Maximal volumes of n-dimensional balls in the p-norm

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Abstract. We revisit the well-known problem of determining the dimension in which a unit ball has maximal volume. We consider balls with respect to the p-norm with arbitrary radius. Given a fixed p, we find all radii for which the volume is maximized in dimension n. Conversely, for a fixed radius, we find all values of p for which the volume is maximal in dimension n.

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1. Introduction

It is well known that the volume of the n-dimensional unit ball is maximal in dimension five, where it begins to decrease, and tends to zero as $n \to \infty$; see e.g. [6]. But there is nothing special about dimension five: For each $n \in \mathbb{N}$, there exists an $r > 0$ such that volume of the Euclidean ball of radius r is maximized in dimension n.

We do not restrict ourselves to Euclidean balls, but consider the p-balls

$$B_p(r) = \{x \in \mathbb{R}^n : |x_1|^p + \cdots + |x_n|^p \leq r^p\},$$

where p is a positive parameter. If $p \geq 1$, then $B_p(r)$ is the ball of radius r with respect to the p-norm $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$. Although $\|\cdot\|_p$ is no longer a norm for $p \in (0, 1)$, the definition of $B_p(r)$ still makes sense, and this family of generalized balls includes some interesting objects such as the region bounded by an astroid or its higher-dimensional counterparts, corresponding to $p = 2/3$.

The volume of the n-dimensional p-ball of radius r is given by the formula

$$V_p^n(r) = (2r)^n \frac{\Gamma(1 + 1/p)^n}{\Gamma(1 + n/p)}, \tag{1.1}$$

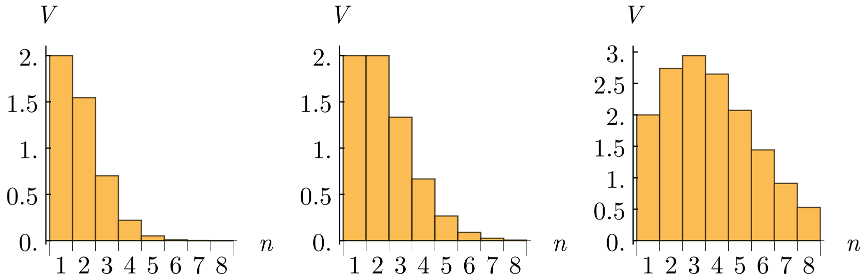

FIGURE 1. Volumes of n-dimensional unit p-balls for $p = \frac{4}{5}$ (left), $p = 1$ (center), and $p = \frac{3}{2}$ (right)

which goes back to Dirichlet and Liouville, see [1, Section 1.8]. Some easily accessible derivations may be found also in [3], [5], [7].

For a given $n \in \mathbb{N}$, our goal is to find all pairs (p, r) for which the p-ball of radius r has maximal volume in dimension n. We will present three results related to this problem. Theorem 1 may be known to experts on special functions, but we provide an elementary proof for completeness. Theorems 2 and 3 seem to be new, despite a fairly large amount of literature devoted to volumes of n-dimensional balls and their qualitative as well as quantitative properties; see e.g. [2], [4] and the references therein.

The only nontrivial property of the gamma function that is needed in the present paper is the following: If $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(x)$ is the logarithmic derivative of the gamma function, then $\psi'(x) > 0$ for all $x \in (0, \infty)$; see [1, Theorem 1.2.5]. In particular, Γ is strictly logarithmically convex on $(0, \infty)$.

2. Main results

Choosing a $p \in (0, \infty)$ and plotting the sequence $\{V_p^n(1)\}_{n=1}^{\infty}$, we obtain pictures such as those in Figure 1. As in the case $p = 2$, the volumes seem to reach a maximum in a certain dimension (possibly $n = 1$), and then begin decreasing to zero. Let us show that this is indeed true, for arbitrary $r > 0$ and $p > 0$.

Theorem 1. *For each $r > 0$ and $p > 0$, the following statements hold:*

1. $\lim_{n \to \infty} V_p^n(r) = 0$.
2. *The sequence $\left\{V_p^{n+1}(r)/V_p^n(r)\right\}_{n=1}^{\infty}$ is decreasing.*
3. *There exists a unique $m \in \mathbb{N}$ such that*

$$V_p^1(r) < V_p^2(r) < \cdots < V_p^m(r), \tag{2.1}$$

$$V_p^m(r) \geq V_p^{m+1}(r) > V_p^{m+2}(r) > \cdots . \tag{2.2}$$

Proof. If $2r\Gamma(1+1/p) \geq 1$, we have the estimate

$$V_p^n(r) = (2r)^n \frac{\Gamma(1+1/p)^n}{\Gamma(1+n/p)} \leq \frac{((2r\Gamma(1+1/p))^p)^{n/p}}{\Gamma(1+\lfloor n/p \rfloor)} \leq \frac{C^{\lfloor n/p \rfloor+1}}{\lfloor n/p \rfloor!} = \frac{C^{k+1}}{k!},$$

where $C = (2r\Gamma(1+1/p))^p$, $k = \lfloor n/p \rfloor$. Since $\lim_{k\to\infty} C^{k+1}/k! = 0$, we get $\lim_{n\to\infty} V_p^n(r) = 0$. The same conclusion obviously holds if $2r\Gamma(1+1/p) < 1$.

Using formula (1.1), it is easy to check that $\left\{V_p^{n+1}(r)/V_p^n(r)\right\}_{n=1}^{\infty}$ is decreasing if and only if

$$\Gamma\left(1+\frac{n+1}{p}\right)/\Gamma\left(1+\frac{n}{p}\right) > \Gamma\left(1+\frac{n}{p}\right)/\Gamma\left(1+\frac{n-1}{p}\right)$$

for each $n \in \mathbb{N}$, $n \geq 2$. Equivalently (taking logarithms of both sides and rearranging the terms), we need

$$\log\Gamma\left(1+\frac{n+1}{p}\right) + \log\Gamma\left(1+\frac{n-1}{p}\right) > 2\log\Gamma\left(1+\frac{n}{p}\right).$$

The last inequality follows immediately from the strict logarithmic convexity of the gamma function.

To get the inequalities (2.1) and (2.2), choose m to be the smallest integer such that $V_p^{m+1}(r)/V_p^m(r) \leq 1$. Such an integer has to exist, for otherwise $\{V_p^n(r)\}_{n=1}^{\infty}$ would be increasing, which contradicts the first statement. □

The second part of Theorem 1 says that $\{V_p^n(r)\}_{n=1}^{\infty}$ is strictly logarithmically concave, while (2.1) and (2.2) say that the sequence is unimodal; the fact that every logarithmically concave sequence with positive terms is necessarily unimodal (except when it is nondecreasing) is well known.

For our purposes, it is important to know that all inequalities in (2.1) and all inequalities except the first one in (2.2) are strict. If $p > 0$ and $r > 0$ are fixed, we can calculate the terms of $\{V_p^n(r)\}_{n=1}^{\infty}$ until we find an integer m such that $V_p^m(r) \geq V_p^{m+1}(r)$. Then we know that the p-ball of radius r has maximal volume in dimension m, and also in dimension $m+1$ if $V_p^m(r) = V_p^{m+1}(r)$. This procedure was used to produce the plot in Figure 2, which corresponds to $r = 1$ and shows the dimension in which $V_p^n(1)$ attains its maximum, depending on the choice of $p > 0$.

In general, we have the following result.

Theorem 2. *If $r \leq \frac{1}{2}$, then the volume of $B_p(r)$ is maximal in dimension 1 for each $p > 0$. Otherwise, if $r > \frac{1}{2}$, there exists an unbounded increasing sequence of positive numbers $\{p_n(r)\}_{n=1}^{\infty}$ with the following properties:*

- *If $p \in (0, p_1(r))$, then the volume of $B_p(r)$ is maximal in dimension 1.*
- *If $p = p_n(r)$ for some $n \in \mathbb{N}$, then the volume of $B_p(r)$ is maximal in dimensions n and $n+1$.*
- *If $p \in (p_{n-1}(r), p_n(r))$ for some $n \in \mathbb{N}$, $n \geq 2$, then the volume of $B_p(r)$ is maximal in dimension n.*

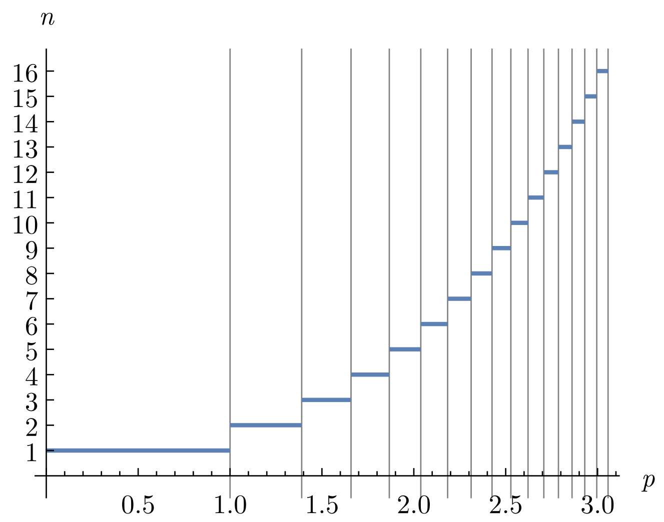

FIGURE 2. For each $p > 0$, the plot shows the dimension n in which the unit p-ball has maximal volume. Vertical lines correspond to those values of p for which the volume is maximized in two adjacent dimensions.

For each $n \in \mathbb{N}$, $p_n(r)$ is the unique $p > 0$ satisfying

$$\frac{\Gamma\left(1+\frac{n+1}{p}\right)}{2r\Gamma\left(1+\frac{n}{p}\right)\Gamma\left(1+\frac{1}{p}\right)} = 1. \tag{2.3}$$

Moreover, $p_n(r)$ is decreasing with respect to r.

Proof. According to the formula (1.1), we have

$$\frac{V_p^n(r)}{V_p^{n+1}(r)} = \frac{\Gamma\left(1+\frac{n+1}{p}\right)}{2r\Gamma\left(1+\frac{n}{p}\right)\Gamma\left(1+\frac{1}{p}\right)} = h_n(1/p, r), \tag{2.4}$$

where $h_n : [0,\infty) \times (0,\infty) \to \mathbb{R}$ is given by

$$h_n(q,r) = \frac{\Gamma\left(1+(n+1)q\right)}{2r\Gamma\left(1+nq\right)\Gamma(1+q)}.$$

The logarithmic derivative of h_n with respect to q is

$$\begin{aligned}
&\frac{\partial}{\partial q}\log h_n(q,r) \\
&= \frac{\partial}{\partial q}\left(\log\Gamma\left(1+(n+1)q\right) - \log(2r) - \log\Gamma\left(1+nq\right) - \log\Gamma(1+q)\right) \\
&= (n+1)\psi(1+(n+1)q) - n\psi(1+nq) - \psi(1+q) \\
&= n\left(\psi(1+(n+1)q) - \psi(1+nq)\right) + \psi(1+(n+1)q) - \psi(1+q) > 0,
\end{aligned}$$

where the last inequality holds because ψ is increasing. Since $\frac{\partial}{\partial q}\log h_n = \frac{\partial h_n}{\partial q}/h_n$ and $h_n > 0$, it follows that $\frac{\partial h_n}{\partial q}(q,r) > 0$ for all $q > 0$ and $r > 0$, which in turn implies that h_n is increasing in q.

It is clear that $h_n(0,r) = \frac{1}{2r}$; let us show that $\lim_{q\to\infty} h_n(q,r) = \infty$. It suffices to consider the limit only with respect to positive integers q (because of the monotonicity of $q \mapsto h_n(q,r)$). In this case we have

$$h_n(q,r) = \frac{1}{2r}\frac{((n+1)q)!}{(nq)!q!}$$
$$= \frac{1}{2r}\frac{nq+q}{q}\cdot\frac{nq+q-1}{q-1}\cdot\frac{nq+q-2}{q-2}\cdots\frac{nq+1}{1} \geq \frac{(n+1)^q}{2r},$$

which goes to infinity as $n \to \infty$. Consequently, for each $r > 0$, the function $h_n(\cdot,r)$ maps $[0,\infty)$ to $[\frac{1}{2r},\infty)$.

If $r \leq \frac{1}{2}$, then $V_p^n(r)/V_p^{n+1}(r) = h_n(1/p,r) > h_n(0,r) = 1/2r \geq 1$. It follows that the sequence $\{V_p^n(r)\}_{n=1}^\infty$ is decreasing, and therefore the p-ball or radius r has maximum volume in dimension 1.

If $r > \frac{1}{2}$, there exists a unique $p_n(r) > 0$ such that $h_n(1/p_n(r),r) = 1$, i.e., $V^n_{p_n(r)}(r) = V^{n+1}_{p_n(r)}(r)$. Hence, for $p = p_n(r)$, the volume of $B_p(r)$ is maximal in dimensions n and $n+1$. If $p > p_n(r)$, then $h_n(1/p_n(r),r) < 1$ and therefore $V_p^n(r) < V_p^{n+1}(r)$. Similarly, if $0 < p < p_n(r)$, then $V_p^n(r) > V_p^{n+1}(r)$.

For each $q > 0$ and $r > 0$, the sequence $\{h_n(q,r)\}_{n=1}^\infty$ is increasing, since $\{\Gamma\left(1+(n+1)q\right)/\Gamma\left(1+nq\right)\}_{n=1}^\infty$ is increasing (cf. the proof of Theorem 1, where we choose $p = 1/q$). We claim that $\{p_n(r)\}_{n=1}^\infty$ is increasing; indeed, the possibility $p_{n+1}(r) \leq p_n(r)$ for some $n \in \mathbb{N}$ leads to a contradiction:

$$1 = h_n(1/p_n(r),r) \leq h_n(1/p_{n+1}(r),r) < h_{n+1}(1/p_{n+1}(r),r) = 1.$$

If $p \in (0,p_1(r))$, then $V_p^1(r) > V_p^2(r)$, and therefore the volume of $B_p(r)$ is maximal in dimension 1. If $p \in (p_{n-1}(r),p_n(r))$ for some $n \in \mathbb{N}$, $n \geq 2$, then $V_p^{n-1}(r) < V_p^n(r)$ and $V_p^n(r) > V_p^{n+1}(r)$. Hence, the volume of $B_p(r)$ is maximal in dimension n.

The sequence $\{p_n(r)\}_{n=1}^\infty$ is unbounded from above: Choose an arbitrary $p > 0$. Then the volume of $B_p(r)$ attains its maximum in a certain dimension n. We necessarily have $p_n(r) \geq p$, for otherwise $V_p^n(r) < V_p^{n+1}(r)$, which contradicts the assumption that $V_p^n(r)$ is maximal. Thus, we see that for each $p > 0$, there exists an $n \in \mathbb{N}$ such that $p_n(r) \geq p$.

Finally, we note that h_n is obviously decreasing in the second variable r. Since $p_n(r)$ satisfies $h_n(1/p_n(r),r) = 1$, we see that $1/p_n(r)$ has to increase if r increases, i.e., $p_n(r)$ has to decrease. □

Returning to Figure 2, Theorem 2 says that the jumps (marked by vertical lines) correspond to the terms of the sequence $\{p_n(1)\}_{n=1}^\infty$, i.e., to solutions of equation (2.3) with $r = 1$. For $n = 1$, we get the solution $p = p_1(1) = 1$, i.e., the unit p-ball has maximal volume in dimension 1 if and only if $p \in (0,1]$ (if $p = 1$, then the maximum is also attained in dimension 2). The remaining values $\{p_n(1)\}_{n=2}^\infty$ can be obtained by numerically solving equation (2.3). For example, the volume of the unit ball is maximized in dimension 5 for all values of p between $p_4(1) \doteq 1.86656$ and $p_5(1) \doteq 2.03793$.

Another question we can ask is the following: If we fix $p > 0$ and $n \in \mathbb{N}$, which radii $r > 0$ have the property that $B_p(r)$ has maximal volume in dimension n? Once we have Theorem 2, the answer is simple.

Theorem 3. *For each $p > 0$, let $\{r_n(p)\}_{n=1}^{\infty}$ be given by*

$$r_n(p) = \frac{\Gamma\left(1 + \frac{n+1}{p}\right)}{2\Gamma\left(1 + \frac{n}{p}\right)\Gamma\left(1 + \frac{1}{p}\right)}, \quad n \in \mathbb{N}. \tag{2.5}$$

Then $\{r_n(p)\}_{n=1}^{\infty}$ is increasing, unbounded, and the following statements hold:

- *If $r \in (0, r_1(p))$, then the volume of $B_p(r)$ is maximal in dimension 1.*
- *If $r = r_n(p)$ for some $n \in \mathbb{N}$, then the volume of $B_p(r)$ is maximal in dimensions n and $n+1$.*
- *If $r \in (r_{n-1}(p), r_n(p))$ for some $n \in \mathbb{N}$, $n \geq 2$, then the volume of $B_p(r)$ is maximal in dimension n.*

Moreover, $r_n(p)$ is decreasing with respect to p.

Proof. $\{r_n(p)\}_{n=1}^{\infty}$ is an increasing sequence, since $\{\Gamma(1+\frac{n+1}{p})/\Gamma(1+\frac{n}{p})\}_{n=1}^{\infty}$ is increasing (see the proof of Theorem 1).

If $n \geq 2$, Theorem 2 says that $B_p(r)$ has maximal volume in dimension n if and only if $p \in [p_{n-1}(r), p_n(r)]$. To find the value of $r > 0$ for which $p = p_{n-1}(r)$, we solve equation (2.3) with n replaced by $n-1$, and get $r = r_{n-1}(p)$. To find the value of $r > 0$ for which $p = p_n(r)$, we solve equation (2.3) and find $r = r_n(p)$. Thus, $B_p(r)$ has maximal volume in dimension $n > 1$ if and only if $r \in [r_{n-1}(p), r_n(p)]$.

Similarly, according to Theorem 2, $B_p(r)$ has maximum volume in dimension 1 for all $r > 0$ such that either $r \in (0, \frac{1}{2}]$, or $r > \frac{1}{2}$ and $p \in (0, p_1(r)]$, i.e., if and only if $r \in (0, r_1(p)]$.

To see that $\{r_n(p)\}_{n=1}^{\infty}$ is unbounded, take an arbitrary $r > 0$. Let $B_p(r)$ have maximal volume in a certain dimension $n \in \mathbb{N}$. Then it follows from the previous part of proof that we necessarily have $r_n(p) \geq r$.

Finally, $r_n(p)$ is decreasing with respect to p, since the function $q \mapsto \Gamma(1 + (n+1)q)/(\Gamma(1 + nq)\Gamma(1+q))$ is increasing, as demonstrated in the proof of Theorem 2. □

An illustration is provided in Figure 3, where we have chosen $p = 2$; the horizontal lines correspond to the terms of the sequence $\{r_n(2)\}_{n=1}^{\infty}$. For example, a Euclidean ball has maximal volume in dimension 5 for all radii r between 0.9375 and 1.01859.

The results obtained so far can be visualized in yet another way: Figure 4 shows the graphs of the functions $p \mapsto r_n(p)$, $p \in (0, \infty)$. The region between the axes and r_1 corresponds to pairs (p, r) for which $B_p(r)$ has maximal volume in dimension 1. Similarly, each region between two adjacent functions r_{n-1} and r_n consists of all pairs (p, r) for which $B_p(r)$ has maximal volume in dimension n. Each vertical line corresponding to a fixed $p > 0$ intersects the

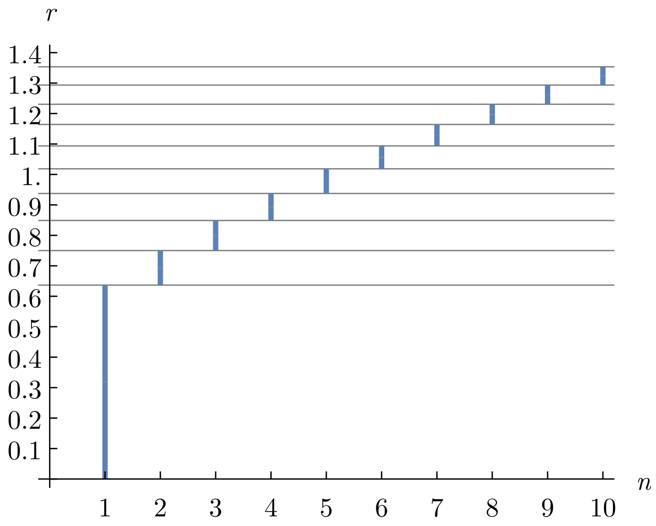

FIGURE 3. For each $n \in \mathbb{N}$, the plot shows all values of r for which the Euclidean ball ($p = 2$) of radius r has maximal volume in dimension n. Horizontal lines correspond to radii for which the volume is maximized in two adjacent dimensions.

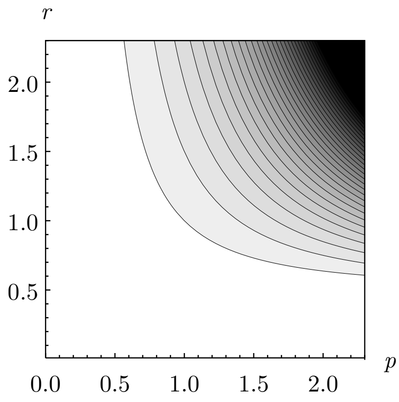

FIGURE 4. Each region represents the pairs (p, r) for which $B_p(r)$ has maximal volume in the same dimension n; darker colors correspond to larger values of n. The curves are graphs of the increasing sequence of functions r_n given by formula (2.5). The least of these functions, r_1, has asymptotes $r = 0.5$ and $p = 0$.

graphs at the points $(p, r_n(p))$, $n \in \mathbb{N}$ (cf. Theorem 2), and each horizontal line corresponding to a fixed $r > \frac{1}{2}$ intersects the graphs at the points $(p_n(r), r)$, $n \in \mathbb{N}$ (cf. Theorem 3).

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