Identities with squares of binomial coefficients*

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Abstract

This paper introduces a method for finding closed forms for certain sums involving squares of binomial coefficients. We use this method to present an alternative approach to a problem of evaluating a different type of sums containing squares of the numbers from Catalan's triangle.

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1 Introduction

In the first part of our paper, we present a method for finding closed forms for the sums

$$S_k(n) := \sum_{l=0}^{n-1} l^k \binom{2n}{l}^2, \quad n \geq 1,$$

$$T_k(n) := \sum_{l=0}^{n-2} l^k \binom{2n-1}{l}^2, \quad n \geq 2$$

(we use the convention $0^0 = 1$), where $k \geq 0$ is a fixed integer. These sums are somewhat tricky in the sense that the standard techniques for hypergeometric summation (see [5]) are not applicable. Indeed, the summation does not run over all possible values of l, which means that methods like Zeilberger's algorithm or Sister Celine's method cannot be used; Gosper's algorithm for indefinite summation fails, too.

In the second part of the paper, we apply our results to evaluate certain sums involving squares of the numbers from Catalan's triangle.

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2 Calculating $S_k(n)$ and $T_k(n)$

The formulas for $S_k(n)$ and $T_k(n)$ in the cases $k = 0$ and $k = 1$ are well-known:

$$S_0(n) = \sum_{l=0}^{n-1} \binom{2n}{l}^2 = \frac{1}{2}\left(\binom{4n}{2n} - \binom{2n}{n}^2\right), \quad n \geq 1,$$

$$T_0(n) = \sum_{l=0}^{n-2} \binom{2n-1}{l}^2 = \frac{1}{2}\binom{4n-2}{2n-1} - \binom{2n-1}{n-1}^2, \quad n \geq 2,$$

$$S_1(n) = \sum_{l=0}^{n-1} l\binom{2n}{l}^2 = n\left(\binom{4n-1}{2n-1} - 3\binom{2n-1}{n-1}^2\right), \quad n \geq 1,$$

$$T_1(n) = \sum_{l=0}^{n-2} l\binom{2n-1}{l}^2 =$$

$$= \frac{2n-1}{2}\left(\binom{4n-3}{2n-2} - 2\binom{2n-1}{n-1}\binom{2n-2}{n-2} - \binom{2n-2}{n-1}^2\right), \quad n \geq 2.$$

These identities follow easily (see e.g. [3]) from the standard formula

$$\sum_{i=0}^{k} \binom{k}{i}^2 = \binom{2k}{k}$$

and the "absorption" identity

$$\binom{k}{i} = \frac{k}{i}\binom{k-1}{i-1}, \quad i \geq 1.$$

The following theorem shows that for $k \geq 2$, there is a formula for S_k which depends on $T_0, \dots, T_{k-2}$:

Theorem 1. *If $k \geq 2$, then*

$$S_k(n) = 4n^2 \sum_{i=0}^{k-2} \binom{k-2}{i} T_i(n), \quad n \geq 1.$$

Proof. We use the absorption identity and binomial theorem:

$$S_k(n) = \sum_{l=1}^{n-1} l^k \binom{2n}{l}^2 = 4n^2 \sum_{l=1}^{n-1} l^{k-2} \binom{2n-1}{l-1}^2 =$$

$$4n^2 \sum_{l=0}^{n-2} (l+1)^{k-2} \binom{2n-1}{l}^2 =$$

$$4n^2 \sum_{l=0}^{n-2} \sum_{i=0}^{k-2} \binom{k-2}{i} l^i \binom{2n-1}{l}^2 = 4n^2 \sum_{i=0}^{k-2} \binom{k-2}{i} T_i(n).$$

□

There is a similar theorem which gives a formula for T_k, $k \geq 2$, in terms of $S_0, \ldots, S_{k-2}$:

Theorem 2. *If $k \geq 2$, then*

$$T_k(n) = (2n-1)^2 \sum_{i=0}^{k-2} \binom{k-2}{i} \left(S_i(n-1) - (n-2)^i \binom{2n-2}{n-2}^2 \right), \quad n \geq 2.$$

Proof.

$$T_k(n) = \sum_{l=1}^{n-2} l^k \binom{2n-1}{l}^2 = (2n-1)^2 \sum_{l=1}^{n-2} l^{k-2} \binom{2n-2}{l-1}^2 =$$

$$(2n-1)^2 \sum_{l=0}^{n-3} (l+1)^{k-2} \binom{2n-2}{l}^2 =$$

$$(2n-1)^2 \sum_{l=0}^{n-3} \sum_{i=0}^{k-2} \binom{k-2}{i} l^i \binom{2n-2}{l}^2 =$$

$$(2n-1)^2 \sum_{i=0}^{k-2} \binom{k-2}{i} \left(\sum_{l=0}^{n-2} l^i \binom{2n-2}{l}^2 - (n-2)^i \binom{2n-2}{n-2}^2 \right) =$$

$$(2n-1)^2 \sum_{i=0}^{k-2} \binom{k-2}{i} \left(S_i(n-1) - (n-2)^i \binom{2n-2}{n-2}^2 \right).$$

□

Since the formulas for S_0, S_1, T_0 and T_1 are known, the previous two theorems can be used to calculate S_k and T_k successively for every k. For example, we obtain

$$S_2(n) = 4n^2 \left(\frac{1}{2} \binom{4n-2}{2n-1} - \binom{2n-1}{n-1}^2 \right)$$

(this agrees with the formula given in [3]),

$$T_2(n) = (2n-1)^2 \left(\frac{1}{2} \left(\binom{4n-4}{2n-2} - \binom{2n-2}{n-1}^2 \right) - \binom{2n-2}{n-2}^2 \right).$$

The complexity of the formulas for S_k and T_k grows with increasing k, but the calculations (including simplification) can be performed using a computer.

Remark 3. A similar approach may be used to evaluate more general sums of the form

$$T_k^m(n) := \sum_{l=0}^{n-2} l^k \binom{2n-m}{l}^2, \quad n \geq m/2,$$

where m is an arbitrary fixed positive or negative integer and $k \geq 0$ as before; note that $T_k = T_k^1$. If m is even, then

$$T_k^m(n) = \sum_{l=0}^{n-2} l^k \binom{2(n-m/2)}{l}^2 =$$

$$\sum_{l=0}^{n-m/2-1} l^k \binom{2(n-m/2)}{l}^2 + \sum_{l=n-m/2}^{n-2} l^k \binom{2(n-m/2)}{l}^2 =$$

$$S_k(n-m/2) + \sum_{p=0}^{m/2-2} (p+n-m/2)^k \binom{2(n-m/2)}{p+n-m/2}^2$$

(we use the convention $\sum_{i=a}^{b} f(i) = -\sum_{i=b}^{a} f(i)$ if $a > b$). Otherwise, if m is odd, then

$$T_k^m(n) = (2n-m)^2 \sum_{l=1}^{n-2} l^{k-2} \binom{2n-m-1}{l-1}^2 =$$

$$(2n-m)^2 \sum_{l=0}^{n-3} (l+1)^{k-2} \binom{2n-m-1}{l}^2 =$$

$$(2n-m)^2 \sum_{l=0}^{n-3} \sum_{p=0}^{k-2} \binom{k-2}{p} l^p \binom{2n-m-1}{l}^2 =$$

$$(2n-m)^2 \sum_{p=0}^{k-2} \binom{k-2}{p} \left(\sum_{l=0}^{n-2} l^p \binom{2n-m-1}{l}^2 - (n-2)^p \binom{2n-m-1}{n-2}^2 \right) =$$

$$(2n-m)^2\sum_{p=0}^{k-2}\binom{k-2}{p}\left(T_p^{m+1}(n)-(n-2)^p\binom{2n-m-1}{n-2}^2\right),$$

i.e. T_k^m can be expressed in terms of $T_0^{m+1},\ldots,T_{k-2}^{m+1}$, whose upper indices are even.

3 A sum related to Catalan's triangle

Catalan's triangle was introduced in [6] by L. W. Shapiro. Its entries are the numbers

$$B_{n,l} := \frac{l}{n}\binom{2n}{n-l}, \quad n, l \in \mathbb{N}, l \le n.$$

The name of the triangle stems from the fact that

$$B_{n,1} = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n} = C_n,$$

i.e. the Catalan numbers appear in the first column of the triangle.

The numbers $B_{n,l}$ appear in several combinatorial problems and identities (see e.g. [6], [3], [2]). The authors of the paper [3] have pointed out a relation of these numbers to a problem of the dynamical behavior of a family of iterative methods applied to quadratic polynomials. They also mentioned the problem of evaluating the sums

$$A_k(n) := \sum_{l=1}^{n} l^k B_{n,l}^2, \quad n \in \mathbb{N},$$

where $k \ge 0$ is a fixed integer. They have obtained closed forms for A_0, A_1 and A_2; we need the first two formulas, which are true for $n \ge 1$:

$$A_0(n) = \sum_{l=1}^{n} B_{n,l}^2 = C_{2n-1} = \frac{1}{2(4n-1)}\binom{4n}{2n},$$

$$A_1(n) = \sum_{l=1}^{n} l B_{n,l}^2 = \frac{n(n+1)}{2} C_{n-1} C_n = \frac{n}{4(2n-1)}\binom{2n}{n}^2.$$

Additional closed forms for $A_3,\ldots,A_7$ were found and proved using the Wilf-Zeilberger method by the same authors in [4]. Finally, general formulas for an arbitrary A_n are given in [1]. We now show a different method of deriving the corresponding formulas.

The following theorem shows that for arbitrary k, the sum A_k can be expressed in terms of $S_0,\ldots,S_{k+2}$:

Theorem 4. *If $k \geq 0$ and $n \geq 1$ are arbitrary integers, then*

$$A_k(n) = \sum_{j=0}^{k+2} \binom{k+2}{j} n^{k-j} (-1)^j S_j(n).$$

Proof.

$$A_k(n) = \sum_{m=1}^{n} m^k B_{n,m}^2 = \frac{1}{n^2} \sum_{m=1}^{n} m^{k+2} \binom{2n}{n-m}^2 = \frac{1}{n^2} \sum_{l=0}^{n-1} (n-l)^{k+2} \binom{2n}{l}^2 =$$

$$\frac{1}{n^2} \sum_{l=0}^{n-1} \sum_{j=0}^{k+2} \binom{k+2}{j} n^{k+2-j} (-1)^j l^j \binom{2n}{l}^2 =$$

$$\sum_{j=0}^{k+2} \binom{k+2}{j} n^{k-j} (-1)^j \sum_{l=0}^{n-1} l^j \binom{2n}{l}^2 = \sum_{j=0}^{k+2} \binom{k+2}{j} n^{k-j} (-1)^j S_j(n).$$

□

This means that given a fixed $k \geq 0$, we have a method for obtaining a closed form for A_k. To avoid laborious calculations, it's again better to use a computer. All the following results were obtained using *Mathematica*. We start with the first few identities for A_k, where k is even:

$$A_2(n) = \frac{n(3n-2)}{2(4n-3)(4n-1)} \binom{4n}{2n},$$

$$A_4(n) = \frac{n\left(15n^3 - 30n^2 + 16n - 2\right)}{2(4n-5)(4n-3)(4n-1)} \binom{4n}{2n},$$

$$A_6(n) = \frac{n(105n^5 - 420n^4 + 588n^3 - 356n^2 + 96n - 10)}{2(4n-7)(4n-5)(4n-3)(4n-1)} \binom{4n}{2n},$$

And now a few formulas for A_k, where k is odd:

$$A_3(n) = \frac{n^2}{4(2n-1)} \binom{2n}{n}^2,$$

$$A_5(n) = \frac{n^2\left(3n^2 - 5n + 1\right)}{4(2n-3)(2n-1)} \binom{2n}{n}^2,$$

$$A_7(n) = \frac{n^2\left(6n^3 - 12n^2 + 6n - 1\right)}{4(2n-3)(2n-1)} \binom{2n}{n}^2.$$

Note that these formulas are written in a slightly different form than in [1] and [4]; our form has the advantage that the right-hand sides are defined for every $n \geq 1$. It should be noted that with increasing k, the time needed to perform the calculation according to Theorem 4 grows rather quickly. The *Mathematica* code which was used to perform all the calculations presented in this paper is available from

`www.karlin.mff.cuni.cz/~slavik/mathematica/identities.nb`

(including more formulas which have been omitted because of their complexity).

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References

[1] X. Chen, W. Chu, *Moments on Catalan numbers*, J. Math. Anal. Appl. 349 (2009), 311-316.

[2] W. J. R. Eplett, *A note about the Catalan triangle*, Discrete Math. 25 (1979), 289–291.

[3] J. M. Gutiérrez, M. A. Hernández, P. J. Miana, N. Romero, *New identities in the Catalan triangle*, J. Math. Anal. Appl. 341 (2008), 52-61.

[4] P. J. Miana, N. Romero, *Computer proofs of new identities in the Catalan triangle*. To appear in Biblioteca de la Revista Matemática Iberoamericana, Proceedings of the Segundas Jornadas de Teoría de Números (2007).
See `www.unirioja.es/dptos/dmc/prienol/english/natalia.html`.

[5] M. Petkovšek, H. S. Wilf, D. Zeilberger, $A = B$, A. K. Peters, 1996.

[6] L. W. Shapiro, *A Catalan triangle*, Discrete Math. 14 (1976), 83-90.