Through the Tiling Glass: Tribonacci Identities

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Abstract

Using combinatorial tiling arguments, we explore identities involving the Tribonacci and Tribonacci-Lucas numbers and related sequences, including the Fibonacci numbers, Narayana's cows sequence, and the Padovan numbers. The tiling framework yields results that both unify and extend existing work. In particular, we derive new identities and provide new combinatorial proofs of several known ones.

Keywords: Tribonacci number, Tribonacci-Lucas number, combinatorial proof, tiling

1. Introduction

Many identities involving the Fibonacci numbers can be proved combinatorially using the fact that F_n counts the number of tilings of a $1 \times n$ rectangle (an n-board, for short) with squares and dominoes. Similarly, the Lucas numbers L_n count the number of tilings of an annular board composed of n cells (with cell n being adjacent to cell 1) using curved squares and dominoes. An excellent classical reference for combinatorial proofs of Fibonacci and Lucas identities based on tilings is [1].

By expanding the set of tiles to include trominoes, we venture down the rabbit hole into the world of Tribonacci numbers T_n, which count the number of tilings of an n-board, and the Tribonacci-Lucas numbers K_n (this name and notation are not completely standard, but we follow [6, 15]), which count all n-bracelets, i.e., tilings of the annular n-board.

There is substantial literature on identities involving the Tribonacci and related sequences, but most sources avoid the combinatorial approach, and instead rely on computational techniques such as generating functions. While such algebraic methods follow their own logic, we find them to be a bit like the Mad Hatter's tea party: perfectly functional in their own world, yet perhaps missing the combinatorial narrative of the tiles that bloom in our garden. An exception is the paper [2], which provides combinatorial proofs of several Tribonacci identities by exploring tilings of a honeycomb strip. In the present paper, we follow the spirit of [1] and focus on combinatorial proofs involving tilings of rectangular strips using squares, dominoes, and trominoes. Our aim is not only to derive new identities, but also to shed new light on known ones through the lens of tiling arguments.

The individual sections of the paper can be read independently, for in this landscape, as the Cheshire Cat might suggest, it matters little which path one chooses first, and their contents are as follows:

- In Section 2, we express T_n in terms of K_n and vice versa.
- The next identities apply not only to T_n and K_n, but to all sequences U_n satisfying the Tribonacci recurrence. In particular, Section 4 establishes the values of $\sum_{k=0}^{n} U_k$, $\sum_{k=0}^{n} U_{2k}$, $\sum_{k=0}^{n} U_{2k+1}$, $\sum_{k=0}^{n} U_{3k}$, $\sum_{k=0}^{n} U_{3k+1}$, $\sum_{k=0}^{n} U_{3k+2}$, and concludes with an Agronomof-type identity.

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- In Section 7, we focus on convolutions of the Tribonacci and Fibonacci numbers. We obtain two identities depending on whether we decide to generalize the Tribonacci numbers by considering U_n as before, or to deal with the generalized Fibonacci (Gibonacci) numbers G_n.
- Sections 13 and 24 evaluate convolutions of Tribonacci-type sequences with Narayana-type sequences that count tilings composed of squares and trominoes, and with Padovan-type sequences that count tilings composed of dominoes and trominoes.
- Finally, in Section 44, we focus on the self-convolutions $\sum_{k=0}^{n} T_k K_{n-k}$ and $\sum_{k=0}^{n} K_k K_{n-k}$.

As shown in [12], the identities in Section 4 (with the exception of the Agronomof-type identity) can also be derived using discrete calculus. Convolution identities such as those in Sections 7–44 are commonly obtained via generating functions; see, for instance, [3, 4, 6, 7, 8, 9]. Nevertheless, we believe that a combinatorial proof provides the best insight into why an identity holds.

2. Relating Tribonacci and Tribonacci-Lucas Numbers

The Tribonacci numbers are defined by the recurrence $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ and the initial values $T_0 = T_1 = T_2 = 1$. Conditioning on the first tile, it is easy to see that T_n counts all tilings of an n-board using squares, dominoes, and trominoes. Many sources, including the OEIS [11], deal with a shifted version of this sequence beginning with 0, 0, 1, ..., see A000073, which means that the indices in various Tribonacci identities are shifted.

The closely related Tribonacci-Lucas numbers K_n count all n-bracelets consisting of curved squares, dominoes, and trominoes. The corresponding OEIS entry is A001644. In particular, we have $K_1 = 1, K_2 = 3$, and $K_3 = 7$ (see Figure 1). For completeness, we now show that the Tribonacci-Lucas numbers satisfy the same recurrence relation as the Tribonacci numbers. The proof is an adaptation of the proof of [1, Combinatorial Theorem 2], which deals with the Lucas numbers. To ensure that the recurrence holds also for $n = 3$, we let $K_0 = 3$. In later proofs, we will interpret this combinatorially by imagining that we have three different 0-bracelets, and we will denote them by $\emptyset_1$, $\emptyset_2$, $\emptyset_3$.

Identity 1. *For $n \geq 3$,*

$$K_n = K_{n-1} + K_{n-2} + K_{n-3}.$$

Proof. The relation holds for $n = 3$, so assume $n \geq 4$. Define the first tile to be the tile that covers cell 1. We condition on the last tile of the bracelet, i.e., the one that precedes the first tile in the counterclockwise direction. There are K_{n-1} tilings that end with a square, K_{n-2} tilings that end with a domino, and K_{n-3} tilings that end with a tromino (note that the positions of all tiles are determined by the placement of the first tile). □

Can we derive a formula linking the Tribonacci and Tribonacci-Lucas numbers together? The following identity is easy to check by verifying that both sides satisfy the same recurrence and share the same initial values. It can also be written in the equivalent form $K_n = T_{n-1} + 2T_{n-2} + 3T_{n-3}$, which is mentioned (with shifted indices) under entry A001644 in the OEIS [11]. The following combinatorial proof inspired by the proof of [1, Identity 32] seems to be new.

Identity 2. *For $n \geq 3$,*

$$K_n = T_n + T_{n-2} + 2T_{n-3}.$$

Proof. The idea is to take an n-bracelet, cut it between two cells and uncurl it into a standard linear board. The details depend on which tiles cover positions $n-1$, n, 1 and 2; see Figure 2.

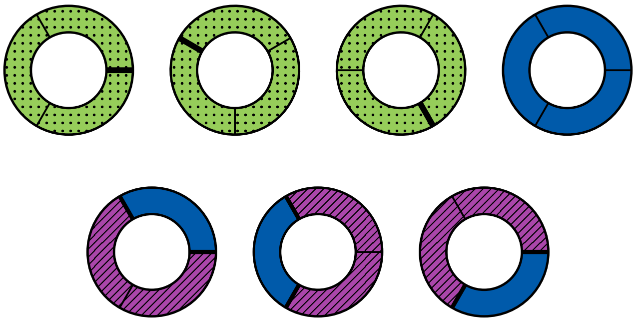

Figure 1: A circular 3-board and its seven possible bracelets, composed of curved squares, dominoes, and trominoes.

Case 1. Two different tiles touch at the boundary of positions n and 1. By breaking the board between them, we can uncurl the n-bracelet into a linear n-board, which yields T_n tilings.

Case 2. A domino covers positions n and 1. In this case, we break the n-bracelet along the edges of that domino. This gives T_{n-2} tilings of a linear board consisting of cells labeled $2, 3, \ldots, n-1$.

Case 3. A tromino covers either positions $n-1, n, 1$ or $n, 1, 2$. The former case results in T_{n-3} tilings of a linear board consisting of cells $2, \ldots, n-2$, while the latter leaves T_{n-3} tilings of a board with cells labeled $3, \ldots, n-1$. ☐

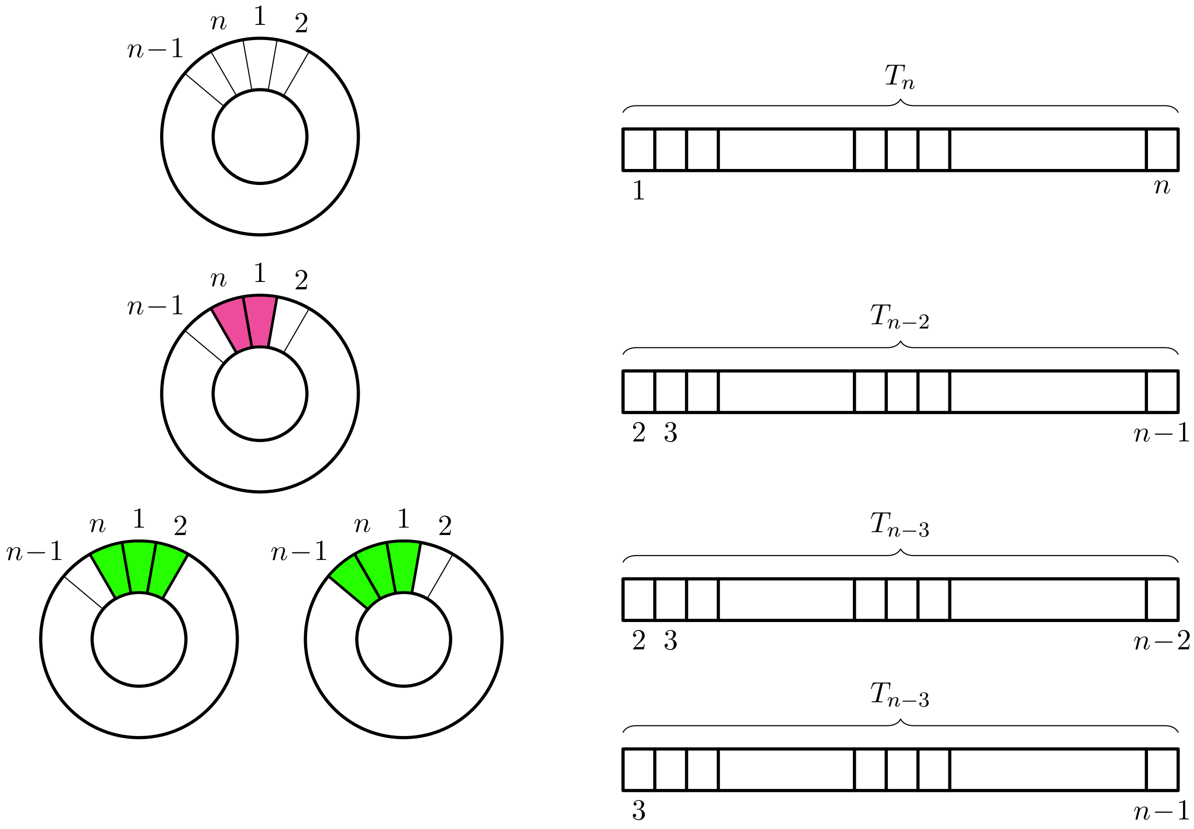

Figure 2: An n-bracelet can be cut between two cells and uncurled into a standard linear board.

Is there an analogue of Identity 2 that would express the Tribonacci numbers in terms of the Tribonacci-Lucas numbers? Sure, we have the following result, which seems to be new.

Identity 3. *For $n \in \mathbb{N}_0$,*

$$T_n = \frac{1}{22}\left(5K_n + K_{n+1} + 2K_{n+2}\right).$$

An algebraic proof is easy: Both sides are sequences satisfying the Tribonacci recurrence, so it suffices to check that they coincide for $n \in \{0, 1, 2\}$, which is a routine calculation.

Due to the factor 22, a combinatorial proof appears challenging, though the reader is welcome to try their luck.

4. Basic identities

To obtain identities that apply to Tribonacci and Tribonacci-Lucas numbers at the same time, we now introduce the generalized Tribonacci numbers by

$$U_n = U_{n-1} + U_{n-2} + U_{n-3}, \quad n \geq 3, \quad U_0 = c, \quad U_1 = a, \quad U_2 = a + b,$$

where a, b, $c \in \mathbb{N}$. For $n \in \mathbb{N}$, the number U_n counts all tilings of an n-board using squares, dominoes, and trominoes, where the first tile has one of a colors if it is a square, one of b colors if it is a domino, and one of c colors if it is a tromino. The remaining tiles do not have colors. The initial values correspond to the fact that there are a tilings (one square) if $n = 1$, $a + b$ tilings (two squares or one domino) if $n = 2$, and $2a + b + c$ tilings (three squares, square and domino, domino and square, tromino) if $n = 3$. From these values, the term U_0 is determined by requiring that the recurrence relation holds for $n = 3$.

The classical Tribonacci numbers correspond to $a = b = c = 1$, while the Tribonacci-Lucas numbers correspond to $a = 1$, $b = 2$, $c = 3$.

The next identity shows how to calculate sums of generalized Tribonacci numbers.

Identity 4. *For $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} U_k = \frac{1}{2}(U_{n+2} + U_n - U_2 + U_0).$$

Proof. How many tilings of an $(n + 3)$-board contain at least one non-square tile?
Answer 1: There are $U_{n+3} - a = U_{n+3} - U_1$ such tilings.
Answer 2: By conditioning on the location of the last non-square tile, we partition the tilings into two cases; see Figure 3.

Case 1. The last tile of length at least two is a domino located at positions $k + 1, k + 2$. If $k \geq 1$, cells 1 through k can be tiled in U_k ways, and cells $k + 3$ through $n + 3$ must be covered only by squares. Summing over all possible positions of the last domino yields $\sum_{k=1}^{n+1} U_k$. If $k = 0$, the number of possibilities is $b = U_2 - U_1$.

Case 2. The last tile of length at least two is a tromino located at positions $k + 1, k + 2$ and $k + 3$. By similar reasoning as in Case 1, if $k \geq 1$, we obtain $\sum_{k=1}^{n} U_k$. If $k = 0$, we get $c = U_0$.

Since both answers count the same set of tilings, we establish the identity

$$U_{n+3} - U_1 = U_2 - U_1 + U_0 + \sum_{k=1}^{n+1} U_k + \sum_{k=1}^{n} U_k.$$

By rearranging the terms and applying the recurrence

$$U_{n+3} - U_{n+1} = U_{n+2} + U_n,$$

we arrive at the desired result. □

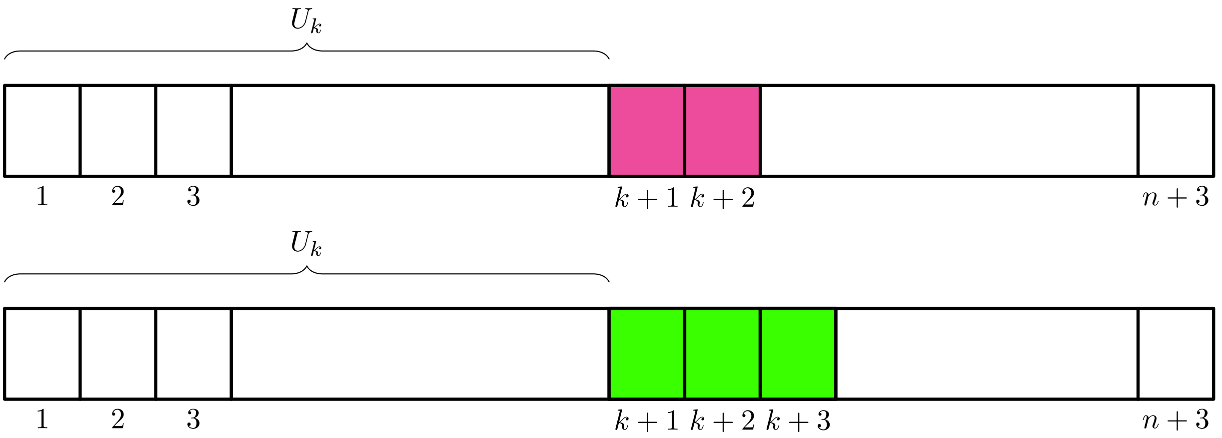

Figure 3: The last tile of length at least two is a domino covering cells $k+1$, $k+2$ or a tromino covering cells $k+1$, $k+2$, $k+3$. In both cases, if $k \geq 1$, there are U_k ways to tile cells 1 through k.

Next, we evaluate the sums $\sum_{k=0}^{n} U_{2k}$ and $\sum_{k=0}^{n} U_{2k+1}$.

Identity 5. *For $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} U_{2k} = \frac{U_{2n+1} + U_{2n} - U_1 + U_0}{2}.$$

Proof. How many tilings of a $(2n+3)$-board exist?
Answer 1: By definition, there are U_{2n+3} such tilings.
Answer 2: We partition the tilings by conditioning on the last tile of odd length, as illustrated in Figure 4. At least one such tile must exist because the total length of the board is odd.

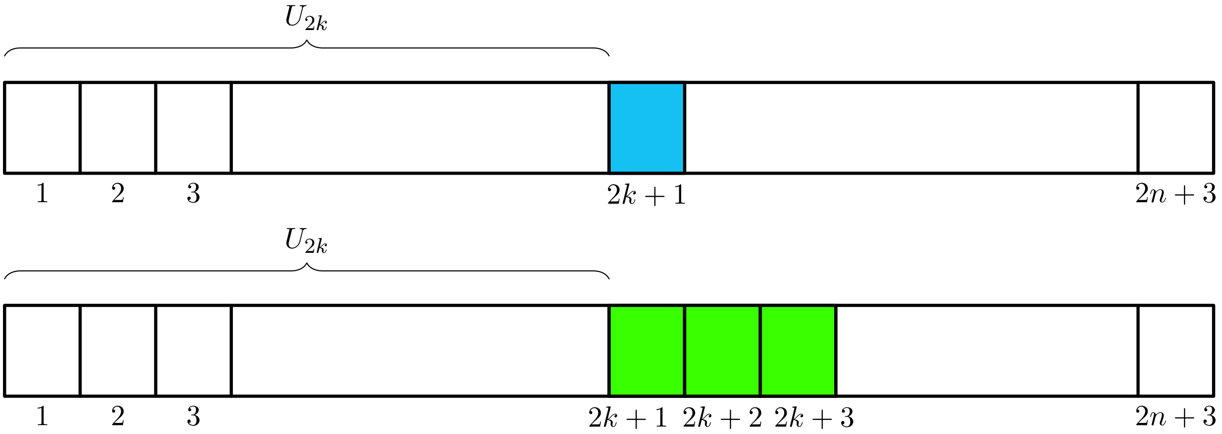

Figure 4: A square is placed at position $2k+1$, preceded by U_{2k} possible tilings if $k \geq 1$. Another possibility is a tromino located at positions $2k+1$, $2k+2$, $2k+3$, which is handled in a similar fashion.

Case 1. The last tile of odd length is a square at cell $2k+1$. If $k \geq 1$, the board to its left can be tiled in U_{2k} ways, while the remaining part of the board to its right must be covered exclusively by dominoes. If $k = 0$, the number of ways is $a = U_1$. Hence, the total number of tilings is $U_1 + \sum_{k=1}^{n+1} U_{2k}$.
Case 2. The last tile of odd length is a tromino occupying cells $2k+1$, $2k+2$, $2k+3$. If $k \geq 1$, the board to its left can be tiled in U_{2k} ways, and all tiles to the right must be dominoes. If $k = 0$, the number of ways is $c = U_0$. Hence, the total number of such tilings is $U_0 + \sum_{k=1}^{n} U_{2k}$.

Since both answers count the same set of tilings, we obtain the identity

$$U_{2n+3} = U_1 + \sum_{k=1}^{n+1} U_{2k} + U_0 + \sum_{k=1}^{n} U_{2k}.$$

By rearranging the terms and applying the recurrence

$$U_{2n+3} - U_{2n+2} = U_{2n+1} + U_{2n},$$

we obtain the final result. □

Identity 6. *For* $n \in \mathbb{N}_0$*,*

$$\sum_{k=0}^{n} U_{2k+1} = \frac{U_{2n+1} + U_{2n+2} + U_1 - U_2}{2}$$

Proof. How many tilings of a $(2n+4)$-board contain at least one non-domino tile?
Answer 1: There are $U_{2n+4} - b = U_{2n+4} - U_2 + U_1$ such tilings.
Answer 2: As before, we condition on the location of the last non-domino tile; see Figure 5.

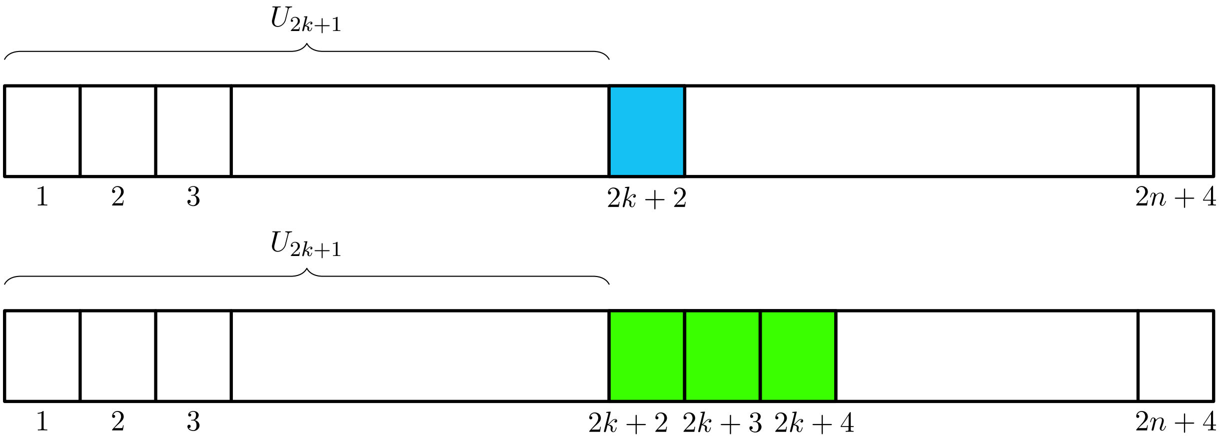

Figure 5: A square is located at position $2k+2$ and a tromino is placed at positions $2k+2$, $2k+3$, $2k+4$. Both odd-length tiles are preceded by U_{2k+1} possible tilings.

Case 1. The last non-domino tile is a square. After this point, the tail of the tiling consists entirely of dominoes and must have an even length. By placing the square at position $2k+2$, there are U_{2k+1} ways to tile the board to its left. Summing over all possible positions of the last square yields $\sum_{k=0}^{n+1} U_{2k+1}$.

Case 2. The last non-domino tile is a tromino, which occupies cells $2k+2$, $2k+3$, $2k+4$. Cells 1 through $2k+1$ can be tiled in U_{2k+1} ways. Allowing k to range over all possible positions and summing the results, we find $\sum_{k=0}^{n} U_{2k+1}$.

Equating these two answers and performing a bit of algebraic tidying, we arrive at the desired identity. □

The next three identities establish the values of $\sum_{k=0}^{n} U_{3k}$, $\sum_{k=0}^{n} U_{3k+1}$, and $\sum_{k=0}^{n} U_{3k+2}$.

Identity 7. *For* $n \in \mathbb{N}_0$*,*

$$\sum_{k=0}^{n} U_{3k} = \frac{1}{2}(U_{3n+2} - U_{3n} + 3U_0 - U_2).$$

Proof. How many tilings of a $3n$-board contain at least one non-tromino tile?
Answer 1: There are $U_{3n} - c = U_{3n} - U_0$ such tilings.
Answer 2: By conditioning on the location of the last non-tromino tile, we obtain two cases; see Figure 6.

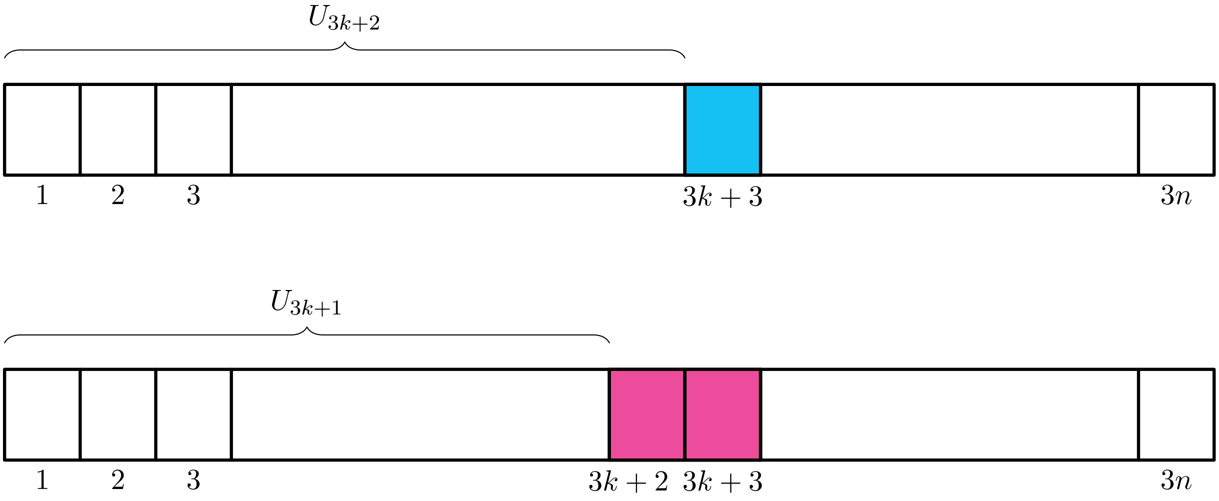

Figure 6: A square is placed at position $3k+3$, with U_{3k+2} ways to tile the board to its left. A domino is located at positions $3k+2$, $3k+3$, and there are U_{3k+1} ways to tile the board to its left.

Case 1. The last non-tromino tile is a square. By placing the square at position $3k+3$, there are U_{3k+2} ways to tile the board to its left. Summing over all possible positions of the last square yields $\sum_{k=0}^{n-1} U_{3k+2}$.
Case 2. The last non-tromino tile is a domino placed at positions $3k+2$, $3k+3$. Cells 1 through $3k+1$ can be tiled in U_{3k+1} ways. Allowing k to range over all possible positions and summing the results, we get $\sum_{k=0}^{n-1} U_{3k+1}$.

Equating these two expressions and applying Identity 4 with n replaced by $3n$, we obtain:

$$\sum_{k=0}^{3n} U_k - \left(\sum_{k=0}^{n-1} U_{3k+2} + \sum_{k=0}^{n-1} U_{3k+1}\right) = \frac{1}{2}(U_{3n+2} + U_{3n} - U_2 + U_0) - U_{3n} + U_0.$$

The final result now follows from an algebraic manipulation. □

Identity 8. *For $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} U_{3k+1} = \frac{1}{2}(U_{3n+2} + U_{3n} - U_2 - U_0 + 2U_1).$$

Proof. How many tilings of a $(3n+1)$-board exist?
Answer 1: By definition, there are U_{3n+1} such tilings.
Answer 2: We condition on the location of the last non-tromino tile, as illustrated in Figure 7. At least one such tile must exist, since the board length is not divisible by three. This partitions our tilings into two cases.

Case 1. The last non-tromino tile is a square at cell $3k+1$. If $k \geq 1$, the board to its left can be tiled in U_{3k} ways, while the remaining part of the board to its right must be covered exclusively by trominoes. If $k = 0$, the number of ways is $a = U_1$. Summing over all possible positions of the

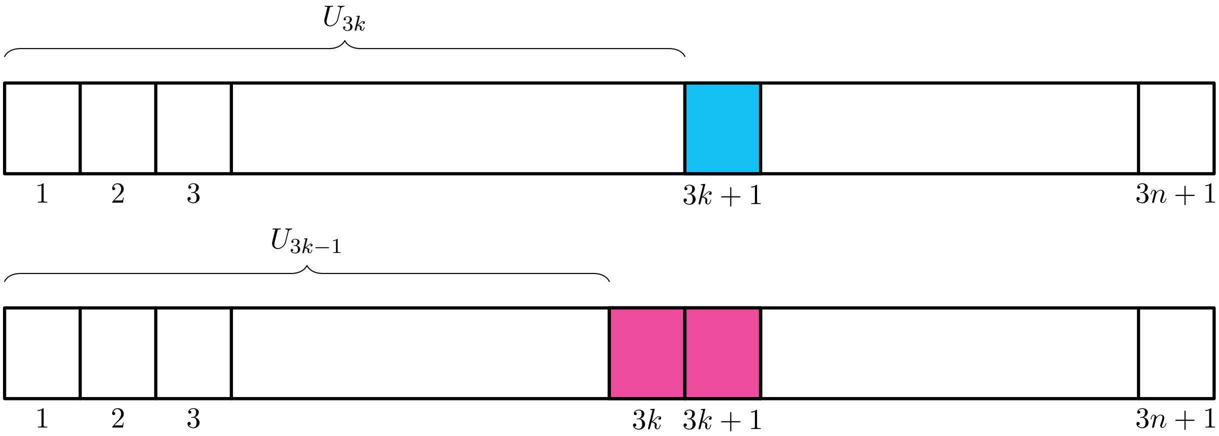

Figure 7: A square is placed at position $3k+1$. If $k \geq 1$, there are U_{3k} ways to tile the board to its left. A domino is treated analogously.

last square yields $U_1 + \sum_{k=1}^{n} U_{3k}$.

Case 2. The last non-tromino tile is a domino placed at positions $3k$, $3k+1$. Cells 1 through $3k-1$ can be tiled in U_{3k-1} ways and the tail of the tiling consists entirely of trominoes. Hence, the total number of tilings is $\sum_{k=1}^{n} U_{3k-1}$.

Since both sides count the same set of tilings, we have

$$U_{3n+1} = U_1 + \sum_{k=1}^{n} U_{3k} + \sum_{k=1}^{n} U_{3k-1}.$$

We decompose the sum $\sum_{j=1}^{3n+1} U_j$ according to residues modulo 3:

$$\sum_{j=1}^{3n+1} U_j = \sum_{k=1}^{n} U_{3k} + \sum_{k=0}^{n} U_{3k+1} + \sum_{k=1}^{n} U_{3k-1}.$$

The desired identity follows by rearranging the terms and applying Identity 4 with n replaced by $3n+1$, i.e.,

$$\sum_{j=1}^{3n+1} U_j = \frac{1}{2}\big(U_{3n+3} + U_{3n+1} - U_2 - U_0\big). \qquad \square$$

Identity 9. *For* $n \in \mathbb{N}_0$*,*

$$\sum_{k=0}^{n} U_{3k+2} = \frac{1}{2}(U_{3n+3} + U_{3n+1} + U_2 - 2U_1 - U_0).$$

Proof. How many tilings of a $(3n+2)$-board exist?
Answer 1: There are U_{3n+2} such tilings.
Answer 2: We condition on the location of the last non-tromino tile, as illustrated in Figure 8. Note that the existence of such a tile is guaranteed. Thus, we obtain two cases.
Case 1. The last non-tromino tile is a square at cell $3k+2$. The board to its left can be tiled in U_{3k+1} ways. Summing over all possible positions of the last square yields $\sum_{k=0}^{n} U_{3k+1}$.
Case 2. The last non-tromino tile is a domino whose right edge is placed at position $3k+2$. If $k \geq 1$, the board to its left can be tiled in U_{3k} ways, while the remaining part of the board to its right must be covered exclusively by trominoes. If $k=0$, the number of ways is $b = U_2 - U_1$.

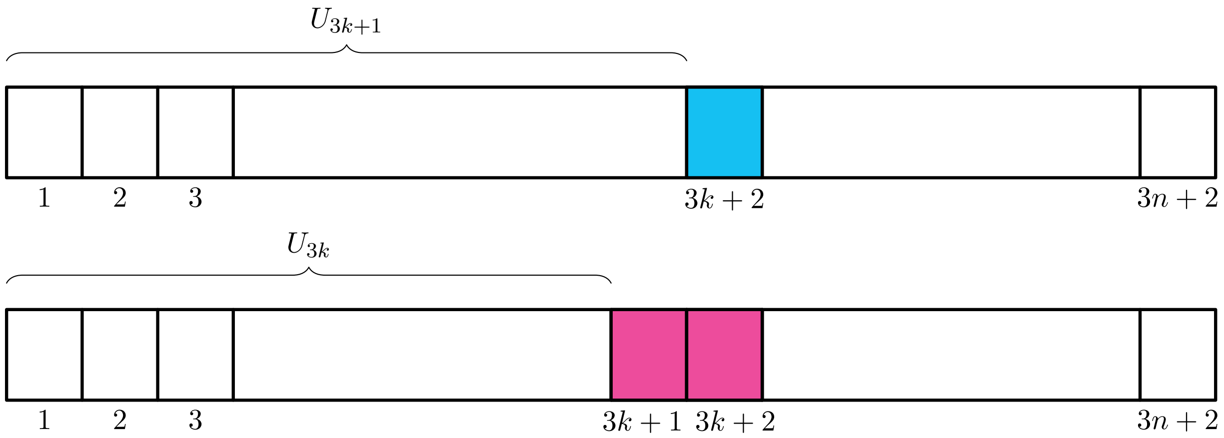

Figure 8: If a square is placed at position $3k+2$, there are U_{3k+1} ways to tile the board to its left. A domino is treated analogously.

Summing over all possible positions of the last domino yields $U_2 - U_1 + \sum_{k=1}^{n} U_{3k}$.

Equating these two counts gives

$$U_{3n+2} = \sum_{k=0}^{n} U_{3k+1} + \sum_{k=1}^{n} U_{3k} + U_2 - U_1.$$

Next, we decompose the sum $\sum_{j=0}^{3n+2} U_j$ according to residue classes modulo 3:

$$\sum_{j=0}^{3n+2} U_j = U_0 + \sum_{k=1}^{n} U_{3k} + \sum_{k=0}^{n} U_{3k+1} + \sum_{k=0}^{n} U_{3k+2}.$$

Finally, we apply Identity 4 with n replaced by $3n+2$, i.e.,

$$\sum_{j=0}^{3n+2} U_j = \frac{1}{2}\big(U_{3n+4} + U_{3n+2} + U_0 - U_2\big).$$

Following a series of algebraic steps, we obtain the identity as presented earlier. ☐

The next result generalizes Agronomof's identity, whose detailed description is available in [14], while combinatorial proofs based on tilings were given in [2, Theorem 6.2] and [12, p. 625]. Here we extend the identity by dealing with the generalized Tribonacci numbers. The result is a special case of [1, Identity 76], but it does not seem as widely known as it deserves, so we include it here.

Identity 10. *For $n, p \geq 2$,*

$$U_{n+p} = U_p T_n + U_{p-1} T_{n-1} + U_{p-2} T_{n-1} + U_{p-1} T_{n-2}.$$

Proof. The left side counts tilings of an $(n+p)$-board. There are four different types corresponding to the terms on the right side. The first term counts tilings that are breakable between positions p and $p+1$. The second term counts tilings with a domino placed over positions p and $p+1$. The final two terms correspond to tilings with a tromino over $p-1$, p, $p+1$ or p, $p+1$, $p+2$, respectively.

The previous argument is correct only for $p \geq 3$, and some care is needed when $p = 2$. In this case, the number of tilings with a tromino placed over positions $p-1$, p, $p+1$ is cT_{n-1}. However, since $c = U_0$, the identity is valid also for $p = 2$. ☐

If we extend the sequences U_n and T_n to negative indices via the Tribonacci recurrence, the previous identity holds for all $n, p \in \mathbb{Z}$. Indeed, if $p \geq 2$ is fixed, then the left side satisfies the Tribonacci relation, and the right side has the same property, because it is a linear combination of sequences satisfying the Tribonacci relation. However, if two sequences satisfying the Tribonacci relation coincide for three consecutive values of n (which they do for all $n \geq 2$), then they coincide for all $n \in \mathbb{Z}$. A similar argument with n fixed shows that the identity is valid for all $p \in \mathbb{Z}$.

In particular, letting $p = 0$, we get

$$\begin{aligned} U_n &= U_0 T_n + (U_{-1} + U_{-2}) T_{n-1} + U_{-1} T_{n-2} \\ &= U_0 T_n + (U_1 - U_0) T_{n-1} + (U_2 - U_1 - U_0) T_{n-2}, \quad n \in \mathbb{Z}. \end{aligned}$$

This relation, which expresses the generalized Tribonacci numbers using the classical ones, is not surprising, since both sides satisfy the Tribonacci relation, and their values for $n \in \{0, 1, 2\}$ coincide.

Remark. The combinatorial proofs of Identities 4–10 in the present section assumed that $U_0 = c$, $U_1 = a$ and $U_2 = a + b$, where a, b, $c \in \mathbb{N}$. However, since both sides of each identity are linear functions of U_0, U_1, U_2 (although we do not write this dependence explicitly), it follows that each identity in fact holds for an arbitrary choice of the initial terms $U_0, U_1, U_2 \in \mathbb{R}$. The same remark also applies to the identities in subsequent sections.

7. Convolutions of Tribonacci and Fibonacci numbers

The Gibonacci (generalized Fibonacci) numbers are defined by $G_n = G_{n-1} + G_{n-2}$ for $n \geq 2$, with initial conditions $G_0 = b$ and $G_1 = a$. For $n \in \mathbb{N}$, the number G_n counts all tilings of an n-board using squares and dominoes, where the first tile has one of a colors in the case of a square and one of b colors in the case of a domino.

For $a = b = 1$, we get the Fibonacci numbers F_n, which count all the tilings of an n-board using squares and dominoes. The choice $a = 1$, $b = 2$ leads to the Lucas numbers L_n, which count n-bracelets consisting of squares and dominoes.

The next result is actually a pair of convolution identities with similar proofs, depending on whether we condition on the first or last tromino, respectively.

Identity 11. *For $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} U_k F_{n-k} = U_{n+3} - G_{n+3} = (U_0 - G_0) T_n + \sum_{k=0}^{n} G_k T_{n-k}.$$

Proof. How many tilings of an $(n+3)$-board contain at least one tromino?

Answer 1: There are $U_{n+3} - G_{n+3}$ such tilings.

Answer 2: We condition on the location of the last tromino. Suppose it covers cells $k{+}1, k{+}2, k{+}3$; see Figure 9. Cells 1 through k can be tiled in U_k ways, and cells $k+4$ through $n+3$ must be covered by squares and dominoes, which can be done in F_{n-k} ways. This result is correct for $k \geq 1$, but some care is needed for $k = 0$. In this case, the number of tilings is cF_n, which is in agreement with $U_0 F_n$. Hence, allowing k to range over all possible positions and summing the results gives $\sum_{k=0}^{n} U_k F_{n-k}$.

Answer 3: We condition on the location of the first tromino. Suppose it covers cells $k{+}1, k{+}2, k{+}3$. Cells 1 through k can be tiled in G_k ways, and cells $k+4$ through $n+3$ in T_{n-k} ways. This result holds for $k \geq 1$; if $k = 0$, the correct count is $cT_n = U_0 T_n$. Hence, allowing k to range over all possible positions and summing the results gives $U_0 T_n + \sum_{k=1}^{n} G_k T_{n-k}$, which equals the right side of the identity. □

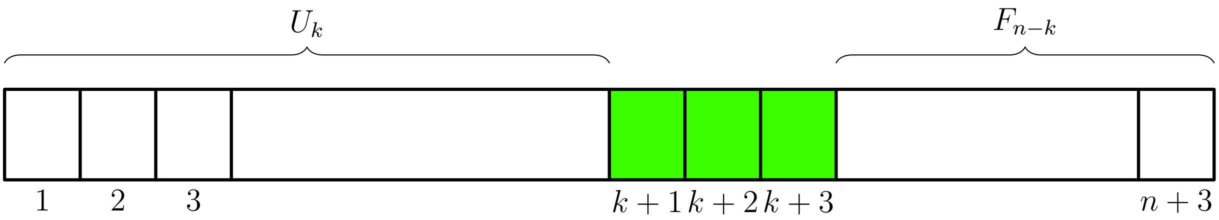

Figure 9: The last tromino covers cells $k+1, k+2, k+3$. The cells to its left can be tiled in U_k ways, while the remaining positions allow for F_{n-k} possible tilings.

When $a = b = c = 1$, Identity 11 reduces to

$$T_{n+3} - F_{n+3} = \sum_{k=0}^{n} T_k F_{n-k},$$

which is stated as an exercise in [1, p. 47]. A proof via a tiling argument is given in [2, Theorem 6.3], while [6, Theorem 2.1] uses generating functions. Both sides of the identity correspond to a shifted version of sequence A000100 from the OEIS [11].

For $a = 1$, $b = 2$, and $c = 3$, Identity 11 yields

$$\sum_{k=0}^{n} K_k F_{n-k} = K_{n+3} - L_{n+3} = T_n + \sum_{k=0}^{n} L_k T_{n-k}.$$

These formulas were proved in [6, Theorems 2.2 and 2.3] via generating functions.

13. Convolutions of Tribonacci and Narayana cows numbers

Next, we introduce the generalized Narayana's cows sequence by

$$O_n = O_{n-1} + O_{n-3}, \quad n \geq 3, \quad O_0 = c, \quad O_1 = a, \quad O_2 = a.$$

For $n \in \mathbb{N}$, the number O_n counts all tilings of an n-board using squares and trominoes, where the initial tile has one of a colors if it is a square, and one of c colors if it is a tromino.

The classical Narayana's cows sequence N_n, see A000930, corresponds to $a = c = 1$, and simply counts the number of uncolored tilings of an n-board using squares and trominoes. It is also natural to investigate the corresponding Lucas-type numbers M_n that count uncolored n-bracelets consisting only of squares and trominoes. An argument similar to the proof of Identity 1 reveals that these numbers satisfy the same recurrence relation, but the initial values now correspond to $a = 1$, $c = 3$. This sequence, which is a shifted version of A001609, does not appear to have a standard name, and we call it the Narayana-Lucas sequence.

Here is the next pair of convolution identities, whose proofs rely on conditioning on the first or last domino, respectively.

Identity 12. *For $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} O_k T_{n-k} + (U_2 - U_1 - O_0) T_n = U_{n+2} - O_{n+2} = \sum_{k=0}^{n} U_k N_{n-k} + (U_2 - U_1 - U_0) N_n.$$

Proof. How many tilings of an $(n+2)$-board contain at least one domino?
Answer 1: There are $U_{n+2} - O_{n+2}$ such tilings.
Answer 2: We condition on the location of the last domino. Suppose it covers positions $k+1$ and $k+2$ as in Figure 10. If $k \geq 1$, the cells to its left can be tiled in U_k ways, while the remaining

positions allow for N_{n-k} possible tilings. If $k=0$, the number of tilings is $bN_n = (U_2 - U_1)N_n$. Summing over all possible positions of the last domino gives $\sum_{k=1}^{n} U_k N_{n-k} + (U_2 - U_1)N_n$, which is equal to the right side of the identity.

Answer 3: We condition on the location of the first domino. Suppose it covers positions $k+1$ and $k+2$. If $k \geq 1$, the cells to its left can be tiled in O_k ways, while the remaining positions allow for T_{n-k} possible tilings. If $k=0$, the number of tilings is $bT_n = (U_2 - U_1)T_n$. Summing over all possible positions of the first domino gives $\sum_{k=1}^{n} O_k T_{n-k} + (U_2 - U_1)T_n$, which is equal to the left side of the identity.

□

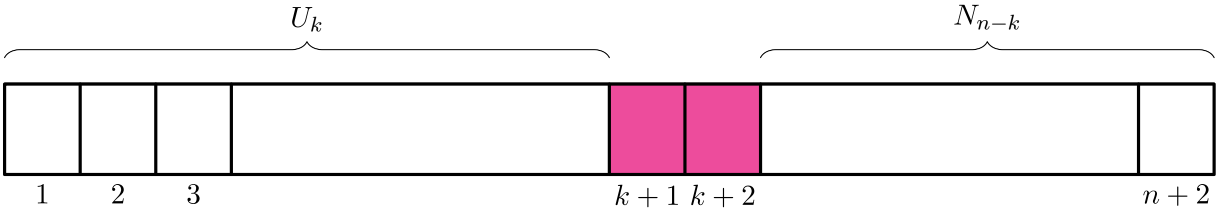

Figure 10: The last domino covers positions $k+1$ and $k+2$. If $k \geq 1$, the cells to its left can be tiled in U_k ways, while the remaining positions allow for N_{n-k} possible tilings.

Remark. The second equality of the previous identity can be expressed solely in terms of U_n and N_n by observing that $O_{n+2} = aN_{n+1} + cN_{n-1}$ (condition on the first tile).

For $a = b = c = 1$, Identity 12 leads to the convolution of the classical Tribonacci and Narayana's cows sequences, namely

$$T_{n+2} - N_{n+2} = \sum_{k=0}^{n} N_k T_{n-k}.$$

This identity is stated as an exercise in [1, p. 47], and a combinatorial proof based on tilings of a honeycomb strip is given in [2, Theorem 6.4]. The identity can also be derived from a general convolution theorem available in [3]. Both sides of the identity correspond to sequence A103321 in the OEIS [11].

For $a = 1$, $b = 2$ and $c = 3$, Identity 12 yields the formulas

$$\sum_{k=0}^{n} M_k T_{n-k} - T_n = K_{n+2} - M_{n+2} = \sum_{k=0}^{n} K_k N_{n-k} - N_n,$$

which seem to be new.

24. Convolutions of Tribonacci and Padovan numbers

Finally, we introduce the generalized Padovan sequence by

$$Q_n = O_{n-2} + Q_{n-3}, \quad n \geq 3, \quad Q_0 = c, \quad O_1 = 0, \quad O_2 = b.$$

For $n \in \mathbb{N}$, the number Q_n counts all the tilings of an n-board using dominoes and trominoes, where the initial tile has one of b colors if it is a domino, and one of c colors if it is a tromino.

The classical Padovan sequence P_n, see A134816, corresponds to $b = c = 1$, and simply counts the number of uncolored tilings of an n-board using dominoes and trominoes. Combinatorial proofs of numerous identities involving the Padovan numbers are available in [5, 13]. Many sources, such as A000931 or [5], deal with shifted versions of this sequence.

The corresponding Lucas-type numbers R_n count uncolored n-bracelets consisting only of dominoes and trominoes. They satisfy the same recurrence relation, but the initial values are now $b = 2$, $c = 3$. The sequence is known as the Perrin sequence A001609, but it would also be natural to call it the Padovan-Lucas sequence.

Here is an additional pair of convolution identities obtained by conditioning on the first or last square, respectively.

Identity 13. *For $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} U_k P_{n-k} + (U_1 - U_0)P_n = U_{n+1} - Q_{n+1} = \sum_{k=0}^{n} Q_k T_{n-k} + (U_1 - Q_0)T_n.$$

Proof. How many tilings of an $(n+1)$-board contain at least one square?

Answer 1: There are $U_{n+1} - Q_{n+1}$ such tilings.

Answer 2: We condition on the location of the last square. Suppose it covers position $k+1$ as in Figure 11. If $k \geq 1$, the cells on its left can be tiled in U_k ways, while the cells on its right can be tiled in P_{n-k} ways. If $k = 0$, the number of tilings is $aP_n = U_1 P_n$. Summing over all possible positions of the last square yields $\sum_{k=1}^{n} U_k P_{n-k} + U_1 P_n$, which is equal to the left side of the identity.

Answer 3: We condition on the location of the first square. Suppose it covers position $k+1$. If $k \geq 1$, the cells on its left can be tiled in Q_k ways, while the cells on its right can be tiled in T_{n-k} ways. If $k = 0$, the number of tilings is $aT_n = U_1 T_n$. Summing over all possible positions of the first square yields $\sum_{k=1}^{n} Q_k T_{n-k} + U_1 T_n$, which is equal to the right side of the identity. ☐

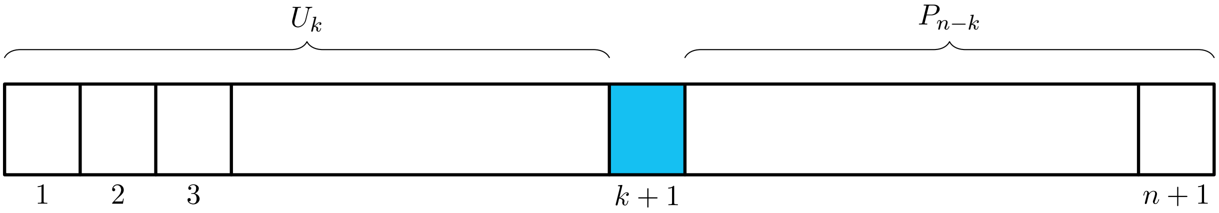

Figure 11: The last square covers position $k+1$. If $k \geq 1$, the cells on its left can be tiled in U_k ways, while the cells on its right can be tiled in P_{n-k} ways.

Remark. One can express the first equality of the previous identity solely in terms of U_n and P_n by observing that $Q_{n+1} = bP_{n-1} + cP_{n-2}$ (condition on the first tile).

For $a = b = c = 1$, Identity 13 leads to the convolution of the Tribonacci and Padovan sequences, namely

$$T_{n+1} - P_{n+1} = \sum_{k=0}^{n} P_k T_{n-k}.$$

This identity is stated as an exercise in [1, p. 47], and a combinatorial proof based on tilings of a honeycomb strip is given in [2, Theorem 6.5]. The identity can also be derived from a general convolution theorem available in [3]. Both sides of the identity correspond to sequence A103322 in the OEIS [11].

For $a = 1$, $b = 2$ and $c = 3$, Identity 13 yields the formulas

$$\sum_{k=0}^{n} K_k P_{n-k} - 2P_n = K_{n+1} - R_{n+1} = \sum_{k=0}^{n} R_k T_{n-k} - 2T_n,$$

which seem to be new.

44. Self-convolutions of Tribonacci numbers

The next identity evaluates the convolution of the Tribonacci and Tribonacci-Lucas numbers. It goes back to [10], where it is written in the form (with shifted indices)

$$\sum_{k=0}^{n-3} T_{k-1}(T_{n-k-1} + T_{n-k-3} + 2T_{n-k-4}) = (n-2)T_{n-2} - T_{n-3}.$$

For a more general version, see [7, Theorem 3.1]. According to Identity 2, the expression in parentheses on the left side corresponds to the Tribonacci-Lucas numbers K_{n-k-1}; this was observed in the paper [4], whose authors derived the identity from a general theorem on convolutions of Fibonacci- and Lucas-type sequences. Here we provide a new combinatorial proof inspired by the proof of [1, Identity 35].

Identity 14. *For $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} T_k K_{n-k} = (n+3)T_n.$$

Proof. The left side of the identity counts all possible pairs (A, B), where A is a tiling of a rectangular board and B is a bracelet whose total length is n. We will show that the right side corresponds to the same count.

We begin by creating nT_n pairs (A, B) by choosing an arbitrary tiling X of an n-board and a number $i \in \{0, \ldots, n-1\}$. Find the largest $j \leq i$ such that X is breakable between positions j and $j+1$. (When $i = 0$, we clearly have $j = 0$.) The tiles in positions $1, \ldots, j$ give rise to a rectangular tiling A, while the remaining tiles produce a bracelet B. Position 1 in B will correspond to position $i+1$ in X. In this way, we have created n pairs (A, B). They are mutually distinct, since we can always reconstruct X and i from (A, B).

The algorithm creates all pairs (A, B) where B is nonempty, but misses the pairs $(A, \emptyset_i)$ where $i \in \{1, 2, 3\}$ and A is a tiling of an n-board.[1] Their count is $3T_n$, which proves the identity. ☐

The next result is a self-convolution identity for the Tribonacci-Lucas numbers. It seems to be new, and the proof is inspired by the proof of [1, Identity 57].

Identity 15. *For $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} K_k K_{n-k} = (n+2)K_n + 3T_n + T_{n-2}.$$

Proof. The left side of the identity counts all possible bracelet pairs (A, B) whose total length is n. We will show that the right side corresponds to the same count.

First, we create nK_n distinct pairs (A, B) by choosing an arbitrary n-bracelet X and a number $i \in \{1, \ldots, n\}$. The idea is to break X after position i, but this may be impossible due to the presence of a tile. So in general, we start at the boundary between positions i and $i+1$, and proceed in a counterclockwise direction until we find a place where the bracelet is breakable, say between positions j and $j+1$. Clearly, the distance between j and i never exceeds 2.

We now split the tiles of X into two groups: All tiles starting with the first[2] until the tile occupying position j will go to group 1, while the remaining tiles will form group 2. Finally, we create a pair (A, B). Bracelet A will contain all tiles from group 1, and bracelet B all tiles from group 2. We preserve the ordering of tiles, and position them as follows:

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[1] Recall that there are three 0-bracelets denoted by $\emptyset_1$, $\emptyset_2$, and $\emptyset_3$.

[2] Recall that the first tile is defined as the tile covering cell 1.

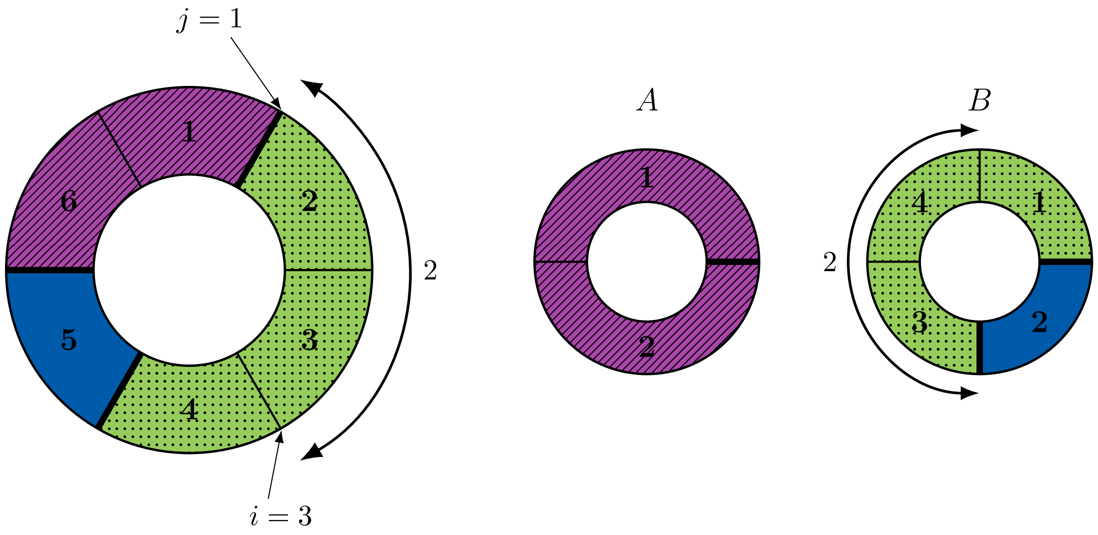

Figure 12: For $i = 3$, the 6-bracelet on the left is transformed into the pair (A, B) on the right. Split occurs after position $j = 1$. The distance between i and j determines the starting position of the first tile in B.

- If both groups are nonempty, the placement of the first tile in A will coincide with its placement in X, and the first tile in B will begin in a position whose distance from 1 in the negative direction is $i - j$. See Figure 12 for an illustration.
- If one of the groups is empty, we will represent it by $\emptyset_1$, $\emptyset_2$, or $\emptyset_3$. If the first group is empty, then B will coincide with X, and we choose $A = \emptyset_i$ (observe that $i \in \{1, 2\}$). If the second group is empty, then A will coincide with X, and we choose $B = \emptyset_{n+1-i}$ (note that $i \in \{n, n-1, n-2\}$).

All pairs created in this way are mutually distinct, since we can reconstruct X and i from (A, B). To get X, it suffices to join A and B, placing the first tile as in A. If both A and B are nonempty, we first find j by looking at the last position occupied by the last tile in A, and then get i by adding the distance between position 1 and the position of the first tile in B. If one of A, B is empty, we find i from the subscript of the corresponding $\emptyset$.

The previous algorithm generates all pairs (A, B) where both bracelets are nonempty, but misses the following pairs where one bracelet is empty:

(a) $(\emptyset_3, B)$, where B is an arbitrary n-bracelet. There are K_n such pairs.

(b) $(\emptyset_i, B)$, where $i \in \{1, 2\}$ and B is an arbitrary n-bracelet breakable between positions 1 and 2. Such a pair could arise only from $X = B$, but since B is breakable after position 1, the first group of tiles cannot be empty. There are $2T_n$ such pairs.

(c) $(\emptyset_2, B)$, where B is an arbitrary n-bracelet breakable between positions 2 and 3, but not between positions 1 and 2 (these cases are already included in the previous item). Hence, there is a domino at positions 1 and 2, or a tromino at positions 1, 2, and 3. There are $T_{n-2} + T_{n-3}$ such pairs.

(d) $(A, \emptyset_3)$, where A is an arbitrary n-bracelet that does not have a tromino covering simultaneously positions $n - 1$, n, and 1. Such a pair could arise only from $i = n - 2$ and $X = A$, but then the last tile of X (a square placed at $n - 1$ or n, a domino placed at $n - 2$, $n - 1$ or $n - 1$, n, or a tromino placed at $n - 2$, $n - 1$, n or $n - 3$, $n - 2$, $n - 1$) goes to the second group, which will become nonempty. There are $K_n - T_{n-3}$ such pairs.

(e) $(A, \emptyset_2)$, where A is an arbitrary n-bracelet breakable between positions n and 1. Such a pair could arise only from $i = n-1$ and $X = A$, but then the tile covering position n goes to the second group, which will become nonempty. There are T_n such pairs.

The number of pairs in (a)–(e) is $2K_n + 3T_n + T_{n-2}$. Hence, the total number including the pairs created earlier is $(n+2)K_n + 3T_n + T_{n-2}$, and the identity is proved. $\square$

Is there a formula for the self-convolution $\sum_{k=0}^{n} T_k T_{n-k}$? Yes, the following result (formulated for a shifted Tribonacci sequence) was obtained using generating functions in [9]:

Identity 16. *For $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} T_k T_{n-k} = \frac{1}{22}\left((5n+15)T_{n+3} - (3n+11)T_{n+2} - (4n+16)T_{n+1}\right).$$

Due to the factor 22, a combinatorial proof appears challenging, though the reader is welcome to try their luck.

81. Conclusion

In this paper, we have journeyed through a combinatorial landscape where identities are not merely algebraic artifacts, but reflections of a tangible tiling logic. Perhaps the rabbit hole goes even deeper. We invite the curious reader to venture further by weaving together different tile sets in even more intricate ways. By looking through the right lens, one might uncover a few more hidden wonders of their own.

Disclosure statement

No conflicts of interest have been reported by the authors.

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