Unbiased Version of Hall's Marriage Theorem in Matrix Form

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Abstract. Inspired by an old result by Georg Frobenius, we show that the unbiased version of Hall's marriage theorem is more transparent when reformulated in the language of matrices. At the same time, we obtain a more general statement applicable to bipartite graphs whose parts need not have the same size.

Given a bipartite graph G, Hall's marriage theorem provides a necessary and sufficient condition for the existence of a matching that covers all vertices in one of the two parts of G. For each vertex set S, let $N(S)$ denote the set of all neighbors of S. Hall's theorem is as follows:

Theorem 1. *For a bipartite graph G with parts V_1 and V_2, the following conditions are equivalent:*

(H1) *G has a matching that covers V_1.*
(H2) *Each set $S \subseteq V_1$ satisfies $|N(S)| \geq |S|$.*

If $|V_1| = |V_2|$, a matching that covers V_1 or V_2 is necessarily a perfect matching. This case was studied in [**3, 8**], where it was shown that condition (H2) might be replaced by one of two alternative conditions. Since these conditions involve vertices from both V_1 and V_2, the authors referred to their results as the "unbiased version" of the marriage theorem.

In this note we show that the results from [**3, 8**] become quite straightforward when translated into the language of matrices. At the same time, we extend them to the case when V_1 and V_2 need not have the same size. We begin by recalling the matrix form of the marriage theorem.

Theorem 2. *For a matrix $A \in \mathbb{R}^{m\times n}$, the following conditions are equivalent:*

(M1) *A has m nonzero entries such that no two lie in the same row or column.*
(M2) *A does not contain a $k \times l$ zero submatrix*[1] *with $k + l = n + 1$.*

To see that Theorem 2 is just a restatement of Theorem 1, observe that a bipartite graph with parts V_1 and V_2 of sizes m and n can be identified with an $m \times n$ matrix A whose element a_{ij} is nonzero if and only if there is an edge connecting the ith vertex of V_1 to the jth vertex of V_2. Condition (H1) is then equivalent to (M1), and condition (H2) is equivalent to (M2). Indeed, a set $S \subseteq V_1$ of size k satisfies $|N(S)| \geq |S|$ if and only if the corresponding k rows of A contain at least k nonzero columns; that is, the number l of columns whose intersection with the given k rows contains only zeros does not exceed $n - k$.

The connection between linear-algebraic results and matchings in bipartite graphs was discovered by Dénes König [**6**]. In 1916, he proved that every regular bipartite multigraph has a perfect matching, and realized that his result immediately gives the

[1] A $k \times l$ submatrix is the intersection of k rows and l columns, which need not be adjacent.

following theorem: If A is a square matrix with nonnegative integer elements such that all rows and columns have the same nonzero sum, then at least one product in the definition of the determinant of A does not vanish (and hence $\det A \neq 0$).

Theorem 2 goes back to a 1917 paper by Georg Frobenius [**4**], who considered the special case $m = n$, and expressed the equivalence of (M1) and (M2) by saying that all products in the definition of the determinant of A vanish if and only if A has a zero $k \times l$ submatrix with $k + l > n$. Frobenius was familiar with the work of König, and showed that his theorem on matrices with identical row and column sums is a consequence of Theorem 2.

Based on this evidence, we can speculate that Frobenius would have been able to reformulate Theorem 2 in terms of matchings in bipartite graphs, and thus obtain Theorem 1. However, it is known that Frobenius did not have a high opinion of graph theory, criticized its use for obtaining results about determinants, and even remarked that König's theorem is only of little value. The history of Hall's theorem and the dispute between Frobenius and König are described in [**1, 7**].

Standard formulations of Hall's theorem involve either matchings in bipartite graphs (as in Theorem 1), or systems of distinct representatives; this version became popular after the publication of Philip Hall's paper [**5**] in 1935, eighteen years after Frobenius's discovery of Theorem 2. Still, the matrix formulation is not forgotten and appears even in modern textbooks, e.g., [**2**, Section 2.1] or [**9**, Section 7.2]. One reason is that it easily leads to Birkhoff's theorem, which says that every doubly stochastic matrix is a convex combination of permutation matrices (see [**9**, p. 626]).

Unlike Theorem 1, the statement of Theorem 2 is unbiased in the sense that rows and columns of A play the same role. To see the relation with the unbiased marriage theorems from [**3, 8**], we make the following observation.

Let $p, q \in \mathbb{N}_0$ satisfy $p + q = n$. If $k, l \in \mathbb{N}$ are such that $k + l = n + 1$, then:

- Either $k > p$, or $l > q$ (for otherwise $k + l \leq p + q = n$).
- Either $k \leq p$, or $l \leq q$ (for otherwise $k + l \geq p + q + 2 = n + 2$).

This simple observation explains why condition (M2) can be replaced by one of two seemingly weaker conditions.

Theorem 3. *For a matrix $A \in \mathbb{R}^{m \times n}$, each of the following conditions is equivalent to* (M2).

(M3) *There exist integers $p, q \in \mathbb{N}_0$ with $p + q = n$ such that A does not contain a $k \times l$ zero submatrix, where $k + l = n + 1$, and $k > p$ or $l > q$.*

(M4) *There exist integers $p, q \in \mathbb{N}_0$ with $p + q = n$ such that A does not contain a $k \times l$ zero submatrix, where $k + l = n + 1$, and $k \leq p$ or $l \leq q$.*

The nonexistence of a $k \times l$ zero submatrix with $k + l = n + 1$ can be rephrased as follows:

- If we select any k rows, then they have at most $n - k$ zero columns, i.e., at least k nonzero columns.
- If we select any l columns, then they have at most $n - l$ zero rows, i.e., at least $m - n + l$ nonzero rows.

Recalling that a bipartite graph with parts V_1 and V_2 can be represented by a matrix whose rows correspond to the vertices of V_1, columns to the vertices of V_2, and nonzero entries to edges, we obtain the following graph-theoretic version of Theorem 3.

Theorem 4. *For a bipartite graph with parts V_1 and V_2, each of the following conditions is equivalent to* (H2).

(H3) *There exist integers $p, q \in \mathbb{N}_0$ satisfying $p + q = |V_2|$ such that each set $S \subseteq V_1$ with $|S| > p$ satisfies $|N(S)| \geq |S|$, and each set $S \subseteq V_2$ with $|S| > q$ satisfies $|N(S)| \geq |V_1| - |V_2| + |S|$.*

(H4) *There exist integers $p, q \in \mathbb{N}_0$ satisfying $p + q = |V_2|$ such that each set $S \subseteq V_1$ with $|S| \leq p$ satisfies $|N(S)| \geq |S|$, and each set $S \subseteq V_2$ with $|S| \leq q$ satisfies $|N(S)| \geq |V_1| - |V_2| + |S|$.*

The results in [**3, 8**] are special cases of Theorem 4 corresponding to $|V_1| = |V_2|$. If $|V_1| > |V_2|$, it is clear that there is no matching that covers V_1, and conditions (H2), (H3), (H4) are false. Hence, Theorems 1 and 4 are interesting only if $|V_1| \leq |V_2|$.

Although Theorem 4 can be obtained by purely graph-theoretic methods (adapting the proof from [**8**]), we believe that our derivation might be more transparent: It shows that Theorem 4 is just a reformulation of Theorem 3, which follows trivially from the matrix version of the marriage theorem.

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REFERENCES

1. Anderson, I. (2013). Combinatorial set theory. In: Wilson, W., Watkins, J. J., eds. *Combinatorics: Ancient and Modern*. Oxford: Oxford Univ. Press, pp. 309–328.
2. Burkard, R., Dell'Amico, M., Martello, S. (2009). *Assignment Problems*. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM).
3. Ehrenborg, R. (2015). An unbiased marriage theorem. *Amer. Math. Monthly*. 122(1): 59.
4. Frobenius, G. (1917). Über zerlegbare Determinanten. *Sitzber. König. Preuss. Akad. Wiss.* 18: 274–277.
5. Hall, P. (1935). On representatives of subsets. *J. Lond. Math. Soc.* 10(1): 26–30.
6. König, D. (1916): Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Math. Ann.* 77(4): 453–465.
7. Schneider, H. (1977). The concepts of irreducibility and full indecomposability of a matrix in the works of Frobenius, König and Markov. *Linear Algebra Appl.* 18(2): 139–162.
8. Shamir, E., Sudakov, B. (2017). Two-sided, unbiased version of Hall's marriage theorem. *Amer. Math. Monthly*. 124(1): 79–80.
9. Strang, G. (1986). *Introduction to Applied Mathematics*. Wellesley, MA: Wellesley-Cambridge Press.

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