} f(s)\mathrm{d}\mu_g(s)$, $t \in [t_0, t_0+T]$, is g-absolutely continuous (we write $F \in \mathcal{AC}_g([t_0, t_0+T])$) and, moreover, $F'_g(t) = f(t)$ for μ_g-a.a. $t \in [t_0, t_0+T)$ (see [13, Theorem 2.4 and Proposition 5.2]).
- If $F\colon [t_0, t_0+T] \to \mathbb{R}$ is g-absolutely continuous, then $F'_g(t)$ exists for μ_g-a.a. $t \in [t_0, t_0+T)$, $F'_g \in \mathcal{L}^1_{\mu_g}([t_0, t_0+T))$, and

$$F(t) = F(t_0) + \int_{[t_0,t)} F'_g(s)\,\mathrm{d}\mu_g(s), \quad t \in [t_0, t_0+T] \tag{5.1}$$

(see [13, Theorem 5.4]).

Given a function $p \in \mathcal{L}^1_{\mu_g}([t_0, t_0+T))$, there exists a unique function $x \in \mathcal{AC}_g([t_0, t_0+T])$ satisfying

$$x'_g(t) = p(t)x(t) \quad \text{for } \mu_g\text{-a.a. } t \in [t_0, t_0+T), \quad x(t_0) = 1.$$

It is called the g-exponential function and denoted by $\exp_g(p, \cdot)$. An explicit formula can be found in [4, 14]. In particular, if $1 + p(t)\Delta^+ g(t) > 0$, $t \in [t_0, t_0+T)$, then $\exp_g(p, t) > 0$ for all $t \in [t_0, t_0+T]$. The g-exponential function is a special case of the generalized exponential function discussed in Section 2: We have $\exp_g(p, t) = e_{\mathrm{d}P}(t, t_0)$, where $P(s) = \int_{t_0}^s p\,\mathrm{d}g$.

Finally, if $F \in \mathcal{AC}_g([t_0, t_0+T])$ and $p \in \mathcal{L}^1_{\mu_g}([t_0, t_0+T))$ is such that $1+p(t)\Delta^+ g(t) > 0$, $t \in [t_0, t_0+T)$, then the map $Q(t) = F(t)/\exp_g(p, t)$, $t \in [t_0, t_0+T]$, is g-absolutely continuous (this can be deduced from [14, Proposition 4.1] and [4, Proposition 5.4]). Furthermore, the quotient rule [14, Proposition 5.2], ensures that for μ_g-a.a. $t \in [t_0, t_0+T)$, we have

$$Q'_g(t) = \frac{F'_g(t)\exp_g(p, t) - F(t)p(t)\exp_g(p, t)}{\exp_g(p, t)(\exp_g(p, t) + p(t)\exp_g(p, t)\Delta^+ g(t))} = \frac{F'_g(t) - p(t)F(t)}{\exp_g(p, t)(1 + p(t)\Delta^+ g(t))}. \tag{5.2}$$

We are now able to prove the following result, which is an improvement of [14, Proposition 4.3]. It is a version of Theorem 3.1 where g is left-continuous, Kurzweil-Stieltjes integrals are replaced by Lebesgue-Stieltjes integrals with respect to μ_g, and the generalized exponential function is replaced by the g-exponential function. Moreover, the proof is based on Stieltjes derivatives. We hope this setting will be more useful to researchers dealing with Stieltjes differential equations.

Theorem 5.1. *Let $L\colon [t_0, t_0+T] \to [0, \infty)$ be such that $L \in \mathcal{L}^1_{\mu_g}([t_0, t_0+T))$. If $u, K\colon [t_0, t_0+T] \to \mathbb{R}$ are such that $L \cdot u, L \cdot K \in \mathcal{L}^1_{\mu_g}([t_0, t_0+T))$ and*

$$u(t) \le K(t) + \int_{[t_0,t)} L(s)u(s)\,\mathrm{d}\mu_g(s), \quad t \in [t_0, t_0+T], \tag{5.3}$$

then

$$u(t) \le K(t) + \int_{[t_0,t)} \frac{K(s)L(s)}{1 + L(s)\Delta^+ g(s)} \frac{\exp_g(L, t)}{\exp_g(L, s)}\,\mathrm{d}\mu_g(s), \quad t \in [t_0, t_0+T]. \tag{5.4}$$

Moreover, if equality holds in (5.3)*, then it also holds in* (5.4)*.*

In addition, if K is bounded on some $[t_0, t] \subset [t_0, t_0+T]$, then

$$u(t) \le \left(\sup_{\xi \in [t_0,t]} K(\xi)\right) \exp_g(L, t). \tag{5.5}$$

Proof. Following the proof of [14, Proposition 4.3], define $U(t) = \int_{[t_0,t)} L(s)u(s)\,\mathrm{d}\mu_g(s)$ and $v(t) = U(t)/\exp_g(L, t)$, $t \in [t_0, t_0+T]$. Then $U, v \in \mathcal{AC}_g([t_0, t_0+T])$ and by (5.2), we know that for μ_g-a.a. $t \in [t_0, t_0+T)$,

$$v'_g(t) = \frac{U'_g(t) - U(t)L(t)}{\exp_g(L, t)(1 + L(t)\Delta^+ g(t))} = \frac{(u(t) - U(t))L(t)}{\exp_g(L, t)(1 + L(t)\Delta^+ g(t))} \le \frac{K(t)L(t)}{\exp_g(L, t)(1 + L(t)\Delta^+ g(t))}, \tag{5.6}$$

where the inequality follows from (5.3). The last function in (5.6) belongs to $\mathcal{L}^1_{\mu_g}([t_0, t_0 + T))$ by a reasoning similar to the one used for the function in (3.4). Hence, since $v(t_0) = 0$, (5.1) ensures that for any $t \in [t_0, t_0 + T]$,

$$U(t) = \exp_g(L,t)v(t) = \exp_g(L,t)\int_{[t_0,t)} v'_g(s)\,\mathrm{d}\mu_g(s) \le \exp_g(L,t)\int_{[t_0,t)} \frac{K(s)L(s)\,\mathrm{d}\mu_g(s)}{\exp_g(L,s)(1+L(s)\Delta^+g(s))},$$

so (5.4) now follows from (5.3). Observe that it is clear from (5.6) that if equality holds in (5.3), then it also holds in (5.4).

To prove (5.5), assume K is bounded on $[t_0, t] \subset [t_0, t_0+T]$. The function $\psi_t(s) = \exp_g(L, t)/\exp_g(L, s)$, $s \in [t_0, t]$, belongs to $\mathcal{AC}_g([t_0, t])$, satisfies $\psi_t(t_0) = \exp_g(L, t)$, and

$$(\psi_t)'_g(s) = -\frac{L(s)\psi_t(s)}{1+L(s)\Delta^+g(s)} \quad \text{for } \mu_g\text{-a.a. } s \in [t_0, t)$$

(this can be deduced from (5.2) since the Stieltjes derivative of a constant function is equal to zero). Hence, it follows from (5.4) that

$$\begin{aligned} u(t) &\le K(t) + \int_{[t_0,t)} \frac{K(s)L(s)\psi_t(s)}{1+L(s)\Delta^+g(s)}\,\mathrm{d}\mu_g(s) \le \left(\sup_{\xi\in[t_0,t]} K(\xi)\right)\left(1 + \int_{[t_0,t)} \frac{L(s)\psi_t(s)}{1+L(s)\Delta^+g(s)}\,\mathrm{d}\mu_g(s)\right) \\ &= \left(\sup_{\xi\in[t_0,t]} K(\xi)\right)\left(1 - \int_{[t_0,t)} (\psi_t)'_g(s)\,\mathrm{d}\mu_g(s)\right) = \left(\sup_{\xi\in[t_0,t]} K(\xi)\right)(1 - (\psi_t(t) - \psi_t(t_0))) \\ &= \left(\sup_{\xi\in[t_0,t]} K(\xi)\right)\psi_t(t_0) = \left(\sup_{\xi\in[t_0,t]} K(\xi)\right)\exp_g(L,t), \end{aligned}$$

which proves (5.5). □

Remark 5.2. The contents and proofs of Theorem 5.1 and [14, Proposition 4.3] are very similar. The fundamental difference between the two results, beyond the fact that Theorem 5.1 does not impose nonnegativity on the functions u and K, comes from (5.6): In [14, Proposition 4.3], the author makes use of the fact that $1 + L(s)\Delta^+g(s) \ge 1$, $s \in [t_0, t_0 + T)$, to get rid of that term in the denominator. This, however, leads to less accurate upper bounds since (5.4) is a sharp upper bound, as shown in Theorem 5.1. In addition, the second part of Theorem 5.1 is more general than the corresponding counterpart of [14, Proposition 4.3] as we simply ask K to be bounded on an interval instead of assuming that $t \mapsto K(t)(1 + L(t)\Delta^+g(t))$ is nondecreasing on that interval.

We now present the corresponding adaptation of Theorem 3.3, which we prove in a similar manner to Theorem 5.1.

Theorem 5.3. *Let $L : [t_0, t_0 + T] \to [0, \infty)$ be such that $L \in \mathcal{L}^1_{\mu_g}([t_0, t_0 + T))$ and $1 - L(t)\Delta^+g(t) > 0$, $t \in [t_0, t_0 + T)$. If $u, K : [t_0, t_0 + T] \to \mathbb{R}$ are such that $L \cdot u, L \cdot K \in \mathcal{L}^1_{\mu_g}([t_0, t_0 + T))$ and*

$$u(t) \le K(t) + \int_{[t,t_0+T)} L(s)u(s)\,\mathrm{d}\mu_g(s), \quad t \in [t_0, t_0 + T], \tag{5.7}$$

then

$$u(t) \le K(t) + \int_{[t,t_0+T)} \frac{K(s)L(s)}{1-L(s)\Delta^+g(s)}\frac{\exp_g(-L,t)}{\exp_g(-L,s)}\,\mathrm{d}\mu_g(s), \quad t \in [t_0, t_0 + T]. \tag{5.8}$$

Moreover, if equality holds in (5.7)*, then it also holds in* (5.8)*.*

In addition, if K is bounded on some $[t, t_0 + T] \subset [t_0, t_0 + T]$, then

$$u(t) \le \left(\sup_{\xi\in[t,t_0+T]} K(\xi)\right)\frac{\exp_g(-L,t)}{\exp_g(-L,t_0+T)}. \tag{5.9}$$

Proof. Define

$$U(t)=\int_{[t,t_0+T)} L(s)u(s)\,\mathrm{d}\mu_g(s)=\int_{[t_0,t_0+T)} L(s)u(s)\,\mathrm{d}\mu_g(s)-\int_{[t_0,t)} L(s)u(s)\,\mathrm{d}\mu_g(s),\quad t\in[t_0,t_0+T],$$

and $v(t)=U(t)/\exp_g(-L,t)$, $t\in[t_0,t_0+T]$. Then $U\in\mathcal{AC}_g([t_0,t_0+T])$ with $U_g'(t)=-L(t)u(t)$ for μ_g-a.a. $t\in[t_0,t_0+T)$ and, as a consequence, $v\in\mathcal{AC}_g([t_0,t_0+T])$. By (5.2),

$$v_g'(t)=\frac{U_g'(t)-(-L(t))U(t)}{\exp_g(-L,t)(1+(-L(t))\Delta^+g(t))}=\frac{-L(t)(u(t)-U(t))}{\exp_g(-L,t)(1-L(t)\Delta^+g(t))}\quad \text{for } \mu_g\text{-a.a. } t\in[t_0,t_0+T).$$

Thus, since $v\in\mathcal{AC}_g([t_0,t_0+T])$, we have that for any $t\in[t_0,t_0+T]$,

$$\begin{aligned} v(t)&=-(v(t_0+T)-v(t))=-\int_{[t,t_0+T)} v_g'(s)\,\mathrm{d}\mu_g(s)=\int_{[t,t_0+T)}\frac{L(s)(u(s)-U(s))\,\mathrm{d}\mu_g(s)}{\exp_g(-L,s)(1-L(s)\Delta^+g(s))}\\ &\le\int_{[t,t_0+T)}\frac{L(s)K(s)}{\exp_g(-L,s)(1-L(s)\Delta^+g(s))}\,\mathrm{d}\mu_g(s), \end{aligned}\tag{5.10}$$

where the inequality is a consequence of (5.7); the integrability of the function on the right-hand side of (5.10) can be proven similarly as the integrability of the function in (5.6). Therefore,

$$U(t)=\exp_g(-L,t)v(t)\le\exp_g(-L,t)\int_{[t,t_0+T)}\frac{L(s)K(s)}{\exp_g(-L,s)(1-L(s)\Delta^+g(s))}\,\mathrm{d}\mu_g(s),\quad t\in[t_0,t_0+T],$$

and (5.8) follows from (5.7). Furthermore, if equality holds in (5.7), we also obtain an equality in (5.10) which, in turn, leads to an equality in (5.8).

Finally, in order to prove (5.9) it is enough to consider the map $\psi_t(s)=\exp_g(-L,t)/\exp_g(-L,s)$, $s\in[t,t_0+T]$, which belongs to $\mathcal{AC}_g([t,t_0+T])$ and satisfies

$$\psi_t(t_0+T)=\frac{\exp_g(-L,t)}{\exp_g(-L,t_0+T)},\qquad (\psi_t)_g'(s)=\frac{L(s)\psi_t(s)}{1-L(s)\Delta^+g(s)}\quad\text{for } g\text{-a.a. } s\in[t,t_0+T).$$

The rest of the proof is analogous to the proof of (5.5) and we omit it. □

Finally, we provide a differential version of Theorem 5.1. Observe that, in this case, the map L is no longer assumed to be nonnegative.

Theorem 5.4. *Let* $u\in\mathcal{AC}_g([t_0,t_0+T])$ *be such that*

$$u_g'(t)\le K(t)+L(t)u(t)\quad \textit{for } g\textit{-a.a. } t\in[t_0,t_0+T),\tag{5.11}$$

where $K,L\in\mathcal{L}^1_{\mu_g}([t_0,t_0+T))$ *and* $1+L(t)\Delta^+g(t)>0$ *for all* $t\in[t_0,t_0+T)$. *Then*

$$u(t)\le u(t_0)\exp_g(L,t)+\int_{[t_0,t)}\frac{K(s)}{1+L(s)\Delta^+g(s)}\frac{\exp_g(L,t)}{\exp_g(L,s)}\,\mathrm{d}\mu_g(s),\quad t\in[t_0,t_0+T].\tag{5.12}$$

Moreover, if equality holds in (5.11)*, then it also holds in* (5.12).

Proof. First, note that the hypotheses ensure that $v(t)=\exp_g(L,t)>0$, $t\in[t_0,t_0+T]$. Furthermore, we have $u/v\in\mathcal{AC}_g([t_0,t_0+T])$ and, by (5.2),

$$\left(\frac{u}{v}\right)_g'(t)=\frac{u_g'(t)-L(t)u(t)}{\exp_g(L,t)(1+L(t)\Delta^+g(t))}\le\frac{K(t)}{\exp_g(L,t)(1+L(t)\Delta^+g(t))}\ \text{for } \mu_g\text{-a.a. } t\in[t_0,t_0+T),\tag{5.13}$$

where the inequality is a consequence of (5.11). The map on the right-hand side of (5.13) is μ_g-integrable on $[t_0,t_0+T)$ by a reasoning analogous to the one used for the function in (3.4). Hence, for $t\in[t_0,t_0+T]$,

$$\frac{u(t)}{v(t)}=\frac{u(t_0)}{v(t_0)}+\int_{[t_0,t)}\left(\frac{u}{v}\right)_g'(s)\,\mathrm{d}\mu_g(s)\le u(t_0)+\int_{[t_0,t)}\frac{K(s)}{\exp_g(L,s)(1+L(s)\Delta^+g(s))}\,\mathrm{d}\mu_g(s),$$

which implies (5.12). If equality holds in (5.11), then it holds in (5.13) and (5.12) as well. □

6 Conclusion

In the present paper, we have established general versions of the Gronwall inequality for Stieltjes integrals, which improve numerous earlier results available in the literature. As an application, we have derived new uniqueness results for measure differential equations and nabla dynamic equations on time scales.

The topic of Stieltjes integral inequalities is far from being exhausted, and we present two ideas for future research:

- Prove a sufficiently general Bihari-type inequality for Stieltjes integrals. In this case, the assumption

$$u(t) \le K(t) + \int_{t_0}^{t} u(s)\,\mathrm{d}P(s) \tag{6.1}$$

  will be replaced by the nonlinear inequality

$$u(t) \le K(t) + \int_{t_0}^{t} \omega(u(s))\,\mathrm{d}P(s), \tag{6.2}$$

  and the goal is to find an a priori estimate for u. A result of this type is available in [23, Theorem 1.40], where it is assumed that P is left-continuous, and K is a constant function. Relaxing or weakening these assumptions would lead to improved well-posedness results for abstract generalized differential equations and measure functional differential equations, see e.g. [24].

- Besides the Gronwall and Bihari inequalities, there is a wealth of additional useful integral inequalities, see e.g. the monograph [21]. A possible project is to investigate which of these inequalities admit generalizations involving Stieltjes integrals.

Acknowledgments

Claudio A. Gallegos was supported by ANID/FONDECYT postdoctorado grant No. 3220147. Ignacio Márquez Albés was funded by the Czech Academy of Sciences (RVO 67985840).

References

[1] D. Anderson, J. Bullock, L. Erbe, A. Peterson, H. Tran, *Nabla dynamic equations*. In M. Bohner, A. Peterson (eds.), *Advances in dynamic equations on time scales*, Birkhäuser, 2003, 47–83.

[2] R. Bellman, *Stability theory of differential equations*, McGraw-Hill Book Co., 1953.

[3] M. Bohner, A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Birkhäuser, 2001.

[4] M. Frigon, R. López Pouso, *Theory and applications of first-order systems of Stieltjes differential equations*, Adv. Nonlinear Anal. 6 (2017), 13–36.

[5] J. Groh, *A nonlinear Volterra–Stieltjes integral equation and a Gronwall inequality in one dimension*, Ill. J. Math. 24 (1980), 244–263.

[6] T. H. Gronwall, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. of Math. 20 (1919), 292–296.

[7] I. Györi, L. Horváth, *Gronwall-Bellman type integral inequalities for abstract Lebesgue integral*, Math. Comput. Modelling 25 (1997), 51–58.

[8] M. Herdegen, S. Herrmann, *Minimal conditions for implications of Gronwall–Bellman type*, J. Math. Anal. Appl. 446 (2017), 1654–1665.

[9] J. V. Herod, *A Gronwall inequality for linear Stieltjes integrals*, Proc. Amer. Math. Soc. 23 (1969), 34–36.

[10] P. Krejčí, G. A. Monteiro, V. Recupero, *Non-convex sweeping processes in the space of regulated functions*, Commun. Pure Appl. Anal. 21 (2022), 2999–3029.

[11] J. Kurzweil, *Generalized ordinary differential equations. Not absolutely continuous solutions*, World Scientific, 2012.

[12] V. Lakshmikantham, S. Leela, *Differential and integral inequalities: Theory and applications. Vol. I: Ordinary differential equations*, Academic Press, 1969.

[13] R. López Pouso, A. Rodríguez, *A new unification of continuous, discrete, and impulsive calculus through Stieltjes derivatives*, Real Anal. Exch. 40 (2014/15), 319–353.

[14] I. Márquez Albés, *Notes on the linear equation with Stieltjes derivatives*, Electron. J. Qual. Theory Differ. Equ. 2021 (2021), article no. 42.

[15] I. Márquez Albés, *Differential problems with Stieltjes derivatives and applications*, Ph.D. thesis, Universidade de Santiago de Compostela, 2021, `https://minerva.usc.es/xmlui/handle/10347/24663`.

[16] I. Márquez Albés, A. Slavík, *The logistic equation in the context of Stieltjes differential and integral equations*, Electron. J. Qual. Theory Differ. Equ. 2023 (2023), article no. 10.

[17] R. M. McLeod, *The generalized Riemann integral*, Mathematical Association of America, 1980.

[18] A. B. Mingarelli, *On a Stieltjes version of Gronwall's inequality*, Proc. Amer. Math. Soc. 82 (1981), 249–252.

[19] G. A. Monteiro, A. Slavík, *Generalized elementary functions*, J. Math. Anal. Appl. 411 (2014), 838–852.

[20] G. A. Monteiro, A. Slavík, M. Tvrdý, *Kurzweil–Stieltjes integral. Theory and applications*, World Scientific, 2019.

[21] B. G. Pachpatte, *Inequalities for differential and integral equations*, Academic Press, 1998.

[22] W. W. Schmaedeke, G. R. Sell, *The Gronwall inequality for modified Stieltjes integrals*, Proc. Amer. Math. Soc. 19 (1968), 1217–1222.

[23] Š. Schwabik, *Generalized ordinary differential equations*, World Scientific, 1992.

[24] A. Slavík, *Well-posedness results for abstract generalized differential equations and measure functional differential equations*, J. Differential Equations 259 (2015), 666–707.