De Bruijn's Short Route to Rényi's Parking Constant

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Abstract. Rényi's parking constant describes the mean density of randomly parked cars in a long street. We present a short elementary derivation of the exact formula for Rényi's constant, which is inspired by de Bruijn's research in number theory and differential equations.

1. INTRODUCTION. A classical parking problem due to Alfréd Rényi [**17**] asks for the expected number of unit-length cars that can be parked randomly in a street of length x. More precisely, suppose the street corresponds to the interval $[0,x]$ with $x>1$. The first car parks randomly with its left end uniformly distributed over $[0,x-1]$. The second car behaves in the same way, but leaves the street if its preferred position overlaps with the position of the first car. The parking process continues as long as the probability that an additional car is able to park is nonzero (see the computer animation in [**23**]). The goal is to calculate $m(x)$, the expected number of cars that will be able to park.

Rényi himself was not really interested in car parking. The English title of his 1958 paper is *On a one-dimensional problem concerning random space filling*, and his parking problem represents a one-dimensional version of the sequential random packing problem. The latter is important in statistical physics and chemistry, and asks for the mean packing density of randomly placed unit spheres. Since the three-dimensional problem is quite difficult, Rényi considered the one-dimensional version as a starting point for further investigations. A wealth of information on the history of sequential packing problems and their applications in various branches of science is available in [**7**] and [**20**].

But let us return to car parking. We imagine the cars to be one-dimensional closed unit intervals, and they are allowed to touch themselves at their endpoints. Alternatively, one could work with disjoint open unit intervals, and this setting leads to the same value of $m(x)$.

To derive an equation for $m(x+1)$, suppose that the first car has parked with its left endpoint at $t\in[0,x]$. Then the remaining empty spaces on both sides have lengths t and $x-t$, respectively. The expected numbers of cars that are able to park there are $m(t)$ and $m(x-t)$, and taking the average over all possible values of t, we obtain

$$\begin{aligned} m(x+1) &= \frac{1}{x}\int_0^x (1+m(t)+m(x-t))\,\mathrm{d}t \\ &= 1+\frac{1}{x}\int_0^x m(t)\,\mathrm{d}t+\frac{1}{x}\int_0^x m(x-t)\,\mathrm{d}t. \end{aligned}$$

Here we assume that $m(x)=0$ for all $x\in[0,1)$. Some care is needed when defining $m(1)$. It is natural to let $m(1)=1$, but then the formula $1+m(t)+m(x-t)$ overcounts the number of cars if $t=1$ or $t=x-1$, because the probability that another car parks exactly in the interval $[0,1]$ or $[x,x+1]$ is zero. However, the integral equation is in fact correct, because the value of $m(1)$ has no influence on the values of the

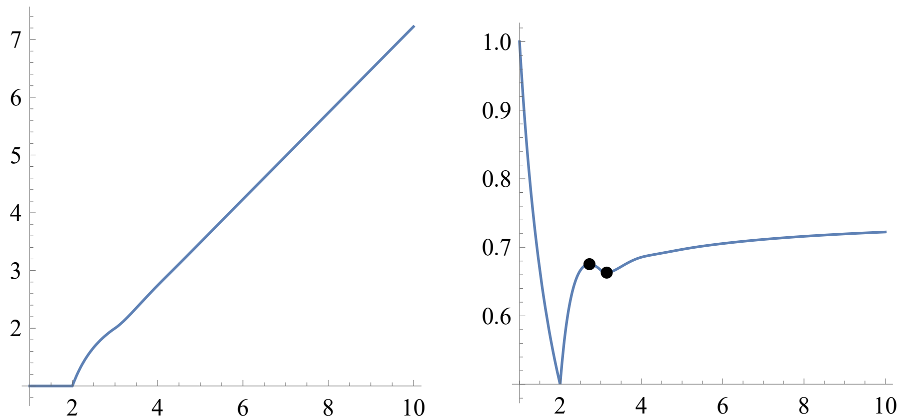

Figure 1. The functions $x \mapsto m(x)$ and $x \mapsto \frac{m(x)}{x}$.

integrals. Further simplification leads to

$$m(x+1) = 1 + \frac{2}{x} \int_0^x m(t)\,\mathrm{d}t, \quad x \geq 1. \tag{1}$$

Note that $m(x) = 1$ for all $x \in [1,2]$, and we can use the previous formula to calculate

$$m(x) = \frac{3x-5}{x-1}, \quad x \in [2,3].$$

In a similar way, we get

$$m(x) = \frac{7x - 4\log(x-2) - 17}{x-1}, \quad x \in [3,4].$$

This can be no longer integrated in terms of elementary functions, so we resort to numerical calculation. It is best to multiply (1) by x and differentiate to get

$$x \cdot m'(x+1) + m(x+1) = 1 + 2m(x), \quad x \geq 1 \tag{2}$$

(the derivative has to be understood as one-sided for $x = 1$). This delay differential equation is easy to solve numerically, for example in *Wolfram Mathematica*; see the left part of Figure 1.

It is obvious that m is nondecreasing, and $m(x) \leq x$ for all $x \geq 1$, since the number of cars never exceeds the length of the interval. On the other hand, we have $m(x) \geq \frac{x}{2}$ for all $x \geq 1$. For $x \in [1,3]$, this follows from the previous explicit formulas. And if the inequality holds for a certain x, then it also holds for $x+2$. Indeed, suppose the first car has already parked, and the remaining empty spaces on both sides have total length $x+1$. If one of them has length less than or equal to 1, we discard it. What remains is either one or two intervals of length greater than one, whose total length is at least x. By the induction hypothesis, the total expected number of cars that can park there is at least $\frac{x}{2}$, which shows that $m(x+2) \geq \frac{x}{2} + 1$.

In fact, it is more interesting to look at the values of $\frac{m(x)}{x}$, which correspond to the expected density of cars; see the right part of Figure 1. This function has a global minimum at $x = 2$, followed by a local maximum at $x = \frac{1}{3}\left(\sqrt{10}+5\right) \approx 2.72076$ and a local minimum at $x \approx 3.14577$ (both local extrema are shown as dots). Then it begins to approach a certain limiting value C, which is known as Rényi's parking constant. Rényi discovered the exact formula

$$C = \int_0^\infty \exp\left(-2\int_0^u \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t\right)\mathrm{d}u, \tag{3}$$

while the numerical value is approximately $C \approx 0.7475979$ (see [**21**]). Thus, for long streets, we expect that randomly parked cars will occupy approximately 75% of the space.

Various authors have devised numerical methods leading to more or less precise estimates of C (see [**14**] and the references therein). With modern computers and software such as *Wolfram Mathematica*, it is straightforward to calculate C to a large number of decimal places directly from (3). But Rényi's parking problem and its modifications, including discrete versions, still remain popular, see e.g., [**4, 5, 8, 19**] and the references therein. Rényi's constant also appears in Finch's delightful survey of mathematical constants [**7**, Section 5.3]. Interestingly, the two-dimensional packing problem, which asks for the mean density of randomly placed unit squares in an $x \times y$ rectangle, remains unsolved. The exact value of the limiting density when $x, y \to \infty$ is unknown, and only numerical results are available. These results indicate that the Palásti conjecture, which predicts the limiting density to be C^2, is probably false.

Rényi's derivation of (3), which is reproduced in [**20**], is not completely elementary. Its first step is to apply the Laplace transform, which converts the delay differential equation (2) to an ordinary differential equation. This equation is solved by the variation of parameters, and the calculation is finished by the use of a Tauberian theorem, which provides a link between the asymptotic behavior of $x \mapsto \int_0^x m$ at infinity, and the behavior of the Laplace transform of m at 0. Different derivations of (3) were proposed in [**10, 13**], but none of them seems to be completely transparent and accessible to a wide readership. An elementary approach is available in [**22**], but it provides only approximate lower and upper bounds for C.

Our goal is to present an alternative and elementary derivation of (3), which is inspired by the work of Nicolaas Govert de Bruijn, in particular his paper [**2**] devoted to the Buchstab function. In Section 2, we present a short and self-contained proof of (3). In Section 3, we compare the proof with de Bruijn's analysis of the Buchstab function, and provide some additional insight and comments. Section 4 contains some further historical remarks.

2. AN ELEMENTARY DERIVATION OF RÉNYI'S CONSTANT. The substitution $n(x) = m(x) + 1$ converts (2) into the homogeneous equation

$$x \cdot n'(x+1) + n(x+1) = 2n(x), \quad x \geq 1,$$

which can be further rewritten as

$$[x \cdot n(x+1)]' = 2n(x), \quad x \geq 1,$$

or equivalently

$$[(x-1)n(x)]' = 2n(x-1), \quad x \geq 2. \tag{4}$$

The initial condition is now $n(x) = m(x) + 1 = 2$ for all $x \in [1, 2]$. Recall that our goal is to calculate $\lim_{x\to\infty} \frac{m(x)}{x}$, but since m and n differ only by a constant, it suffices to evaluate $\lim_{x\to\infty} \frac{n(x)}{x}$. We will achieve this goal in three steps:

1. Observe that solutions of the differential equation (4), which contains a delayed argument, are closely related to solutions of another differential equation with an advanced argument.
2. Find a suitable solution of the latter equation and investigate its asymptotic behavior.
3. Combine the previous steps and analyze the asymptotic behavior of $x \mapsto \frac{n(x)}{x}$.

The relation between differential equations with delayed and advanced arguments is a general phenomenon that will be discussed in Section 3. But, for the purpose of calculating Rényi's constant, it suffices to prove the following simple result.

Proposition 1. *If* $n : [1, \infty) \to \mathbb{R}$ *is a solution of* (4) *and* $h : [1, \infty) \to \mathbb{R}$ *is a solution of the equation*

$$\frac{1}{2}(x-1)h'(x-1) + h(x) = 0, \quad x \geq 2, \tag{5}$$

then the function $\langle n, h\rangle : [1, \infty) \to \mathbb{R}$ *given by*

$$\langle n, h\rangle(x) = 2\int_{x-1}^{x} n(t)h(t)\,\mathrm{d}t + (x-1)n(x)h(x-1) \tag{6}$$

is constant on $[2, \infty)$.

Proof. Differentiation yields

$$\begin{aligned}\langle n, h\rangle'(x) = {} & 2n(x)h(x) - 2n(x-1)h(x-1) + n(x)h(x-1) \\ & +(x-1)n'(x)h(x-1) + (x-1)n(x)h'(x-1).\end{aligned}$$

Replacing $n(x-1)$ by $\frac{1}{2}[(x-1)n(x)]' = \frac{1}{2}n(x) + \frac{1}{2}(x-1)n'(x)$, we get

$$\langle n, h\rangle'(x) = 2n(x)h(x) + (x-1)n(x)h'(x-1),$$

which is zero because of (5). ■

In the next proposition, we find a solution of equation (5) with the help of the function

$$\operatorname{Ein} u = \int_0^u \frac{1 - \mathrm{e}^{-t}}{t}\,\mathrm{d}t, \quad u \in [0, \infty). \tag{7}$$

Having in mind the formula (3) that we want to prove, the appearance of the previous integral in our calculations should come as no surprise, because our ultimate goal is to show that $\lim_{x\to\infty} \frac{n(x)}{x} = \int_0^\infty \mathrm{e}^{-2\operatorname{Ein} u}\,\mathrm{d}u$.

The integral in (7) exists and is finite, since the integrand has limit 1 for $t \to 0+$. Note that Ein is a nonnegative function and $\lim_{u\to\infty} \operatorname{Ein} u = \infty$, because the integrand is greater than $\frac{1}{2t}$ for all sufficiently large t.

Proposition 2. *The function*

$$h(x) = \int_0^\infty u \cdot \mathrm{e}^{-u \cdot x - 2\,\mathrm{Ein}\,u}\,\mathrm{d}u, \quad x \in [1, \infty), \tag{8}$$

is a solution of equation (5)*. Moreover, it satisfies* $h(x) \sim \frac{1}{x^2}$ *for* $x \to \infty$*.*

Proof. The integral in the definition of h exists and is finite, because the integrand is less than or equal to $u \cdot \mathrm{e}^{-u}$. Differentiation under the integral sign gives

$$h'(x) = -\int_0^\infty u^2 \cdot \mathrm{e}^{-u \cdot x - 2\,\mathrm{Ein}\,u}\,\mathrm{d}u, \quad x \geq 1$$

(this differentiation is justified, since $0 \leq u^2 \cdot \mathrm{e}^{-u \cdot x - 2\,\mathrm{Ein}\,u} \leq u^2 \cdot \mathrm{e}^{-u}$). Therefore, for $x \geq 2$,

$$\begin{aligned} \frac{1}{2}(x-1)h'(x-1) + h(x) &= \frac{1}{2}(1-x) \int_0^\infty u^2 \cdot \mathrm{e}^{-u \cdot (x-1) - 2\,\mathrm{Ein}\,u}\,\mathrm{d}u \\ + \int_0^\infty u \cdot \mathrm{e}^{-u \cdot x - 2\,\mathrm{Ein}\,u}\,\mathrm{d}u &= \frac{1}{2} \int_0^\infty \mathrm{e}^{-u \cdot (x-1) - 2\,\mathrm{Ein}\,u} \left((1-x)u^2 + 2u \cdot \mathrm{e}^{-u} \right) \mathrm{d}u \\ &= \frac{1}{2} \left[u^2 \cdot \mathrm{e}^{-u \cdot (x-1) - 2\,\mathrm{Ein}\,u} \right]_{u=0}^{\infty} = 0. \end{aligned}$$

To prove the second statement, we calculate (performing the change of variables $t = x \cdot u$)

$$\begin{aligned} \lim_{x \to \infty} \left(x^2 \cdot h(x) \right) &= \lim_{x \to \infty} \int_0^\infty x^2 \cdot u \cdot \mathrm{e}^{-u \cdot x - 2\,\mathrm{Ein}\,u}\,\mathrm{d}u \\ &= \lim_{x \to \infty} \int_0^\infty t \cdot \mathrm{e}^{-t} \cdot \mathrm{e}^{-2\,\mathrm{Ein}(t/x)}\,\mathrm{d}t \\ &= \int_0^\infty \lim_{x \to \infty} \left(t \cdot \mathrm{e}^{-t} \cdot \mathrm{e}^{-2\,\mathrm{Ein}(t/x)} \right) \mathrm{d}t = \int_0^\infty t \cdot \mathrm{e}^{-t}\,\mathrm{d}t = 1. \end{aligned}$$

The interchange of the order of the limit and the integral can be justified using the dominated convergence theorem (note that $0 \leq t \cdot \mathrm{e}^{-t} \cdot \mathrm{e}^{-2\,\mathrm{Ein}(t/x)} \leq t \cdot \mathrm{e}^{-t}$). ∎

We are now ready for the calculation of Rényi's constant.

Theorem 3. *If* $n : [1, \infty) \to \mathbb{R}$ *is the solution of* (4) *satisfying* $n(x) = 2$ *for all* $x \in [1, 2]$*, then*

$$\lim_{x \to \infty} \frac{n(x)}{x} = \int_0^\infty \mathrm{e}^{-2\,\mathrm{Ein}\,u}\,\mathrm{d}u = \int_0^\infty \exp\left(-2 \int_0^u \frac{1 - \mathrm{e}^{-t}}{t}\,\mathrm{d}t \right) \mathrm{d}u.$$

Proof. Let h be given by (8). Since $\langle n, h \rangle$ introduced in (6) is a constant function, we have

$$\langle n, h \rangle(2) = \lim_{x \to \infty} \langle n, h \rangle(x).$$

Using the definition of $\langle n,h\rangle$ and the fact that $n(x)=2$ for all $x\in[1,2]$, we get

$$4\int_1^2 h(t)\,\mathrm{d}t+2h(1)=\lim_{x\to\infty}\left(2\int_{x-1}^x n(t)h(t)\,\mathrm{d}t+(x-1)n(x)h(x-1)\right).$$

Since $h(t)\sim\frac{1}{t^2}$ for $t\to\infty$, there exists a $t_0\geq 1$ such that $0\leq h(t)\leq\frac{2}{t^2}$ for all $t\geq t_0$. Recall also that $n(t)=m(t)+1\leq t+1\leq 2t$ for all $t\geq 1$. Therefore, if $x\geq t_0+1\geq 2$, we have

$$0\leq\int_{x-1}^x n(t)h(t)\,\mathrm{d}t\leq\int_{x-1}^x\frac{4}{t}\,\mathrm{d}t=4\log\frac{x}{x-1},$$

which shows that $\lim_{x\to\infty}\int_{x-1}^x n(t)h(t)\,\mathrm{d}t=0$. Consequently,

$$\begin{aligned}4\int_1^2 h(t)\,\mathrm{d}t+2h(1)&=\lim_{x\to\infty}(x-1)n(x)h(x-1)\\&=\lim_{x\to\infty}\frac{n(x)}{x}\cdot x\cdot(x-1)\cdot h(x-1)=\lim_{x\to\infty}\frac{n(x)}{x},\end{aligned}$$

since $h(x-1)\sim\frac{1}{(x-1)^2}$ for $x\to\infty$. Using the definition of h and interchanging the order of integration, we get

$$\begin{aligned}\lim_{x\to\infty}\frac{n(x)}{x}&=4\int_1^2\left(\int_0^\infty u\cdot\mathrm{e}^{-u\cdot t-2\operatorname{Ein}u}\,\mathrm{d}u\right)\mathrm{d}t+2h(1)\\&=4\int_0^\infty u\cdot\mathrm{e}^{-2\operatorname{Ein}u}\cdot\left(\int_1^2\mathrm{e}^{-u\cdot t}\,\mathrm{d}t\right)\mathrm{d}u+2h(1)\\&=4\int_0^\infty\mathrm{e}^{-2\operatorname{Ein}u}\cdot(\mathrm{e}^{-u}-\mathrm{e}^{-2u})\,\mathrm{d}u+2\int_0^\infty u\cdot\mathrm{e}^{-u}\cdot\mathrm{e}^{-2\operatorname{Ein}u}\,\mathrm{d}u\\&=\int_0^\infty\mathrm{e}^{-2\operatorname{Ein}u}\cdot\left(4\mathrm{e}^{-u}-4\mathrm{e}^{-2u}+2u\cdot\mathrm{e}^{-u}-1\right)\mathrm{d}u+\int_0^\infty\mathrm{e}^{-2\operatorname{Ein}u}\,\mathrm{d}u.\end{aligned}$$

To conclude the proof, it suffices to show that the first integral on the right-hand side vanishes. Indeed, the integrand has a primitive function

$$F(u)=\mathrm{e}^{-2\operatorname{Ein}u}\cdot(-2u\cdot\mathrm{e}^{-u}+u)$$

(the reader is invited to check this by differentiation, using the fact that $(\operatorname{Ein}u)'=\frac{1-\mathrm{e}^{-u}}{u}$). Obviously, $F(0)=0$. Moreover,

$$\begin{aligned}\operatorname{Ein}u&=\int_0^1\frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t+\int_1^u\frac{1}{t}\,\mathrm{d}t-\int_1^u\frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t\\&=\alpha+\log u-\int_1^u\frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t,\end{aligned}\tag{9}$$

where $\alpha=\int_0^1\frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t$, and therefore

$$\lim_{u\to\infty}F(u)=\mathrm{e}^{-2\alpha}\lim_{u\to\infty}\left(\frac{1}{u^2}\cdot\exp\left(2\int_1^u\frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t\right)\cdot(-2u\cdot\mathrm{e}^{-u}+u)\right)=0.$$

The last equality follows from the fact that the term with the integral is bounded:

$$0 \le \int_1^u \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t \le \int_1^u \mathrm{e}^{-t}\,\mathrm{d}t = \mathrm{e}^{-1} - \mathrm{e}^{-u} \le \mathrm{e}^{-1}. \qquad \blacksquare$$

Remark 4. The function Ein introduced in (7) is known as the complementary or modified exponential integral. It can be also expressed as the power series

$$\operatorname{Ein} u = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} u^k}{k \cdot k!}$$

(which is easily obtained from (7) using the Taylor series for e^{-t} and term-by-term integration), or in the form

$$\operatorname{Ein} u = \Gamma(0,u) + \gamma + \log u \tag{10}$$

(see [**15**, Section 6.2]), where $\Gamma(0,u) = \int_u^\infty \frac{\mathrm{e}^{-t}}{t}$ is a special case of the incomplete gamma function, and $\gamma \approx 0.577216$ is Euler's constant. Thus, Rényi's constant is often expressed in the form

$$C = \mathrm{e}^{-2\gamma} \int_0^\infty \frac{\mathrm{e}^{-2\Gamma(0,u)}}{u^2}\,\mathrm{d}u.$$

Formula (10) could have been used in the final part of our derivation in place of (9), but we have intentionally avoided (10) to keep the proof self-contained.

3. THE BUCHSTAB FUNCTION AND ADJOINT EQUATIONS. The proof presented in Section 2 is elementary, but the reader might feel that Propositions 1 and 2, came out of blue sky. Our next goal is to shed a bit more light on the whole calculation, convince the reader that it is not a mere collection of ad-hoc tricks, and provide some historical background.

De Bruijn's paper [**2**], which served as an inspiration for Section 2, deals with the Buchstab function $\omega : [1,\infty) \to \mathbb{R}$, which is well known in number theory. It satisfies

$$[x \cdot \omega(x)]' = \omega(x-1), \quad x \ge 2, \tag{11}$$

and $\omega(x) = \frac{1}{x}$ for $x \in [1,2]$; see Figure 2. The final part of [**2**] contains a novel short proof of the identity $\lim_{x\to\infty} \omega(x) = \mathrm{e}^{-\gamma}$. It is based on the fact that if h is an arbitrary solution of the differential equation

$$x \cdot h'(x-1) + h(x) = 0, \tag{12}$$

then the expression

$$\int_{x-1}^{x} \omega(t) h(t)\,\mathrm{d}t + x \cdot \omega(x) h(x-1) \tag{13}$$

does not depend on x; this is de Bruijn's version of our Proposition 1. He referred to (12) as the adjoint equation to (11), and said that (13) represents an "invariant inner

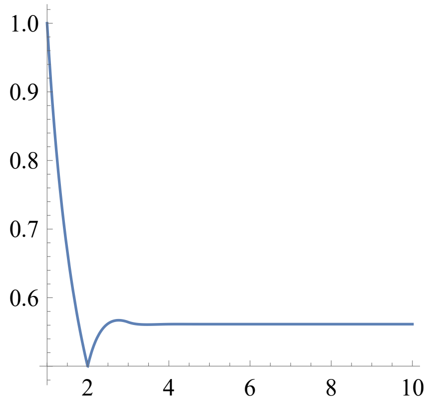

Figure 2. The Buchstab function ω.

product” of solutions to the two equations. Next, he stated (without using the notation Ein and without any further explanation) that

$$h(x) = \int_0^\infty e^{-u\cdot(x+1)-\mathrm{Ein}\,u}\,\mathrm{d}u$$

is a solution of the adjoint equation (12), which satisfies $h(x) \sim \frac{1}{x}$ for $x \to \infty$; this was our inspiration for Proposition 2. Finally, he evaluated the inner product (13) for $x = 2$ and $x \to \infty$, integrated by parts, and obtained the desired identity $\lim_{x\to\infty} \omega(x) = \mathrm{e}^{-\gamma}$.

De Bruijn was aware that his method is useful not only for the investigation of the Buchstab function, but also for similar problems. In the end of his 1950 paper [**2**], he revealed that the method originated in his unpublished studies of the equation $F'(x) = \mathrm{e}^{\alpha x+\beta}F(x-1)$. These appeared in the 1953 paper [**3**], where he considered a more general linear “differential-difference” operator Λ of the form

$$\Lambda f(x) = w(x)f'(x) + p(x)f(x) - q(x)f(x-1). \tag{14}$$

He introduced the adjoint operator Λ^* by

$$\Lambda^* g(x) = -[w(x)g(x)]' + p(x)g(x) - q(x+1)g(x+1), \tag{15}$$

as well as the inner product of f and g given by

$$\int_{x-1}^{x} q(t+1)f(t)g(t+1)\,\mathrm{d}t + w(x)f(x)g(x). \tag{16}$$

It is a simple exercise to check that (16) does not depend on x whenever f, g are solutions of the differential equations $\Lambda f = 0$ and $\Lambda^* g = 0$, respectively.

At first glance, it seems that (16) is not completely consistent with (6) and (13), where both functions in the integrand are evaluated at the same point t. However, this is just a matter of notation. Indeed, the Buchstab differential equation (11) corresponds to

$$0 = \Lambda\omega(x) = x\cdot\omega'(x) + \omega(x) - \omega(x-1),$$

and comparison with (14) shows that $w(x) = x, p(x) = 1, q(x) = 1$. Then, according to (15) and (16), the adjoint differential equation is

$$0 = \Lambda^* g(x) = -[x \cdot g(x)]' + g(x) - g(x+1) = -x \cdot g'(x) - g(x+1),$$

while the inner product of ω and g is

$$\int_{x-1}^{x} \omega(t) g(t+1)\, \mathrm{d}t + x \cdot \omega(x) g(x).$$

These results coincide with (12) and (13) if we denote $h(x) = g(x+1)$.

Similarly, the Rényi differential equation (4) corresponds to

$$0 = \Lambda n(x) = (x-1)n'(x) + n(x) - 2n(x-1),$$

i.e., we have $w(x) = x - 1$, $p(x) = 1$, $q(x) = 2$. The adjoint differential equation is

$$\begin{aligned} 0 = \Lambda^* g(x) &= -[(x-1)g(x)]' + g(x) - 2g(x+1) \\ &= -(x-1)g'(x) - 2g(x+1), \end{aligned} \tag{17}$$

while the inner product of n and g is

$$\int_{x-1}^{x} 2n(t) g(t+1)\, \mathrm{d}t + (x-1) n(x) g(x). \tag{18}$$

These results coincide with (5) and (6) if we denote $h(x) = g(x+1)$.

There exist other number-theoretic papers dealing with the Buchstab function, differential-difference equations, and their adjoints. A remarkably clear exposition can be found in [**12**]. The authors consider delay differential equations of the form

$$x \cdot G'(x) = -a \cdot G(x) - b \cdot G(x-1), \tag{19}$$

where a, $b \in \mathbb{R}$. The corresponding adjoint equation is then defined as

$$(x \cdot g(x))' = a \cdot g(x) + b \cdot g(x+1), \tag{20}$$

and the authors mention that it always has a solution satisfying $g(x) \sim x^{a+b-1}$ for $x \to \infty$. If $a + b < 1$, the solution is given by the formula

$$g(x) = \frac{1}{\Gamma(1-a-b)} \int_0^\infty \frac{e^{-x \cdot u + b \cdot \mathrm{Ein}\, u}}{u^{a+b}}\, \mathrm{d}u. \tag{21}$$

Proofs of these statements are available in [**11**]. Our Proposition 2 is a special case of these results: Starting with the Rényi differential equation (4) and letting $G(x) = n(x+1)$, we obtain (19) with $a = 1$ and $b = -2$. The adjoint equation (20) is then just a shifted version of (5), and the solution (21) matches (8).

The above-mentioned number-theoretic papers do not seem to be well known among experts in differential equations. However, there are two nice papers [**9**] and [**16**] published in the last decade, which focus primarily on differential equations themselves, and use the concept of adjoint equations to study the asymptotic behavior of the general delay differential equation $x'(t) = p(t)x(t-r)$.

4. CONCLUSION. Was de Bruijn aware of the connection between his methods and the parking problem? Rényi's paper [**17**] appeared only in 1958, five years after de Bruijn's paper [**3**]. It was written in Hungarian and followed by Russian and English summaries. An English translation [**18**] was published in the fourth volume of *Selected Translations in Mathematical Statistics and Probability* in 1963. De Bruijn lived a long life until 2012, but as far as we know, the parking problem is never mentioned in his publications. Still, there is convincing evidence that he was familiar with the problem. Besides calculating the value of C, Rényi obtained the asymptotic estimate

$$m(x) = C \cdot (x + 1) - 1 + O(x^{-n}), \tag{22}$$

which holds for all $n \geq 1$. He also remarked [**17**, p. 127]: "N. G. de Bruijn pointed out that using his method the estimation of the remainder term can be made still sharper."

But this is not the end of the story. One of de Bruijn's Ph.D. students at the Technical University Eindhoven was J. J. A. Beenakker, whose 1966 doctoral thesis [**1**] contains a comprehensive study of the delay differential equation $\alpha \cdot x \cdot f'(x) + f(x - 1) = 0$, including an asymptotic analysis of its solutions. The thesis was motivated by problems in analytic number theory, but the last four-page chapter is devoted to Rényi's parking problem! Beenakker knew Rényi's asymptotic estimate (22), as well as the improved estimate

$$m(x) = C \cdot (x + 1) - 1 + O\left(\left(\frac{2\mathrm{e}}{x}\right)^{x-3/2}\right)$$

due to A. Dvoretzky and H. Robbins [**6**]. Using the methods developed earlier in his thesis, Beenakker obtained the finer estimate

$$m(x) = C \cdot (x + 1) - 1 + O\left(\left(\frac{2\mathrm{e}}{x \cdot \log x}\right)^{x}\right).$$

The full details are rather involved, but it is clear that his calculation is also based on the adjoint equation (17), as well as on the inner product (18). Beenakker's thesis remained his only mathematical publication, and his work is now nearly forgotten. In fact, we discovered the thesis only when the present article was almost finished.

In view of these facts, we think it is fully justified to refer to the proof given in Section 2 as to "de Bruijn's route to Rényi's parking constant." We believe it is simpler than Rényi's original approach, and moreover provides a nice illustration of the duality between differential equations with delayed and advanced arguments.

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