# Kurzweil and McShane product integration in Banach algebras

Antonín Slavík
Charles University in Prague, Faculty of Mathematics and Physics
Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail: slavik@karlin.mff.cuni.cz

**Abstract**

We study product integrability of functions with values in unital Banach algebras. The product integrals are understood in the sense of Kurzweil, McShane, or Riemann. In particular, we introduce new concepts of strong Kurzweil and McShane product integrals, and investigate their properties. We also provide necessary and sufficient conditions for product integrability of functions with countably many discontinuities.

**Keywords:** product integral, strong product integral, gauge integral, Bochner integral, unital Banach algebra, abstract linear differential equation

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## 1 Introduction

The concept of product integration goes back to V. Volterra (see e.g. [21, 17]), who studied linear systems of differential equations of the form

$$\begin{aligned} W'(t) &= A(t)W(t), \qquad t \in [a,b], \\ W(a) &= I, \end{aligned} \tag{1.1}$$

where $I$ is the identity matrix, $A : [a,b] \to \mathbb{R}^{n\times n}$ is a given continuous function and $W : [a,b] \to \mathbb{R}^{n\times n}$ is the unknown function. To find the solution (whose existence and uniqueness is easy to prove), Volterra considered products of the form

$$(I + A(\xi_m)(t_m - t_{m-1}))(I + A(\xi_{m-1})(t_{m-1} - t_{m-2})) \cdots (I + A(\xi_1)(t_1 - t_0)), \tag{1.2}$$

where $a = t_0 < t_1 < \ldots < t_m = b$ and $\xi_i \in [t_{i-1}, t_i]$, $i \in \{1, \ldots, m\}$, is a tagged partition of $[a,b]$. As the lengths of the subintervals $[t_{i-1}, t_i]$ approach zero, the value of the product (1.2) tends to a matrix which is called the product integral of $A$ over $[a,b]$; let us denote it by $\prod_a^b (I + A(t)\,\mathrm{d}t)$. Now, if we consider the indefinite product integral $W(t) = \prod_a^t (I + A(s)\,\mathrm{d}s)$, it can be shown that $W$ is the solution of (1.1).
Volterra observed that this procedure still works if $A$ is no longer continuous but merely Riemann integrable; in this case, the indefinite product integral satisfies

$$W(t) = I + \int_a^t A(s)W(s)\,\mathrm{d}s, \qquad t \in [a,b],$$

where the integral on the right-hand side is the Riemann integral. Consequently, we have $W'(t) = A(t)W(t)$ almost everywhere in $[a,b]$. Later, the concept of product integral was extended to Lebesgue integrable functions $A$ (see [13, 2, 17]). On the other hand, P. R. Masani realized that the definition of the product integral makes sense if $A$ takes values in an arbitrary unital Banach algebra (see [12, 17]). For example, if $X$ is a Banach space and $A$ has values in the Banach algebra $\mathcal{L}(X)$ of all bounded linear operators on $X$, then the product integral of $A$ provides the unique solution of the operator equation (1.1).

It is well known that the Henstock-Kurzweil integral (also known as the gauge integral) generalizes the integrals of Riemann, Lebesgue, and Newton (see e.g. [6]). Hence, it seems natural to replace the original Riemann-type definition of product integral by a gauge-type definition; this idea was pursued by J. Jarník, J. Kurzweil [11] and Š. Schwabik [14], who also introduced the related concept of McShane product integral [15]. These authors developed a complete theory of Kurzweil and McShane product integrals for matrix-valued functions, but also realized that the proofs of their main results are not applicable in infinite dimension.

The aim of this paper is to provide a satisfactory theory of Kurzweil and McShane product integration in infinite-dimensional Banach algebras. We show that in infinite dimension, the two product integrals cannot be expected to have exactly the same properties as in finite dimension; for example, the indefinite product integrals need not be differentiable almost everywhere. On the other hand, we introduce the concepts of strong Kurzweil and McShane product integrals, whose properties are completely analogous to the finite-dimensional product integrals. In infinite dimension, the class of strongly Kurzweil/McShane product integrable functions is a proper subset of the class of Kurzweil/McShane product integrable functions, but is still wide enough for practical purposes. For example, a function with countably many discontinuities is strongly Kurzweil product integrable if and only if it is Kurzweil product integrable.

The paper is organized as follows. Section 2 summarizes some preliminaries, namely the definitions of the Kurzweil and McShane integrals and their strong counterparts, and the properties of the exponential and logarithm functions in Banach algebras. In Section 3, we recall the definitions of the Kurzweil and McShane product integrals, introduce their strong versions, and develop some of their basic properties. Section 4 is devoted to the properties of the indefinite strong product integrals, which are then used to show that strong McShane product integrability coincides with Bochner integrability. In Section 5, we develop necessary and sufficient conditions for product integrability of functions with countably many discontinuities. The final Section 6 contains a list of some open problems.

A detailed account of the history of product integration theory can be found in [17]. For applications of product integrals to differential equations in the real and complex domain, probability theory, dynamic equations on time scales, etc., see [2, 4, 17, 18].

# 2 Preliminaries

Let us recall some definitions of integrability for vector-valued functions.

A tagged partition of an interval $[a,b]$ is a collection of point-interval pairs $D = (\xi_i, [t_{i-1}, t_i])_{i=1}^m$, where $a = t_0 < t_1 < \cdots < t_m = b$ and $\xi_i \in [t_{i-1}, t_i]$ for every $i \in \{1, \ldots, m\}$. We refer to $t_0, \ldots, t_m$ as the division points of $D$, while $\xi_1, \ldots, \xi_m$ are the tags of $D$. If we relax the assumption $\xi_i \in [t_{i-1}, t_i]$ and replace it by $\xi_i \in [a,b]$, then the collection $D$ is called a free tagged partition.

Given a function $\delta : [a,b] \to \mathbb{R}^+$ (called a gauge on $[a,b]$), a free tagged partition is called $\delta$-fine if

$$[t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)), \quad i \in \{1, \ldots, m\}.$$

Let $X$ be a Banach space. A function $f : [a,b] \to X$ is called Henstock-Kurzweil integrable if there is a vector

$S_f \in X$ with the following property: To each $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\left\| \sum_{i=1}^{m} f(\xi_i)(t_i - t_{i-1}) - S_f \right\| < \varepsilon \tag{2.1}$$

for every $\delta$-fine tagged partition of $[a,b]$. In this case, $S_f$ is called the Henstock-Kurzweil integral of $f$ over $[a,b]$, and is denoted by $\int_a^b f(t)\,\mathrm{d}t$.

If (2.1) holds for all $\delta$-fine free tagged partitions of $[a,b]$, then $f$ is called McShane integrable over $[a,b]$. The McShane integral $S_f$ will again be denoted by $\int_a^b f(t)\,\mathrm{d}t$.

The definition of Riemann integrability is obtained from the definition of Henstock-Kurzweil integrability if the gauge $\delta$ is assumed to be constant on $[a,b]$. In this case, the integral $\int_a^b f(t)\,\mathrm{d}t$ is called the Riemann (or Graves) integral.

A function $f : [a,b] \to X$ is called strongly Henstock-Kurzweil integrable if there is a function $F : [a,b] \to X$ with the following property: To each $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{m} \| f(\xi_i)(t_i - t_{i-1}) - (F(t_i) - F(t_{i-1})) \| < \varepsilon \tag{2.2}$$

for every $\delta$-fine tagged partition of $[a,b]$. In this case, we define the strong Henstock-Kurzweil integral of $f$ over $[a,b]$ as $\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a)$. This integral is also known as the Henstock-Lebesgue integral or variational Henstock integral.

If (2.2) holds for all $\delta$-fine free tagged partitions of $[a,b]$, then $f$ is called strongly McShane integrable over $[a,b]$. The strong McShane integral is again defined as $\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a)$. Some authors refer to this integral as the variational McShane integral.

For the basic properties of these integrals, see [16]. We also need the concept of Bochner integrability (see e.g. [16, Chapter 1]), in particular the following facts:

- A strongly measurable function $f : [a,b] \to X$ is Bochner integrable if and only if the function $\|f\|$ is Lebesgue integrable.
- A function $f : [a,b] \to X$ is Bochner integrable if and only if there exists an absolutely continuous function $F : [a,b] \to X$ such that $F' = f$ almost everywhere in $[a,b]$; in that case, we have $\int_a^b f = F(b) - F(a)$ (see [9, Theorem 1.4.6]).
- A function $f : [a,b] \to X$ is strongly McShane integrable if and only if it is Bochner integrable; in this case, the values of the integrals coincide (see e.g. [1, 3]).

Note that in infinite-dimensional spaces, Riemann integrability does not necessarily imply Bochner integrability (see Example 4.4).

In the rest of the paper, we assume that $X$ is a complete normed algebra with a unit element $I$ whose norm is 1, i.e., it is a unital Banach algebra.

We occasionally use the following identity, which can be found in [15, Lemma 11].

**Lemma 2.1.** *For every $n \in \mathbb{N}$ and all $x_1, \dots, x_n, y_1, \dots, y_n \in X$, we have*

$$x_n \cdots x_1 - y_n \cdots y_1 = \sum_{j=1}^{n} x_n \cdots x_{j+1}(x_j - y_j) y_{j-1} \cdots y_1. \tag{2.3}$$

Recall that in a unital Banach algebra $X$, we may introduce the exponential and logarithm function as follows:

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = I + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots, \quad x \in X,$$

$$\log x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-I)^n}{n}, \quad \|x - I\| < 1.$$

These functions have similar properties as in the familiar case when $X = \mathbb{R}^{n\times n}$, in particular:

1. The exponential and logarithm are continuous functions.
2. For every $x \in X$, $\exp x$ is an invertible element and its inverse is $\exp(-x)$.
3. If $x$, $y \in X$ are such that $xy = yx$, then $\exp(x+y) = \exp x \exp y$ and $\log xy = \log x + \log y$ if all three logarithms are defined.
4. $\exp(\log x) = x$ if $\|x - I\| < 1$, and $\log(\exp x) = x$ if $\|x\| < \log 2$.
5. We have the estimates

$$\|\exp x\| \le \exp \|x\|, \tag{2.4}$$

$$\|\exp x - \exp y\| \le \|x - y\| \exp(\max(\|x\|, \|y\|)), \tag{2.5}$$

$$\|\log x - \log y\| \le \frac{\|x-y\|}{1 - \max(\|x - I\|, \|y - I\|)}. \tag{2.6}$$

The first one follows immediately from the definition of the exponential function, and the second one can be found in [15, Lemma 13]. Let us prove the third one; according to (2.3), we have

$$\|(x-I)^n - (y-I)^n\| = \left\| \sum_{k=1}^{n} (x-I)^{n-k}(x-y)(y-I)^{k-1} \right\|$$

$$\le \|x-y\| \sum_{k=1}^{n} \|x-I\|^{n-k} \|y-I\|^{k-1} \le \|x-y\| \cdot n \cdot \max(\|x-I\|, \|y-I\|)^{n-1},$$

and therefore

$$\|\log x - \log y\| = \left\| \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-I)^n - (y-I)^n}{n} \right\| \le \sum_{n=1}^{\infty} \frac{\|(x-I)^n - (y-I)^n\|}{n}$$

$$\le \|x-y\| \sum_{n=1}^{\infty} \max(\|x-I\|, \|y-I\|)^{n-1} = \frac{\|x-y\|}{1 - \max(\|x-I\|, \|y-I\|)}.$$

# 3 Product integrals and their strong versions

We start with the definitions of the Riemann, McShane and Kurzweil product integrals $\prod_a^b (I + A(t)\,\mathrm{d}t)$. Let us make the following agreement: If $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in X$, then the symbol $\prod_{i=m}^{1} x_i$ stands for the product $x_m x_{m-1} \cdots x_1$.

**Definition 3.1.** A function $A : [a, b] \to X$ is called Kurzweil product integrable, if there exists an invertible element $P_A \in X$ with the following property: For each $\varepsilon > 0$, there exists a gauge $\delta : [a, b] \to \mathbb{R}^+$ such that

$$\left\| \prod_{i=m}^{1} (I + A(\xi_i)(t_i - t_{i-1})) - P_A \right\| < \varepsilon \tag{3.1}$$

for all $\delta$-fine tagged partitions of $[a, b]$. In this case, $P_A$ is called the Kurzweil product integral of $A$ and will be denoted by $\prod_a^b (I + A(t)\,\mathrm{d}t)$. This integral is also known under the name Perron product integral.
If (3.1) holds for all $\delta$-fine free tagged partitions of $[a, b]$, then $A$ is called McShane product integrable over $[a, b]$. The McShane product integral $P_A$ will again be denoted by $\prod_a^b (I + A(t)\,\mathrm{d}t)$.
The definition of Riemann product integrability is obtained from the definition of Kurzweil product integrability if the gauge $\delta$ is assumed to be constant on $[a, b]$. In this case, the integral $\prod_a^b (I + A(t)\,\mathrm{d}t)$ is called the Riemann product integral.

**Remark 3.2.** Let us mention two basic properties which are common to all three types of product integrals:

- If $\prod_a^b (I + A(t)\,\mathrm{d}t)$ exists, and if $c \in (a, b)$, then $\prod_a^c (I + A(t)\,\mathrm{d}t)$ and $\prod_c^b (I + A(t)\,\mathrm{d}t)$ exist as well, and

  $$\prod_a^b (I + A(t)\,\mathrm{d}t) = \prod_c^b (I + A(t)\,\mathrm{d}t) \prod_a^c (I + A(t)\,\mathrm{d}t).$$

  Conversely, if the product integrals on the right-hand side exist for a certain $c \in (a, b)$, then the product integral on the left-hand side exists as well and the equality holds.

- If $\prod_a^b (I + A(t)\,\mathrm{d}t)$ exists, then the functions $t \mapsto \prod_a^t (I + A(s)\,\mathrm{d}s)$ and $t \mapsto \prod_t^b (I + A(s)\,\mathrm{d}s)$ are continuous on $[a, b]$.

More information on Riemann product integrals, including the proofs of the two properties from Remark 3.2, can be found in [12, 17]. Kurzweil product integrals were introduced in [11], where they were called "Perron product integrals" and denoted by $(\mathrm{PP})\int_a^b (I + A(t)\,\mathrm{d}t)$; additional results can be found in [14, 20]. McShane product integrals were introduced in [15], where they were referred to as "Bochner product integrals"; further results were obtained in [20]. In these sources, the Kurzweil and McShane product integrals are studied in the special cases $X = \mathbb{R}^{n\times n}$ or $X = \mathcal{L}(Y)$, where $Y$ is a Banach space; however, the proofs of the two properties from Remark 3.2 remain valid in all unital Banach algebras.

**Remark 3.3.** Next, we summarize some more specific properties of the Riemann, McShane and Kurzweil product integrals:

- Riemann or McShane product integrability implies Kurzweil product integrability; this follows immediately from the definitions.

- A function is Riemann product integrable if and only if it is Riemann integrable; see [12, Section 5] or [17, Section 5.5].

- In Definition 3.1, we were assuming that $P_A$ is an invertible element of $X$. For the Riemann product integral, it turns out that this assumption is not necessary, i.e., if the products in (3.1) have a limit $P_A$, then it is always invertible; see [12, Theorem 14.1] or [17, Theorem 5.4.6]. On the other hand, this is no longer true for Kurzweil or McShane product integrals (cf. [20, Example 9]).

- For the Kurzweil product integral, we have the following Hake-type theorem: Assume that $\prod_a^t(I + A(s)\,\mathrm{d}s)$ exists for all $t \in [a,b)$. If $\lim_{t\to b-}\prod_a^t(I+A(s)\,\mathrm{d}s)$ exists and is invertible, then $\prod_a^b(I+A(s)\,\mathrm{d}s)$ exists as well and is equal to the limit. See [14, Theorem 1.13] (the proof still works in unital Banach algebras).

In [11, Theorem 2.4], J. Jarník and J. Kurzweil pointed out that in the finite-dimensional case $X = \mathbb{R}^{n\times n}$, the indefinite Kurzweil product integral $W(t) = \prod_a^t(I + A(s)\,\mathrm{d}s)$ has the following property:
*For every $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that*

$$\sum_{i=1}^{m} \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| < \varepsilon$$

*for every $\delta$-fine tagged partition of $[a,b]$.*
This property plays a key role in their proof that $W'(t) = A(t)W(t)$ almost everywhere in $[a,b]$. However, the original proof of the above-mentioned property is no longer applicable in infinite dimension; it depends on [11, Lemma 2.2], which holds if and only if the dimension is finite (see [19]). Moreover, one can easily construct examples of functions whose indefinite Kurzweil product integral is not differentiable almost everywhere; see Example 4.4.
A similar situation is known from the Henstock-Kurzweil integration theory; in infinite dimension, the indefinite Henstock-Kurzweil integral need not be differentiable almost everywhere, but one can overcome this problem by working with the strong Henstock-Kurzweil integral instead (see e.g. [16]). These facts provide a motivation for introducing the strong Kurzweil and McShane product integrals as follows.

**Definition 3.4.** A function $A : [a,b] \to X$ is called strongly Kurzweil product integrable if there is a function $W : [a,b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a,b]$, both $W$ and $W^{-1}$ are bounded, and for every $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{m} \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| < \varepsilon \tag{3.2}$$

for every $\delta$-fine tagged partition of $[a,b]$. In this case, we define the strong Kurzweil product integral as $\prod_a^b(I + A(t)\,\mathrm{d}t) = W(b)W(a)^{-1}$.
If (3.2) holds for all $\delta$-fine free tagged partitions of $[a,b]$, then $A$ is called strongly McShane product integrable over $[a,b]$. The strong McShane product integral is again defined as $\prod_a^b(I + A(t)\,\mathrm{d}t) = W(b)W(a)^{-1}$.

As we will see later in the paper, the properties of strong product integrals are quite similar to the properties of finite-dimensional product integrals.

Note that if $A$ is strongly product integrable, then the function $W$ from Definition 3.4 is not unique (any function obtained from $W$ by multiplying it from the right by an arbitrary invertible element of $X$ has the same properties as $W$). However, the next theorem shows that strong product integrability implies ordinary product integrability, and if $W$ is an arbitrary function satisfying the conditions from Definition 3.4, then $W(b)W(a)^{-1}$ is the ordinary product integral of $A$; this justifies the correctness of Definition 3.4, and also explains why we use the same symbol for product integrals and strong product integrals. (On the other hand, product integrability does not necessarily imply strong product integrability; see Example 4.4.) The proof is inspired by the proof of [11, Theorem 2.7].

**Theorem 3.5.** *If $A : [a,b] \to X$ is strongly Kurzweil/McShane product integrable, it is also Kurzweil/McShane product integrable and the values of the integrals coincide.*

*Proof.* Let us prove the statement concerning Kurzweil product integrals; the proof of the McShane counterpart is a straightforward modification. Consider the function $W$ from Definition 3.4. There exists a constant $M > 0$ such that $\|W(t)\| \le M$ and $\|W(t)^{-1}\| \le M$ for all $t \in [a,b]$. Take an arbitrary $\varepsilon \in (0, \frac{1}{M^2})$. There exists a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\sum_{i=1}^m \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| < \varepsilon$$

for every $\delta$-fine tagged partition of $[a,b]$. Consequently,

$$\sum_{i=1}^m \|W(t_i)^{-1}(I + A(\xi_i)(t_i - t_{i-1}))W(t_{i-1}) - I\| < M^2\varepsilon < 1.$$

Now, we need the following estimate from [11, Lemma 2.1]: If $y_1, \dots, y_m \in X$ are such that $\sum_{i=1}^m \|y_i\| \le 1$, then

$$\left\|(I + y_m)\cdots(I + y_1) - I - \sum_{i=1}^m y_i\right\| \le \left(\sum_{i=1}^m \|y_i\|\right)^2.$$

By letting $y_i = W(t_i)^{-1}(I + A(\xi_i)(t_i - t_{i-1}))W(t_{i-1}) - I$, $i \in \{1, \dots, m\}$, we get

$$\left\|W(t_m)^{-1}\left(\prod_{i=m}^{1}(I + A(\xi_i)(t_i - t_{i-1}))\right)W(t_0) - I\right\| = \left\|\prod_{i=m}^{1} W(t_i)^{-1}(I + A(\xi_i)(t_i - t_{i-1}))W(t_{i-1}) - I\right\|$$

$$= \|(I + y_m)\cdots(I + y_1) - I\| \le \sum_{i=1}^m \|y_i\| + \left(\sum_{i=1}^m \|y_i\|\right)^2 < M^2\varepsilon + M^4\varepsilon^2.$$

It follows that

$$\left\|\prod_{i=m}^{1}(I + A(\xi_i)(t_i - t_{i-1})) - W(b)W(a)^{-1}\right\| = \left\|\prod_{i=m}^{1}(I + A(\xi_i)(t_i - t_{i-1})) - W(t_m)W(t_0)^{-1}\right\| < M^4\varepsilon + M^6\varepsilon^2$$

for every $\delta$-fine tagged partition of $[a,b]$, which proves that the Kurzweil product integral exists and equals $W(b)W(a)^{-1}$. □

**Remark 3.6.** The strong Kurzweil and McShane product integrals have the following properties:

- If the strong integral $\prod_a^b(I + A(t)\,\mathrm{d}t)$ exists, and if $c \in (a,b)$, then the strong integrals $\prod_a^c(I + A(t)\,\mathrm{d}t)$ and $\prod_c^b(I + A(t)\,\mathrm{d}t)$ exist as well, and

$$\prod_a^b(I + A(t)\,\mathrm{d}t) = \prod_c^b(I + A(t)\,\mathrm{d}t)\prod_a^c(I + A(t)\,\mathrm{d}t). \tag{3.3}$$

  Indeed, strong product integrability on subintervals is a direct consequence of Definition 3.4, while the relation (3.3) follows from Theorem 3.5 and Remark 3.2.

- If $A$ is strongly product integrable, then (3.2) holds with $W(t) = \prod_a^t(I + A(s)\,\mathrm{d}s)$, i.e., it reduces to

$$\sum_{i=1}^m \left\|I + A(\xi_i)(t_i - t_{i-1}) - \prod_{t_{i-1}}^{t_i}(I + A(s)\,\mathrm{d}s)\right\| < \varepsilon.$$

  The reason is that for every $i \in \{1, \dots, m\}$, $A$ is strongly product integrable on $[t_{i-1}, t_i]$, and it follows from Theorem 3.5 that $\prod_{t_{i-1}}^{t_i}(I + A(s)\,\mathrm{d}s) = W(t_i)W(t_{i-1})^{-1}$.

- If the strong integral $\prod_a^b (I + A(t)\,\mathrm{d}t)$ exists, then the functions $t \mapsto \prod_a^t (I + A(s)\,\mathrm{d}s)$ and $t \mapsto \prod_t^b (I + A(s)\,\mathrm{d}s)$ are continuous on $[a, b]$. This is a direct consequence of Theorem 3.5 and Remark 3.2.

  According to the previous two facts, the function $W$ from Definition 3.4 is necessarily continuous.

- If $X = \mathbb{R}^{n\times n}$ and $A$ is Kurzweil/McShane product integrable, then it is also strongly Kurzweil/McShane product integrable (see [20, Theorem 14]).

Next, we recall the definitions of the exponential product integrals and introduce their strong counterparts. The following two definitions are motivated by the fact that

$$\exp(A(\xi_i)(t_i - t_{i-1})) = I + A(\xi_i)(t_i - t_{i-1}) + O(\|A(\xi_i)\|^2 (t_i - t_{i-1})^2),$$

and it seems plausible that the higher-order terms do not contribute to the value of the product integral.

**Definition 3.7.** A function $A : [a, b] \to X$ is called Kurzweil exponentially product integrable, if there exists an invertible element $P_A \in X$ with the following property: For each $\varepsilon > 0$, there exists a gauge $\delta : [a, b] \to \mathbb{R}^+$ such that

$$\left\| \prod_{i=m}^{1} \exp(A(\xi_i)(t_i - t_{i-1})) - P_A \right\| < \varepsilon \tag{3.4}$$

for all $\delta$-fine tagged partitions of $[a, b]$. In this case, $P_A$ is called the Kurzweil exponential product integral of $A$ and will be denoted by $\prod_a^b \exp(A(t)\,\mathrm{d}t)$.
If (3.4) holds for all $\delta$-fine free tagged partitions of $[a, b]$, then $A$ is called McShane exponentially product integrable over $[a, b]$. The McShane exponential product integral $P_A$ will again be denoted by $\prod_a^b \exp(A(t)\,\mathrm{d}t)$.
The definition of Riemann product integrability is obtained from the definition of Kurzweil exponential product integrability if the gauge $\delta$ is assumed to be constant on $[a, b]$. In this case, the integral $\prod_a^b \exp(A(t)\,\mathrm{d}t)$ is called the Riemann exponential product integral.

**Definition 3.8.** A function $A : [a, b] \to X$ is called strongly Kurzweil exponentially product integrable if there is a function $W : [a, b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a, b]$, both $W$ and $W^{-1}$ are bounded, and for every $\varepsilon > 0$, there is a gauge $\delta : [a, b] \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{m} \| \exp(A(\xi_i)(t_i - t_{i-1})) - W(t_i)W(t_{i-1})^{-1} \| < \varepsilon \tag{3.5}$$

for every $\delta$-fine tagged partition of $[a, b]$. In this case, we define the strong Kurzweil exponential product integral as $\prod_a^b \exp(A(t)\,\mathrm{d}t) = W(b)W(a)^{-1}$.
If (3.5) holds for all $\delta$-fine free tagged partitions of $[a, b]$, then $A$ is called strongly McShane exponentially product integrable over $[a, b]$. The strong McShane exponential product integral is again defined as $\prod_a^b \exp(A(t)\,\mathrm{d}t) = W(b)W(a)^{-1}$.

**Remark 3.9.** It is not our intention to develop a systematic theory of exponential product integrals; let us collect only some basic facts that will be needed later.

- If $A$ is Riemann integrable, then the Riemann product integral $\prod_a^b \exp(A(t)\,\mathrm{d}t)$ exists and equals $\prod_a^b (I + A(t)\,\mathrm{d}t)$; see [12, Lemma 24.1] or [17, Theorem 3.2.2].

- If $A$ is Bochner integrable, then the McShane product integrals $\prod_a^b \exp(A(t)\,\mathrm{d}t)$ and $\prod_a^b (I + A(t)\,\mathrm{d}t)$ exist and are equal to each other; see [15, Theorems 14 and 16]. Moreover, the corresponding indefinite product integral $W(t) = \prod_a^t (I + A(s)\,\mathrm{d}s) = \prod_a^t \exp(A(s)\,\mathrm{d}s)$ satisfies

$$W(t) = I + \int_a^t A(s)W(s)\,\mathrm{d}s, \quad t \in [a,b],$$

where the integral on the right-hand side is the Bochner integral; for a proof of this fact, see [2, Theorem 1.2 and Section 1.8] (the original proof for $X = \mathbb{R}^{n\times n}$ remains valid in unital Banach algebras).

- If $A$ is strongly Kurzweil/McShane exponentially product integrable, then it is also Kurzweil/McShane exponentially product integrable and the values of the integrals coincide. The proof is the same as the proof of Theorem 3.5; it is enough to replace all terms of the form $I + A(\xi_i)(t_i - t_{i-1})$ by $\exp(A(\xi_i)(t_i - t_{i-1}))$.

- If $X = \mathbb{R}^{n\times n}$ and $A$ is Kurzweil/McShane exponentially product integrable, then it is also strongly Kurzweil/McShane exponentially product integrable (see [20, Theorem 14]).

**Lemma 3.10.** *For every function $A : [a,b] \to X$ and every $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that*

$$\sum_{i=1}^m \|I + A(\xi_i)(t_i - t_{i-1}) - \exp(A(\xi_i)(t_i - t_{i-1}))\| < \varepsilon$$

*for every $\delta$-fine free tagged partition of $[a,b]$.*

*Proof.* Take a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\|A(\xi)\|\delta(\xi) < \frac{1}{2}, \qquad \|A(\xi)\|^2\delta(\xi) < \frac{\varepsilon}{2(b-a)e}$$

for all $\xi \in [a,b]$. Then, for every $\delta$-fine free tagged partition of $[a,b]$, we get

$$\begin{aligned}&\|I + A(\xi_i)(t_i - t_{i-1}) - \exp(A(\xi_i)(t_i - t_{i-1}))\| \le \|A(\xi_i)\|^2(t_i - t_{i-1})^2 \exp(\|A(\xi_i)\|(t_i - t_{i-1}))\\ &\le 2\delta(\xi_i)\|A(\xi_i)\|^2(t_i - t_{i-1})\exp(2\delta(\xi_i)\|A(\xi_i)\|) < \frac{\varepsilon(t_i - t_{i-1})}{b-a},\end{aligned}$$

and consequently

$$\sum_{i=1}^m \|I + A(\xi_i)(t_i - t_{i-1}) - \exp(A(\xi_i)(t_i - t_{i-1}))\| < \sum_{i=1}^m \frac{\varepsilon(t_i - t_{i-1})}{b-a} = \varepsilon. \qquad \square$$

**Corollary 3.11.** *A function $A : [a,b] \to X$ is strongly Kurzweil/McShane product integrable if and only if $A$ is strongly Kurzweil/McShane exponentially product integrable. In this case, the values of the product integral and exponential product integral coincide.*

In general, product integrals are much harder to calculate than ordinary integrals. However, if the values of $A : [a,b] \to X$ commute with each other, it turns out that

$$\prod_a^b (I + A(t)\,\mathrm{d}t) = \exp\left(\int_a^b A(t)\,\mathrm{d}t\right)$$

whenever at least one of the integrals exists. For Riemann-type integrals, this theorem was proved by Masani; see [12, Theorem 24.3] or [17, Theorem 5.5.11]. The corresponding statement for Kurzweil-type integrals and $X = \mathbb{R}^{n\times}$ can be found in [11, Theorem 3.7]. Here we focus on strong Kurzweil and McShane product integrals.

**Theorem 3.12.** *Consider a function $A : [a,b] \to X$ such that*

$$A(t_1)A(t_2) = A(t_2)A(t_1) \quad \textit{for all } t_1, t_2 \in [a,b]. \tag{3.6}$$

*If $A$ is strongly Henstock-Kurzweil/McShane integrable, then it is strongly Kurzweil/McShane product integrable. In this case, we have*

$$\prod_a^b (I + A(t)\,\mathrm{d}t) = \exp\left(\int_a^b A(t)\,\mathrm{d}t\right).$$

*Proof.* We prove the statement concerning the Kurzweil integrals; the McShane version can be obtained in an analogous way. Clearly, it is enough to prove that $A$ is strongly Kurzweil exponentially product integrable. Let $W(t) = \exp\left(\int_a^t A(s)\,\mathrm{d}s\right)$, $t \in [a,b]$. If $[x,y] \subset [a,b]$, then

$$\begin{aligned}W(y)W(x)^{-1} &= \exp\left(\int_a^y A(s)\,\mathrm{d}s\right)\exp\left(\int_a^x A(s)\,\mathrm{d}s\right)^{-1} = \exp\left(\int_a^y A(s)\,\mathrm{d}s\right)\exp\left(-\int_a^x A(s)\,\mathrm{d}s\right)\\ &= \exp\left(\int_a^y A(s)\,\mathrm{d}s - \int_a^x A(s)\,\mathrm{d}s\right) = \exp\left(\int_x^y A(s)\,\mathrm{d}s\right),\end{aligned}$$

because assumption (3.6) implies that $\int_a^y A(s)\,\mathrm{d}s$ and $\int_a^x A(s)\,\mathrm{d}s$ commute. (Think of the integrals as limits of integral sums, which obviously commute.) Let $M = \sup_{t\in[a,b]} \left\|\int_a^t A(s)\,\mathrm{d}s\right\|$. Then we have $\|\int_x^y A(s)\,\mathrm{d}s\| = \|\int_a^y A(s)\,\mathrm{d}s - \int_a^x A(s)\,\mathrm{d}s\| \le 2M$ whenever $[x,y] \subset [a,b]$.
Take an arbitrary $\varepsilon > 0$. There exists a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\sum_{i=1}^m \left\|A(\xi_i)(t_i - t_{i-1}) - \int_{t_{i-1}}^{t_i} A(s)\,\mathrm{d}s\right\| < \varepsilon$$

for every $\delta$-fine tagged partition of $[a,b]$. Without loss of generality, we can assume the gauge is chosen so that $2\|A(\xi)\|\delta(\xi) \le 1$ for all $\xi \in [a,b]$.
Using the estimate (2.5), we get

$$\begin{aligned}\sum_{i=1}^m \left\|\exp(A(\xi_i)(t_i - t_{i-1})) - W(t_i)W(t_{i-1})^{-1}\right\| &= \sum_{i=1}^m \left\|\exp(A(\xi_i)(t_i - t_{i-1})) - \exp\left(\int_{t_{i-1}}^{t_i} A(s)\,\mathrm{d}s\right)\right\|\\ \le \sum_{i=1}^m \left\|A(\xi_i)(t_i - t_{i-1}) - \int_{t_{i-1}}^{t_i} A(s)\,\mathrm{d}s\right\| &\exp\left(\max\left(\|A(\xi_i)\|(t_i - t_{i-1}), \left\|\int_{t_{i-1}}^{t_i} A(s)\,\mathrm{d}s\right\|\right)\right)\\ \le \sum_{i=1}^m \left\|A(\xi_i)(t_i - t_{i-1}) - \int_{t_{i-1}}^{t_i} A(s)\,\mathrm{d}s\right\| &\exp\left(\max\left(1, 2M\right)\right) < \exp\left(\max\left(1, 2M\right)\right)\varepsilon\end{aligned}$$

for every $\delta$-fine tagged partition of $[a,b]$. This proves that $A$ is strongly Kurzweil product integrable and $\prod_a^b \exp(A(t)\,\mathrm{d}t) = W(b)W(a)^{-1} = \exp\left(\int_a^b A(t)\,\mathrm{d}t\right)$. □

**Theorem 3.13.** *Consider a function $A : [a,b] \to X$ such that* (3.6) *holds. If $A$ is strongly Kurzweil/McShane product integrable, then it is strongly Kurzweil-Henstock/McShane integrable. In this case, we have*

$$\prod_a^b (I + A(t)\,\mathrm{d}t) = \exp\left(\int_a^b A(t)\,\mathrm{d}t\right).$$

*Proof.* We prove the statement concerning the Kurzweil integrals; the McShane counterpart can be obtained in an analogous way. We use the fact that strong product integrability implies strong exponential product integrability. Let $W(t) = \prod_a^t \exp(A(s)\,\mathrm{d}s)$, $t \in [a,b]$. Denote $M = \sup_{t\in[a,b]} \|W(t)^{-1}\|$. Since $W$ is (uniformly) continuous, there exists a $\Delta > 0$ such that $[x,y] \subset [a,b]$ and $\|x-y\| < \Delta$ implies

$$\|W(x) - W(y)\| < \frac{1}{2M}.$$

Without loss of generality, we can assume that $b - a < \Delta$ (otherwise, we can split $[a,b]$ into subintervals whose length is smaller than $\Delta$). Now, if $[x,y] \subset [a,b]$, we have

$$\left\|\prod_x^y \exp(A(s)\,\mathrm{d}s) - I\right\| = \|W(y)W(x)^{-1} - I\| \le \|W(y) - W(x)\| \cdot \|W(y)^{-1}\| < \frac{1}{2},$$

$$\left\|\left(\prod_x^y \exp(A(s)\,\mathrm{d}s)\right)^{-1} - I\right\| = \|W(x)W(y)^{-1} - I\| \le \|W(x) - W(y)\| \cdot \|W(y)^{-1}\| < \frac{1}{2}.$$

Let $F(t) = \log\left(\prod_a^t \exp(A(s)\,\mathrm{d}s)\right)$. If $[x,y] \subset [a,b]$, then

$$\begin{aligned}
F(y) - F(x) &= \log\left(\prod_a^y \exp(A(s)\,\mathrm{d}s)\right) - \log\left(\prod_a^x \exp(A(s)\,\mathrm{d}s)\right) \\
&= \log\left(\prod_a^y \exp(A(s)\,\mathrm{d}s)\right) + \log\left(\prod_a^x \exp(A(s)\,\mathrm{d}s)\right)^{-1} \\
&= \log\left(\left(\prod_a^y \exp(A(s)\,\mathrm{d}s)\right)\left(\prod_a^x \exp(A(s)\,\mathrm{d}s)\right)^{-1}\right) = \log\left(\prod_x^y \exp(A(s)\,\mathrm{d}s)\right)
\end{aligned}$$

because assumption (3.6) implies that $\prod_a^y \exp(A(s)\,\mathrm{d}s)$ and $(\prod_a^x \exp(A(s)\,\mathrm{d}s))^{-1}$ commute. (Think of the product integrals as the limits of products from Definition 3.7; these products obviously commute.)
Take an arbitrary $\varepsilon > 0$. There exists a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\sum_{i=1}^m \left\|\exp(A(\xi_i)(t_i - t_{i-1})) - \prod_{t_{i-1}}^{t_i} \exp(A(s)\,\mathrm{d}s)\right\| < \varepsilon$$

for every $\delta$-fine tagged partition of $[a,b]$. Without loss of generality, assume the gauge $\delta$ is chosen so that

$$2\|A(\xi)\|\delta(\xi)\exp(2\|A(\xi)\|\delta(\xi)) \le \frac{1}{2}, \qquad \xi \in [a,b],$$

which in turn means that

$$\|\exp(A(\xi_i)(t_i - t_{i-1})) - I\| \le \|A(\xi)\|(t_i - t_{i-1})\exp(\|A(\xi)\|(t_i - t_{i-1})) < \|A(\xi)\|2\delta(\xi_i)\exp(\|A(\xi)\|2\delta(\xi_i)) \le \frac{1}{2},$$

and also

$$\|A(\xi_i)\|(t_i - t_{i-1}) < \|A(\xi_i)\|2\delta(\xi_i) \le \frac{1}{2} < \log 2.$$

Using the estimate (2.6), we get

$$\sum_{i=1}^m \|A(\xi_i)(t_i - t_{i-1}) - (F(t_i) - F(t_{i-1}))\| = \sum_{i=1}^m \left\|\log\left(\exp(A(\xi_i)(t_i - t_{i-1}))\right) - \log\left(\prod_{t_{i-1}}^{t_i} \exp(A(s)\,\mathrm{d}s)\right)\right\|$$

$$\leq \sum_{i=1}^{m} \frac{\left\|\exp(A(\xi_i)(t_i - t_{i-1})) - \prod_{t_{i-1}}^{t_i} \exp(A(s)\,\mathrm{d}s)\right\|}{1 - \max\left(\|\exp(A(\xi_i)(t_i - t_{i-1})) - I\|, \left\|\prod_{t_{i-1}}^{t_i} \exp(A(s)\,\mathrm{d}s) - I\right\|\right)}$$

$$\leq 2\sum_{i=1}^{m} \left\|\exp(A(\xi_i)(t_i - t_{i-1})) - \prod_{t_{i-1}}^{t_i} \exp(A(s)\,\mathrm{d}s)\right\| < 2\varepsilon,$$

i.e., $A$ is strongly Henstock-Kurzweil integrable and $\int_a^b A(t)\,\mathrm{d}t = F(b) - F(a) = \log\left(\prod_a^b \exp(A(s)\,\mathrm{d}s)\right)$. $\square$

Our next goal is to show that every Bochner integrable function is strongly McShane product integrable. In the following lemma, the symbol $\mu$ stands for the Lebesgue measure in $\mathbb{R}$.

**Lemma 3.14.** *Assume that $A : [a, b] \to X$ is Bochner integrable. Then for every $\varepsilon > 0$, there is a gauge $\delta : [a, b] \to \mathbb{R}^+$ such that*

$$\sum_{i=1}^{r}\sum_{j=1}^{s} \|\exp(A(t_i)\mu(J_i \cap L_j)) - \exp(A(s_j)\mu(J_i \cap L_j))\| < \varepsilon$$

*for each pair of $\delta$-fine free tagged partitions $(t_i, J_i)_{i=1}^r$ and $(s_j, L_j)_{j=1}^s$.*

*Proof.* Let $M = \exp\left(1 + \int_a^b \|A(t)\|\,\mathrm{d}t\right)$. Take an arbitrary $\varepsilon > 0$. Every Bochner integrable function is strongly McShane integrable, and therefore satisfies the so-called condition $\mathcal{S}^*\mathcal{M}$ (see [16, Lemma 3.6.12]), i.e., there is a gauge $\delta : [a, b] \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{r}\sum_{j=1}^{s} \|A(t_i) - A(s_j)\|\mu(J_i \cap L_j) < \frac{\varepsilon}{M}$$

for each pair of $\delta$-fine free tagged partitions $(t_i, J_i)_{i=1}^r$ and $(s_j, L_j)_{j=1}^s$. Since $\|A\|$ is Bochner and therefore also McShane integrable, we can assume that $\delta$ is chosen in such a way that

$$\left|\sum_{i=1}^{r} \|A(t_i)\|\mu(J_i) - \int_a^b \|A(t)\|\,\mathrm{d}t\right| < 1$$

for every $\delta$-fine free tagged partition $(t_i, J_i)_{i=1}^r$ of $[a, b]$.
Consequently, using the estimate (2.5), we get

$$\sum_{i=1}^{r}\sum_{j=1}^{s} \|\exp(A(t_i)\mu(J_i \cap L_j)) - \exp(A(s_j)\mu(J_i \cap L_j))\|$$

$$\leq \sum_{i=1}^{r}\sum_{j=1}^{s} \|A(t_i) - A(s_j)\|\mu(J_i \cap L_j)\exp(\max(\|A(t_i)\|\mu(J_i \cap L_j), \|A(s_j)\|\mu(J_i \cap L_j)))$$

$$\leq \left(\sum_{i=1}^{r}\sum_{j=1}^{s} \|(A(t_i) - A(s_j))\mu(J_i \cap L_j)\|\right) \exp\left(\max\left(\sum_{i=1}^{r} \|A(t_i)\|\mu(J_i), \sum_{j=1}^{s} \|A(s_j)\|\mu(L_j)\right)\right)$$

$$\leq \left(\sum_{i=1}^{r}\sum_{j=1}^{s} \|(A(t_i) - A(s_j))\mu(J_i \cap L_j)\|\right) \exp\left(1 + \int_a^b \|A(t)\|\,\mathrm{d}t\right) < \varepsilon. \qquad \square$$

**Theorem 3.15.** *If $A : [a, b] \to X$ is Bochner integrable, then it is strongly McShane product integrable.*

*Proof.* By Corollary 3.11, it is enough to prove that $A$ is strongly McShane exponentially product integrable. From Remark 3.9, we already know that $A$ is McShane exponentially product integrable. Let $W(t) = \prod_a^t \exp(A(s)\,\mathrm{d}s)$, $t \in [a,b]$. Denote $M = \exp\left(1 + \int_a^b \|A(t)\|\,\mathrm{d}t\right)$. Take an arbitrary $\varepsilon > 0$ and let $\delta : [a,b] \to \mathbb{R}^+$ be the corresponding gauge from Lemma 3.14. Since $\|A\|$ is Bochner integrable and therefore also McShane integrable, we can assume that $\delta$ is chosen is such a way that if $(\eta_i, [u_{i-1}, u_i])_{i=1}^m$ is a $\delta$-fine free tagged partition of $[a,b]$, then

$$\left|\sum_{i=1}^m \|A(\eta_i)\|(u_i - u_{i-1}) - \int_a^b \|A(t)\|\,\mathrm{d}t\right| < 1. \tag{3.7}$$

Consider an arbitrary $\delta$-fine free tagged partition $(\xi_i, [t_{i-1}, t_i])_{i=1}^k$ of $[a,b]$.
For every $i \in \{1,\dots,k\}$, $A$ is Bochner integrable and therefore also exponentially McShane product integrable on $[t_{i-1}, t_i]$. Hence, there exists a $\delta$-fine free tagged partition $(\xi_j^i, [t_{j-1}^i, t_j^i])_{j=1}^{l_i}$ of $[t_{i-1}, t_i]$ such that

$$\left\|\prod_{j=l^i}^{1} \exp(A(\xi_j^i)\Delta t_j^i) - \prod_{t_{i-1}}^{t_i} \exp(A(t)\,\mathrm{d}t)\right\| < \frac{\varepsilon}{k}, \quad i \in \{1,\dots,k\}, \tag{3.8}$$

where $\Delta t_j^i = t_j^i - t_{j-1}^i$.
Note that the collections $(\xi_i, [t_{j-1}^i, t_j^i])$ and $(\xi_j^i, [t_{j-1}^i, t_j^i])$, where $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,l^i\}$, are $\delta$-fine free tagged partitions of $[a,b]$.
Using the identity (2.3), we obtain the estimate

$$\begin{aligned}
&\left\|\prod_{j=l^i}^{1} \exp(A(\xi_i)\Delta t_j^i) - \prod_{j=l^i}^{1} \exp(A(\xi_j^i)\Delta t_j^i)\right\| \\
&\le \sum_{j=1}^{l^i} \left(\prod_{p=l^i}^{j+1} \exp(\|A(\xi_i)\|\Delta t_p^i)\right) \|\exp(A(\xi_i)\Delta t_j^i) - \exp(A(\xi_j^i)\Delta t_j^i)\| \left(\prod_{p=j-1}^{1} \exp(\|A(\xi_p^i)\|\Delta t_p^i)\right) \\
&\le \exp\left(\sum_{p=1}^{l^i} \|A(\xi_p^i)\|\Delta t_p^i\right) \exp\left(\sum_{p=1}^{l^i} \|A(\xi_i)\|\Delta t_p^i\right) \sum_{j=1}^{l^i} \|\exp(A(\xi_i)\Delta t_j^i) - \exp(A(\xi_j^i)\Delta t_j^i)\| \\
&\le \exp\left(\sum_{i=1}^{k}\sum_{p=1}^{l^i} \|A(\xi_i)\|\Delta t_p^i\right) \exp\left(\sum_{i=1}^{k}\sum_{p=1}^{l^i} \|A(\xi_p^i)\|\Delta t_p^i\right) \sum_{j=1}^{l^i} \|\exp(A(\xi_i)\Delta t_j^i) - \exp(A(\xi_j^i)\Delta t_j^i)\| \\
&\le M^2 \sum_{j=1}^{l^i} \|\exp(A(\xi_i)\Delta t_j^i) - \exp(A(\xi_j^i)\Delta t_j^i)\|,
\end{aligned}$$

where the last inequality is a consequence of (3.7). According to the conclusion of Lemma 3.14, we get

$$\sum_{i=1}^{k}\left\|\prod_{j=l^i}^{1} \exp(A(\xi_i)\Delta t_j^i) - \prod_{j=l^i}^{1} \exp(A(\xi_j^i)\Delta t_j^i)\right\| \le M^2 \sum_{i=1}^{k}\sum_{j=1}^{l^i} \|\exp(A(\xi_i)\Delta t_j^i) - \exp(A(\xi_j^i)\Delta t_j^i)\| < M^2\varepsilon.$$

By combining the previous estimate with (3.8), we obtain

$$\begin{aligned}
\sum_{i=1}^{k} \left\|\exp(A(\xi_i)(t_i - t_{i-1})) - W(t_i)W(t_{i-1})^{-1}\right\| &= \sum_{i=1}^{k}\left\|\prod_{j=l^i}^{1} \exp(A(\xi_i)\Delta t_j^i) - \prod_{t_{i-1}}^{t_i} \exp(A(t)\,\mathrm{d}t)\right\| \\
\le \sum_{i=1}^{k}\left\|\prod_{j=l^i}^{1} \exp(A(\xi_i)\Delta t_j^i) - \prod_{j=l^i}^{1} \exp(A(\xi_j^i)\Delta t_j^i)\right\| &+ \sum_{i=1}^{k}\left\|\prod_{j=l^i}^{1} \exp(A(\xi_j^i)\Delta t_j^i) - \prod_{t_{i-1}}^{t_i} \exp(A(t)\,\mathrm{d}t)\right\| < \varepsilon(M^2+1),
\end{aligned}$$

which proves that $A$ is strongly McShane exponentially product integrable. $\square$

## 4 Indefinite strong product integrals

In this section, we investigate the properties of the indefinite strong Kurzweil and McShane product integrals. Also, we prove that strong McShane product integrability is equivalent to Bochner integrability. The proofs of Theorems 4.1, 4.6, and 4.7 are fairly straightforward adaptations of the proofs of Theorems 3.1, 3.3, and 3.4 from [11], but we provide them here for reader's convenience.

**Theorem 4.1.** *Assume that $A:[a,b]\to X$ is strongly Kurzweil product integrable and $W(t)=\prod_a^t(I+A(s)\,\mathrm{d}s)$, $t\in[a,b]$. Then there exists a set $Z\subset[a,b]$ of measure zero with the following property: For every $\varepsilon>0$ and $t\in[a,b]\backslash Z$, there is a $\delta(t)>0$ such that*

$$\|I+A(t)(y-x)-W(y)W(x)^{-1}\|\le\varepsilon(y-x) \tag{4.1}$$

*whenever $[x,y]\subset(t-\delta(t),t+\delta(t))\cap[a,b]$.*

*Proof.* Let $Z\subset[a,b]$ be the set of all $t\in[a,b]$ for which (4.1) does not hold. For every $t\in Z$, there exist $\eta(t)>0$ and sequences $\{x_l(t)\}_{l=1}^\infty$ and $\{y_l(t)\}_{l=1}^\infty$ such that

$$x_l(t)\le t\le y_l(t),\quad \lim_{l\to\infty}(y_l(t)-x_l(t))=0,$$

$$\|I+A(t)(y_l(t)-x_l(t))-W(y_l(t))W(x_l(t))^{-1}\|\ge\eta(t)(y_l(t)-x_l(t)). \tag{4.2}$$

Clearly, $Z=\bigcup_{n=1}^\infty Z_n$. Assume that $Z$ has outer Lebesgue measure $\mu^*(Z)>0$. Hence, there is an $r\in\mathbb{N}$ such that $\mu^*(Z_r)>0$. Consider an arbitrary gauge $\delta:[a,b]\to\mathbb{R}^+$. For every $t\in Z_r$, find $l_0(t)\in\mathbb{N}$ such that

$$t-\delta(t)<x_l(t)\le t\le y_l(t)<t+\delta(t)$$

for all $l\ge l_0(t)$. The collection of intervals $\{[x_l(t),y_l(t)];\ t\in Z_r, l\ge l_0(t)\}$ is a Vitali cover of the set $Z_r$. By the Vitali covering theorem, it contains a finite subsystem of intervals $([\xi_j,\eta_j])_{j=1}^s$ for which

$$\tau_j-\delta(\tau_j)<\xi_j\le\tau_j\le\eta_j<\tau_j+\delta(\tau_j),\quad \tau_j\in Z_r,\quad j\in\{1,2,\dots,s\},$$

$$\eta_j\le\xi_{j+1},\quad j\in\{1,2,\dots,s-1\},$$

$$\mu^*\Big(Z_r\setminus\bigcup_{j=1}^s[\xi_j,\eta_j]\Big)<\frac{1}{2}\mu^*(Z_r).$$

Consequently,

$$\sum_{j=1}^s(\eta_j-\xi_j)\ge\mu^*\Big(Z_r\cap\bigcup_{j=1}^s[\xi_j,\eta_j]\Big)>\frac{1}{2}\mu^*(Z_r).$$

This inequality together with (4.2) yields

$$\sum_{j=1}^s\|I+A(\tau_j)(\eta_j-\xi_j)-W(\eta_j)W(\xi_j)^{-1}\|\ge\frac{1}{r}\sum_{j=1}^s(\eta_j-\xi_j)>\frac{1}{2r}\mu^*(Z_r).$$

Since the expression on the right-hand side is a constant independent on the choice of the gauge $\delta$, we get a contradiction with the assumption that $A$ is strongly Kurzweil product integrable. □

**Theorem 4.2.** *Assume that $A:[a,b]\to X$ is strongly Kurzweil product integrable and $W(t)=\prod_a^t(I+A(s)\,\mathrm{d}s)$, $t\in[a,b]$. Then $W'(t)=A(t)W(t)$ almost everywhere on $[a,b]$.*

*Proof.* Consider an arbitrary $\varepsilon > 0$. By Theorem 4.1 there exists a $Z \subset [a,b]$ of measure zero such that for every $t \in [a,b] \setminus Z$ there is $\delta(t) > 0$ such that

$$\|I + A(t)(y-t) - W(y)W(t)^{-1}\| \le \varepsilon(y-t)$$

whenever $y \in (t, t+\delta(t))$. Dividing by $(y-t)$, we get

$$\left\| A(t) - \frac{W(y)-W(t)}{y-t} W(t)^{-1} \right\| \le \varepsilon,$$

and consequently

$$\left\| A(t)W(t) - \frac{W(y)-W(t)}{y-t} \right\| \le \varepsilon \|W(t)\|,$$

i.e., the right-sided derivative of $W$ at the point $t$ exists and equals $A(t)W(t)$. A similar reasoning shows that also the left-sided derivative of $W$ at $t$ exists and equals $A(t)W(t)$. ☐

**Theorem 4.3.** *Every strongly Kurzweil product integrable function is strongly measurable.*

*Proof.* If $A : [a,b] \to X$ is strongly Kurzweil product integrable, then the indefinite Kurzweil product integral $W(t) = \prod_a^t (I + A(s)\,\mathrm{d}s)$, $t \in [a,b]$, satisfies $W' = AW$ almost everywhere on $[a,b]$. $W$ and $W^{-1}$ are continuous, and therefore strongly measurable. $W'$ is also strongly measurable because it is the derivative of a strongly measurable function (a verification of this fact, which is analogous to the proof for real-valued functions, can be found e.g. in the proof of [9, Theorem 1.4.6]). Now, $A = W'W^{-1}$ is almost everywhere equal to a product of two strongly measurable functions, and therefore $A$ is also strongly measurable. ☐

**Example 4.4.** Consider the space $X$ of all bounded real functions defined on $[0,1]$ equipped with the supremum norm. For two functions $f, g \in X$, let $fg \in X$ be the function obtained by pointwise multiplication of $f$ and $g$. Clearly, $X$ is a unital Banach algebra.
Let $A : [0,1] \to X$ be given by $A(t) = \chi_{[0,t]}$. L. M. Graves observed in [7] that $A$ is nowhere continuous, but Riemann integrable. In [5, Example 12], R. Gordon noted that $A$ is not strongly measurable. From the viewpoint of product integration theory, it is useful to observe the following facts:

- $A$ is Riemann product integrable (because it is Riemann integrable) and therefore also Kurzweil product integrable.
- $A$ is neither Bochner integrable nor strongly Kurzweil/McShane product integrable (because it is not strongly measurable).
- $A$ is McShane product integrable.

Let us verify the last claim. First, we show that $A$ is McShane integrable (cf. [3, Example 3.1], which is concerned with a similar function $t \mapsto \chi_{[t,1]}$) and $\int_0^1 A(t)\,\mathrm{d}t = S$, where $S(\tau) = 1-\tau$ for $\tau \in [0,1]$. For a given $\varepsilon > 0$, let $\delta(t) = \varepsilon$ for all $t \in [0,1]$. Now, consider an arbitrary $\delta$-fine free tagged partition $(\xi_i, [t_{i-1}, t_i])_{i=1}^m$ of $[0,1]$. We have

$$\left\| \sum_{i=1}^m A(\xi_i)(t_i - t_{i-1}) - S \right\| = \sup_{\tau \in [0,1]} \left| \sum_{i=1}^m A(\xi_i)(\tau)(t_i - t_{i-1}) - S(\tau) \right| = \sup_{\tau \in [0,1]} \left| \sum_{i;\, \xi_i \ge \tau} (t_i - t_{i-1}) - (1-\tau) \right|$$

Note that $\xi_i \ge \tau$ implies $t_{i-1} > \xi_i - \varepsilon \ge \tau - \varepsilon$, and therefore

$$\sum_{i;\, \xi_i \ge \tau} (t_i - t_{i-1}) < 1 - (\tau - \varepsilon) = 1 - \tau + \varepsilon.$$

A lower bound for the same sum is obtained by finding an upper bound for $\sum_{i;\,\xi_i<\tau}(t_i-t_{i-1})$. Now $\xi_i<\tau$ implies $t_i<\xi_i+\varepsilon<\tau+\varepsilon$. It follows that $\sum_{i;\,\xi_i<\tau}(t_i-t_{i-1})<\tau+\varepsilon$, and consequently

$$\sum_{i;\,\xi_i\geq\tau}(t_i-t_{i-1})=1-\sum_{i;\,\xi_i<\tau}(t_i-t_{i-1})>1-(\tau+\varepsilon)=1-\tau-\varepsilon.$$

This proves

$$\left\|\sum_{i=1}^{m}A(\xi_i)(t_i-t_{i-1})-S\right\|=\sup_{\tau\in[0,1]}\left|\sum_{i;\,\xi_i\geq\tau}(t_i-t_{i-1})-(1-\tau)\right|\leq\varepsilon, \tag{4.3}$$

or $\int_0^1 A(t)\,\mathrm{d}t=S$.

Next, we claim that $A$ is McShane product integrable with $\prod_0^1(I+A(t)\,\mathrm{d}t)=\exp S$. Take an arbitrary $\varepsilon>0$ and consider the corresponding gauge $\delta:[0,1]\to\mathbb{R}^+$ from Lemma 3.10. Without loss of generality, assume that $\delta(\xi)\leq\varepsilon$ for all $\xi\in[0,1]$. For every $\delta$-fine free tagged partition $(\xi_i,[t_{i-1},t_i])_{i=1}^m$ of $[0,1]$, we get

$$\left\|\prod_{i=m}^{1}(I+A(\xi_i)(t_i-t_{i-1}))-\exp S\right\|\leq\left\|\prod_{i=m}^{1}(I+A(\xi_i)(t_i-t_{i-1}))-\prod_{i=m}^{1}\exp(A(\xi_i)(t_i-t_{i-1}))\right\|$$
$$+\left\|\prod_{i=m}^{1}\exp(A(\xi_i)(t_i-t_{i-1}))-\exp S\right\|.$$

An estimate for the first term can be obtained using identity (2.3):

$$\left\|\prod_{i=m}^{1}(I+A(\xi_i)(t_i-t_{i-1}))-\prod_{i=m}^{1}\exp(A(\xi_i)(t_i-t_{i-1}))\right\|$$
$$\leq\sum_{i=1}^{m}\left(\prod_{j=m}^{i+1}(1+\|A(\xi_j)\|(t_j-t_{j-1}))\right)\|I+A(\xi_i)(t_i-t_{i-1})-\exp(A(\xi_i)(t_i-t_{i-1}))\|\left(\prod_{j=i-1}^{1}\exp(\|A(\xi_j)\|(t_j-t_{j-1}))\right)$$
$$\leq\exp\left(\sum_{j=1}^{m}\|A(\xi_j)\|(t_j-t_{j-1})\right)\sum_{i=1}^{m}\|I+A(\xi_i)(t_i-t_{i-1})-\exp(A(\xi_i)(t_i-t_{i-1}))\|\leq(\exp 1)\,\varepsilon$$

To estimate the second term, we use inequality (2.5) together with (4.3), as well as the fact that $X$ is a commutative algebra:

$$\left\|\prod_{i=m}^{1}\exp(A(\xi_i)(t_i-t_{i-1}))-\exp S\right\|=\left\|\exp\left(\sum_{i=1}^{m}A(\xi_i)(t_i-t_{i-1})\right)-\exp S\right\|$$
$$\leq\left\|\sum_{i=1}^{m}A(\xi_i)(t_i-t_{i-1})-S\right\|\exp\left(\max\left(\left\|\sum_{i=1}^{m}A(\xi_i)(t_i-t_{i-1})\right\|,\|S\|\right)\right)\leq\varepsilon\exp 1.$$

This confirms that the McShane product integral $\prod_0^1(I+A(t)\,\mathrm{d}t)$ exists and equals $\exp S$.

**Definition 4.5.** A function $W:[a,b]\to X$ is said to satisfy the strong Luzin condition on $[a,b]$ if for every $\varepsilon>0$ and $Z\subset[a,b]$ of measure zero, there exists a function $\delta:Z\to\mathbb{R}^+$ such that

$$\sum_{j=1}^{m}\|W(v_j)-W(u_j)\|<\varepsilon$$

for every collection of point-interval pairs $(\tau_j,[u_j,v_j])_{j=1}^m$ with $[u_j,v_j]\subset[a,b]$, $\tau_j\in Z$, and $\tau_j-\delta(\tau_j)<u_j\leq\tau_j\leq v_j<\tau_j+\delta(\tau_j)$ for all $j\in\{1,2,\ldots,m\}$.

We remark that every function which satisfies the strong Luzin condition is necessarily continuous.

**Theorem 4.6.** *If $A : [a, b] \to X$ is strongly Kurzweil product integrable, then the indefinite product integral $W(t) = \prod_a^t (I + A(s)\,\mathrm{d}s)$, $t \in [a, b]$, satisfies the strong Luzin condition on $[a, b]$.*

*Proof.* Let $M = \sup_{t\in[a,b]} \|W(t)\|$. Take arbitrary $\varepsilon > 0$ and $Z \subset [a, b]$ of measure zero. For every $i \in \mathbb{N}$, let $Z_i = \{t \in Z;\, i - 1 \le \|A(t)\| < i\}$. Since $Z_i$ has measure zero, it can be enclosed in an open set of an arbitrarily small measure. Hence, there exists a function $\delta_i : Z_i \to \mathbb{R}^+$ such that

$$\mu\left(\bigcup_{t\in Z_i} (t - \delta_i(t), t + \delta_i(t))\right) \le \frac{\varepsilon}{i2^i}.$$

Using the strong product integrability of $A$, we find a gauge $\delta : [a, b] \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{m} \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| < \varepsilon \tag{4.4}$$

for every $\delta$-fine tagged partition of $[a, b]$. Without loss of generality, we can assume that $\delta(t) \le \delta_i(t)$ whenever $t \in Z_i$. Take an arbitrary collection of point-interval pairs $(\tau_j, [u_j, v_j])_{j=1}^m$ satisfying $[u_j, v_j] \subset [a, b]$, $\tau_j \in Z$, and $\tau_j - \delta(\tau_j) < u_j \le \tau_j \le v_j < \tau_j + \delta(\tau_j)$ for all $j \in \{1, 2, \ldots, m\}$. Then

$$\sum_{j=1}^{m} \|A(\tau_j)\|(v_j - u_j) = \sum_{i=1}^{\infty} \sum_{j;\, \tau_j\in Z_i} \|A(\tau_j)\|(v_j - u_j) \le \sum_{i=1}^{\infty} i\frac{\varepsilon}{i2^i} = \varepsilon.$$

It follows from (4.4) that

$$\sum_{j=1}^{m} \|I - W(v_j)W(u_j)^{-1}\| \le \sum_{j=1}^{m} \|I + A(\tau_j)(v_j - u_j) - W(v_j)W(u_j)^{-1}\| + \sum_{j=1}^{m} \|A(\tau_j)\|(v_j - u_j) < 2\varepsilon,$$

and therefore

$$\sum_{j=1}^{m} \|W(v_j) - W(u_j)\| \le \sum_{j=1}^{m} \|W(v_j)W(u_j)^{-1} - I\| \cdot \|W(u_j)\| \le 2M\varepsilon,$$

which confirms that $W$ satisfies the strong Luzin condition. □

**Theorem 4.7.** *Consider a function $A : [a, b] \to X$. Assume there is a function $W : [a, b] \to X$ which satisfies the strong Luzin condition, $W(t)^{-1}$ exists for all $t \in [a, b]$, and $W'(t) = A(t)W(t)$ for every $t \in [a, b]\backslash Z$, where $\mu(Z) = 0$. Then $A$ is strongly Kurzweil product integrable and $\prod_a^b (I + A(t)\,\mathrm{d}t) = W(b)W(a)^{-1}$.*

*Proof.* The strong Luzin condition implies that $W$ is continuous. Hence, $W^{-1}$ is continuous as well. Let $M = \sup_{t\in[a,b]} \|W(t)^{-1}\|$. Take an arbitrary $\varepsilon > 0$.
For every $t \in [a, b]\backslash Z$, there exists a $\Delta > 0$ such that

$$\|W(y) - W(t) - W'(t)(y - t)\| \le \varepsilon(y - t), \quad y \in [t, t + \Delta) \cap [a, b],$$

$$\|W(t) - W(x) - W'(t)(t - x)\| \le \varepsilon(t - x), \quad x \in (t - \Delta, t] \cap [a, b].$$

Consequently,

$$\|W(y) - W(x) - W'(t)(y - x)\| \le \|W(y) - W(t) - W'(t)(y - t)\| + \|W(t) - W(x) - W'(t)(t - x)\| \le \varepsilon(y - x)$$

whenever $x, y \in [a,b]$ and $t - \Delta < x \le t \le y < t + \Delta$. This means that

$$\lim_{\substack{x,y\to t,\\ x\neq y,\, t\in[x,y]}} \frac{W(y)-W(x)}{y-x} = W'(t),$$

and thus

$$\lim_{\substack{x,y\to t,\\ x\neq y,\, t\in[x,y]}} \frac{W(y)W(x)^{-1}-I}{y-x} = \lim_{\substack{x,y\to t,\\ x\neq y,\, t\in[x,y]}} \left(\frac{W(y)-W(x)}{y-x}W(x)^{-1}\right) = W'(t)W(t)^{-1} = A(t).$$

It follows that for every $t \in [a,b]\backslash Z$ and $\varepsilon > 0$, there exists a $\delta(t) > 0$ such that

$$\left\|\frac{W(y)W(x)^{-1}-I}{y-x} - A(t)\right\| < \frac{\varepsilon}{b-a}$$

whenever $x, y \in [a,b]$, $t - \delta(t) < x \le t \le y < t + \delta(t)$, and $x < y$. Hence, we also have

$$\left\|W(y)W(x)^{-1} - I - A(t)(y-x)\right\| < \frac{\varepsilon(y-x)}{b-a}$$

if $x, y \in [a,b]$ and $t - \delta(t) < x \le t \le y < t + \delta(t)$.
Next, we extend the domain of $\delta$ to the whole interval $[a,b]$ by taking the function $\delta : Z \to \mathbb{R}^+$ from Definition 4.5. Moreover, using the same argument as in the proof of Theorem 4.6, we can assume that $\delta$ is chosen in such a way that

$$\sum_{j=1}^{l} \|A(\tau_j)\|(v_j - u_j) < \varepsilon$$

for every collection of point-interval pairs $(\tau_j, [u_j, v_j])_{j=1}^{l}$ satisfying $[u_j, v_j] \subset [a,b]$, $\tau_j \in Z$, and $\tau_j - \delta(\tau_j) < u_j \le \tau_j \le v_j < \tau_j + \delta(\tau_j)$ for all $j \in \{1, 2, \dots, l\}$.
Now, for every $\delta$-fine tagged partition of $[a,b]$, we obtain

$$\sum_{i=1}^{m} \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| \le \sum_{i;\, \xi_i\in Z} \|A(\xi_i)\|(t_i - t_{i-1}) + \sum_{i;\, \xi_i\in Z} \|W(t_i) - W(t_{i-1})\|\cdot\|W(t_{i-1})^{-1}\|$$

$$+ \sum_{i;\, \xi_i\in[a,b]\backslash Z} \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| < \varepsilon + \varepsilon M + \sum_{i;\, \xi_i\in[a,b]\backslash Z} \frac{\varepsilon(t_i - t_{i-1})}{b-a} \le \varepsilon(M+2).$$

Hence, $A$ is strongly Kurzweil product integrable and $\prod_a^b (I + A(t)\,\mathrm{d}t) = W(b)W(a)^{-1}$. □

As an immediate consequence of Theorems 4.2, 4.6 and 4.7, we obtain the following characterization of strongly Kurzweil product integrable functions.

**Corollary 4.8.** *For every function $A : [a,b] \to X$, the following conditions are equivalent:*

1. *$A$ is strongly Kurzweil product integrable.*
2. *There is a function $W : [a,b] \to X$ which satisfies the strong Luzin condition, $W(t)^{-1}$ exists for all $t \in [a,b]$, and $W'(t) = A(t)W(t)$ for every $t \in [a,b]\backslash Z$, where $\mu(Z) = 0$.*

**Remark 4.9.** From Corollary 4.8, we deduce that the strong Kurzweil product integral has the following properties:

- In Remark 3.6, we already noticed that strong product integrability on $[a,b]$ implies strong product integrability on every subinterval. Now, consider an arbitrary $c\in(a,b)$. If $A:[a,b]\to X$ is strongly Kurzweil product integrable on $[a,c]$ and $[c,b]$, then $A$ is strongly Kurzweil product integrable on $[a,b]$. To see this, it is enough to let

$$W(t)=\begin{cases}\prod_a^t(I+A(s)\,\mathrm{d}s), & t\in[a,c],\\ \prod_c^t(I+A(s)\,\mathrm{d}s)\prod_a^c(I+A(s)\,\mathrm{d}s), & t\in(c,b],\end{cases}$$

and apply Corollary 4.8. (A different proof can be found in [10].)

- If $A_1,A_2:[a,b]\to X$ are such that $A_1=A_2$ almost everywhere and $A_1$ is strongly Kurzweil product integrable, then $A_2$ is strongly Kurzweil product integrable and $\prod_a^b(I+A_2(t)\,\mathrm{d}t)=\prod_a^b(I+A_1(t)\,\mathrm{d}t)$.

- We have the following Hake-type theorem: Assume that the strong product integral $\prod_a^t(I+A(s)\,\mathrm{d}s)$ exists for all $t\in[a,b)$. If $\lim_{t\to b-}\prod_a^t(I+A(s)\,\mathrm{d}s)$ exists and is invertible, then $\prod_a^b(I+A(s)\,\mathrm{d}s)$ exists as well and is equal to the limit. (For a different proof, see [10].)

In practice, it can be difficult to verify whether a given function $W$ satisfies the strong Luzin condition. The next theorem shows that if the relation $W'(t)=A(t)W(t)$ holds on $[a,b]\backslash Z$ with a countable set $Z$, then it is enough to assume that $W$ is continuous.

**Theorem 4.10.** *Consider a function $A:[a,b]\to X$. Assume there is a continuous function $W:[a,b]\to X$ such that $W(t)^{-1}$ exists for all $t\in[a,b]$, and $W'(t)=A(t)W(t)$ for all $t\in[a,b]\backslash Z$, where $Z$ is countable. Then $A$ is strongly Kurzweil product integrable and $\prod_a^b(I+A(t)\,\mathrm{d}t)=W(b)W(a)^{-1}$.*

*Proof.* Denote $Z=\{z_1,z_2,\dots\}$. Take an arbitrary $\varepsilon>0$. For every $n\in\mathbb{N}$, let $\delta(z_n)>0$ be such that

$$\|A(z_n)\|(y-x)<\frac{\varepsilon}{2^n}$$

and

$$\|W(y)W(x)^{-1}-I\|<\frac{\varepsilon}{2^n}$$

for every interval $[x,y]\subset(z_n-\delta(z_n),z_n+\delta(z_n))\cap[a,b]$. (Note that $(x,y)\mapsto W(y)W(x)^{-1}$ is a product of two continuous functions, and therefore a continuous function on $[a,b]\times[a,b]$, which equals $I$ when $x=y$.) Next, consider an arbitrary $t\in[a,b]\backslash Z$. As in the proof of the previous theorem, there exists a $\delta(t)>0$ such that

$$\big\|W(y)W(x)^{-1}-I-A(t)(y-x)\big\|<\frac{\varepsilon(y-x)}{b-a}$$

whenever $[x,y]\subset(t-\delta(t),t+\delta(t))\cap[a,b]$.
The function $\delta$ is now defined on $[a,b]$. For an arbitrary $\delta$-fine tagged partition of $[a,b]$, we get

$$\sum_{i=1}^m\|W(t_i)W(t_{i-1})^{-1}-I-A(\xi_i)(t_i-t_{i-1})\|$$

$$=\sum_{\substack{i\in\{1,\dots,m\},\\ \xi_i\notin Z}}\|W(t_i)W(t_{i-1})^{-1}-I-A(\xi_i)(t_i-t_{i-1})\|+\sum_{\substack{i\in\{1,\dots,m\},\\ \xi_i\in Z}}\|W(t_i)W(t_{i-1})^{-1}-I-A(\xi_i)(t_i-t_{i-1})\|$$

$$<\varepsilon+\sum_{\substack{i\in\{1,\dots,m\},\\ \xi_i\in Z}}\big(\|W(t_i)W(t_{i-1})^{-1}-I\|+\|A(\xi_i)\|(t_i-t_{i-1})\big)\le\varepsilon+2\sum_{n=1}^\infty\frac{\varepsilon}{2^n}=3\varepsilon.$$

Note that both $W$ and $W^{-1}$ are continuous, and therefore bounded. Hence, $A$ is strongly Kurzweil product integrable and $\prod_a^b(I+A(t)\,\mathrm{d}t)=W(b)W(a)^{-1}$. □

**Example 4.11.** If $A(t) = A$ for all $t \in [a,b]$, then the strong Kurzweil product integral $\prod_a^b (I + A(t)\,\mathrm{d}t)$ exists and equals $\exp((b-a)A)$; to see this, it is enough to apply Theorem 4.10 with $W(t) = \exp((t-a)A)$. According to Remark 4.9, the same result holds if $A(t) = A$ almost everywhere on $[a,b]$.

We now turn to the properties of the strong McShane product integrals. The proof of the following theorem is a simple adaptation of the proof of [20, Theorem 18].

**Theorem 4.12.** *If $A$ is strongly McShane product integrable, then the indefinite product integral $W(t) = \prod_a^t (I + A(s)\,\mathrm{d}s)$, $t \in [a,b]$, is absolutely continuous on $[a,b]$.*

*Proof.* Let $M = \sup_{t\in[a,b]} \|W(t)\|$. Take an arbitrary $\varepsilon > 0$. There is a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{m} \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| < \frac{\varepsilon}{2M}$$

for every $\delta$-fine free tagged partition of $[a,b]$. We fix one of these $\delta$-fine free tagged partitions and let

$$\sigma = \frac{\varepsilon}{2M(\max_{i=1,\dots,m} \|A(\xi_i)\| + 1)}.$$

Consider an arbitrary collection of non-overlapping intervals $([u_j, v_j])_{j=1}^r$ in $[a,b]$, where $\sum_{j=1}^r (v_j - u_j) < \sigma$. By subdividing the intervals $[u_j, v_j]$ if necessary, it can be assumed that for every $j \in \{1,\dots,r\}$, we have $[u_j, v_j] \subset [t_{i-1}, t_i]$ for a certain $i \in \{1,\dots,m\}$; let $\tau_j = \xi_i$. Then

$$\sum_{j=1}^{r} \|W(v_j) - W(u_j)\| \le \sum_{j=1}^{r} \|I - W(v_j)W^{-1}(u_j)\| \cdot \|W(u_j)\|$$

$$\le M \sum_{j=1}^{r} \|I + A(\tau_j)(v_j - u_j) - W(v_j)W^{-1}(u_j)\| + M \max_{i=1,\dots,q} \|A(\xi_i)\| \sum_{j=1}^{r} (v_j - u_j) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which proves that $W$ is absolutely continuous. □

**Lemma 4.13.** *Consider a function $A : [a,b] \to X$ and suppose there is an absolutely continuous function $W : [a,b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a,b]$, and $W'(t) = A(t)W(t)$ almost everywhere on $[a,b]$. Then $A$ is Bochner integrable.*

*Proof.* $W$ is continuous, and therefore strongly measurable. Consequently, its derivative $W'$, which is defined almost everywhere on $[a,b]$ and can be extended to $[a,b]$ in an arbitrary way, is also strongly measurable. Now, $A = W'W^{-1}$ is almost everywhere equal to a product of two strongly measurable functions, and therefore $A$ is also strongly measurable. Next, note that $AW$ is Bochner integrable (since it has an absolutely continuous primitive $W$). Hence, $\|AW\|$ is Lebesgue integrable. Because $W^{-1}$ is continuous, it follows that $\|W^{-1}\|$ is bounded and measurable on $[a,b]$. Consequently, $\|A\| = \|AWW^{-1}\| \le \|AW\| \cdot \|W^{-1}\|$ is Lebesgue integrable, which in turn means that $A$ is Bochner integrable. □

**Theorem 4.14.** *For every function $A : [a,b] \to X$, the following conditions are equivalent:*

1. *$A$ is strongly McShane product integrable.*
2. *There is an absolutely continuous function $W : [a,b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a,b]$, and $W'(t) = A(t)W(t)$ for every $t \in [a,b]\backslash Z$, where $\mu(Z) = 0$.*
3. *$A$ is Bochner integrable.*

4. *A is strongly measurable and $\|A\|$ is Kurzweil product integrable.*

*Proof.* The implication $1 \Rightarrow 2$ follows from Theorems 4.2 and 4.12, implication $2 \Rightarrow 3$ from Lemma 4.13, and implication $3 \Rightarrow 1$ from Theorem 3.15. Let us prove $4 \Rightarrow 3$: Because $\|A\|$ is real-valued and Kurzweil product integrable, it is strongly Kurzweil product integrable. By Theorem 3.13, $\|A\|$ is Henstock-Kurzweil integrable. For nonnegative functions, Henstock-Kurzweil integrability implies Lebesgue integrability (see [6, Theorem 9.13]). Taking into account that $A$ is strongly measurable, we deduce that $A$ is Bochner integrable. Finally, we prove the implication $3 \Rightarrow 4$: Bochner integrability of $A$ implies that $A$ is strongly measurable, and also that $\|A\|$ is Lebesgue integrable, which in turn means that $\|A\|$ is Kurzweil product integrable. $\square$

**Example 4.15.** If $A(t) = A$ almost everywhere on $[a, b]$, we know from Example 4.11 that the strong Kurzweil product integral $\prod_a^b(I+A(t)\,\mathrm{d}t)$ exists and equals $\exp\left((b-a)A\right)$. For the same reason, the Kurzweil product integral $\prod_a^b(1+\|A(t)\|\,\mathrm{d}t)$ exists and equals $\exp\left((b-a)\|A\|\right)$. Hence, by Theorem 4.14, the strong McShane product integral $\prod_a^b(I + A(t)\,\mathrm{d}t)$ exists as well.
A simple consequence is that for step functions with finitely many steps, the strong Kurzweil and strong McShane product integrals always exist and are easy to calculate. Indeed, if there is a partition $a = t_0 < t_1 < \cdots < t_m = b$ and $A(t) = A_i \in X$ for all $t \in (t_{i-1}, t_i)$, then

$$\prod_a^b (I + A(t)\,\mathrm{d}t) = \prod_{i=m}^{1} \prod_{t_{i-1}}^{t_i} (I + A(t)\,\mathrm{d}t) = \prod_{i=m}^{1} \exp(A_i(t_i - t_{i-1})).$$

# 5 Functions with countably many discontinuities

Our next goal is to obtain necessary and sufficient conditions for product integrability of functions with countably many discontinuities. This class includes every right regulated or left regulated function because these functions have only countably many discontinuities (see [8, Lemma 2.6]).

**Theorem 5.1.** *If $A : [a, b] \to X$ has countably many discontinuities, then the following conditions are equivalent:*

1. *A is Riemann product integrable.*
2. *A is bounded.*

*If any of these conditions is satisfied, there is a Lipschitz-continuous function $W : [a, b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a, b]$, and $W'(t) = A(t)W(t)$ for all $t \in [a, b]\backslash Z$, where $Z \subset [a, b]$ is countable.*

*Proof.* Recall that $A$ is Riemann product integrable if and only if $A$ is Riemann integrable. The implication $1 \Rightarrow 2$ follows from the fact that every Riemann integrable function is bounded. On the other hand, every bounded function which is almost everywhere continuous is Riemann integrable (see [7, Theorem 1]); this verifies the implication $2 \Rightarrow 1$. To prove the final statement, let

$$W(t) = \prod_a^t (I + A(s)\,\mathrm{d}s), \quad t \in [a, b].$$

Then (see [12, Theorem 23.1] or [17, Theorem 5.6.1])

$$W(t) = I + \int_a^t A(s)W(s)\,\mathrm{d}s, \quad t \in [a, b],$$

where the integral on the right-hand side is the Riemann integral. The set $Z$ of all discontinuities of $A$ is countable, and we have $W'(t) = A(t)W(t)$ for all $t \in [a,b]\backslash Z$. Both $A$ and $W$ are bounded; let $L = \sup_{t\in[a,b]} \|A(t)W(t)\|$. Consequently,

$$\|W(y) - W(x)\| = \left\| \int_x^y A(s)W(s)\,\mathrm{d}s \right\| \le L|y-x|$$

whenever $x, y \in [a,b]$; this proves that $W$ is Lipschitz-continuous. □

**Theorem 5.2.** *Assume that $A : [a,b] \to X$ has countably many discontinuities and is Kurzweil product integrable. Let*

$$W(t) = \prod_a^t (I + A(s)\,\mathrm{d}s), \quad t \in [a,b].$$

*Then $W'(t) = A(t)W(t)$ for all $t \in [a,b]\backslash Z$, where $Z \subset [a,b]$ is countable.*

*Proof.* Consider an arbitrary $t \in (a,b)$ such that $A$ is continuous at $t$. There exists a $\delta > 0$ such that $A$ is bounded on $[t-\delta, t+\delta] \subset [a,b]$. Consequently, $A$ is Riemann product integrable on $[t-\delta, t+\delta]$. Also, we have

$$W(s) = \prod_{t-\delta}^s (I + A(u)\,\mathrm{d}u) \prod_a^{t-\delta} (I + A(u)\,\mathrm{d}u), \quad s \in [t-\delta, t+\delta].$$

The first product integral is understood in Riemann's sense; hence, at every point $s$ where $A$ is continuous, $W'(s)$ exists and equals (see the proof of Theorem 5.1)

$$W'(s) = A(s) \prod_{t-\delta}^s (I + A(u)\,\mathrm{d}u) \prod_a^{t-\delta} (I + A(u)\,\mathrm{d}u).$$

In particular, for $s = t$ we get

$$W'(t) = A(t) \prod_{t-\delta}^t (I + A(u)\,\mathrm{d}u) \prod_a^{t-\delta} (I + A(u)\,\mathrm{d}u) = A(t)W(t).$$

This proves that $W'(t) = A(t)W(t)$ for all $t \in [a,b]\backslash Z$, where $Z \subset [a,b]$ is countable. □

**Theorem 5.3.** *If $A : [a,b] \to X$ has countably many discontinuities, then the following conditions are equivalent:*

1. *$A$ is strongly Kurzweil product integrable.*
2. *$A$ is Kurzweil product integrable.*
3. *There is a continuous function $W : [a,b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a,b]$, $W$ is differentiable on $[a,b]$ with the possible exception of a countable set $Z$, and $W'(t) = A(t)W(t)$ for all $t \in [a,b]\backslash Z$.*

*Proof.* The implication $1 \Rightarrow 2$ follows from Theorem 3.5, the implication $2 \Rightarrow 3$ from Theorem 5.2, and $3 \Rightarrow 1$ is a consequence of Theorem 4.10. □

**Theorem 5.4.** *If $A : [a,b] \to X$ has countably many discontinuities, then the following conditions are equivalent:*

1. *A is Bochner integrable.*

2. *A is strongly McShane product integrable.*

3. *There is an absolutely continuous function $W : [a,b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a,b]$, and $W'(t) = A(t)W(t)$ for all $t \in [a,b]\backslash Z$, where $Z \subset [a,b]$ is countable.*

*Proof.* The equivalence 1 $\Leftrightarrow$ 2 follows from Theorem 4.14.
If $A$ is strongly McShane product integrable, consider the indefinite McShane integral

$$W(t) = \prod_a^t (I + A(s)\,\mathrm{d}s), \quad t \in [a,b].$$

It follows from Theorem 5.2 that $W' = AW$ on $[a,b]\backslash Z$, where $Z$ is countable. Also, $W$ is absolutely continuous by Theorem 4.12; this proves the implication 2 $\Rightarrow$ 3. Finally, the implication 3 $\Rightarrow$ 1 follows from Lemma 4.13. $\square$

Additional criteria concerning product integrability of functions with countably many discontinuities (especially right regulated functions) can be found in the recent paper [10].

**Remark 5.5.** In general (even for functions with countably many discontinuities), Henstock-Kurzweil integrability and Kurzweil product integrability do not imply each other; see [11, Example 4.7]. However, we have seen that in some special cases, product integrability is equivalent to integrability. Let us recall that

- Riemann product integrability is equivalent to Riemann integrability,
- strong McShane product integrability is equivalent to Bochner (or strong McShane) integrability,
- for functions whose values commute with themselves, strong Kurzweil product integrability is equivalent to strong Henstock-Kurzweil integrability.

In these situations, one can use existing criteria of integrability, such as those presented in [8], to study product integrability.

# 6 Open problems

We conclude the paper with a list of several open problems:

- If we restrict ourselves to functions with countably many discontinuities, is it true that McShane product integrability implies strong McShane product integrability (i.e., Bochner integrability)? If not, is there any other simple characterization of McShane product integrable functions with countably many discontinuities? Note that McShane product integrability implies Kurzweil product integrability, which in turn implies strong Kurzweil product integrability (see Theorem 5.3). Hence, the question reduces to the problem whether the indefinite McShane product integral is absolutely continuous.

- Since strong McShane product integrability coincides with Bochner integrability, it is clear that the class of strongly McShane product integrable functions is a vector space. However, it is not at all obvious whether this statement holds for Kurzweil/McShane product integrable functions and strongly Kurzweil product integrable functions. A partial result in this direction can be found in [10], where it is shown that the sum of a strongly Kurzweil product integrable function and a Bochner integrable function is strongly Kurzweil product integrable.

- For $X = \mathbb{R}^{n\times n}$, exponential Kurzweil/McShane product integrability is equivalent to Kurzweil/McShane product integrability. Is this still true in infinite-dimensional Banach algebras? We only know that the answer is affirmative for strong product integrals.

- Does Riemann product integrability imply McShane product integrability?

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