Discrete-space systems of partial dynamic equations and discrete-space wave equation

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Abstract

We study the well-posedness of initial-value problems for systems of partial dynamic equations with discrete space and arbitrary time domain. We also present the superposition principle for infinite linear combinations of solutions. As an example, we consider the discrete-space wave equation, which is equivalent to a pair of first-order equations. We provide a general method for finding fundamental solutions and illustrate it on several examples of time scales.

Keywords: wave equation; lattice equation; time scale; well-posedness; superposition principle; fundamental solution

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1 Introduction

In contrast to ordinary dynamic equations, the theory of partial dynamic equations is far less developed and the literature is rather scarce; see e.g. [1, 9, 10, 13]. This paper is a contribution to the recent studies of partial dynamic equations on discrete-space domains (also known as lattice equations). After the transport equation considered in [20, 22] and various diffusion-type equations investigated in [2, 7, 17, 18], the next natural candidate is the one-dimensional discrete-space wave equation

$$u^{\Delta\Delta}(x,t) = c^2(u(x+1,t) - 2u(x,t) + u(x-1,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{T}, \tag{1.1}$$

as well as its N-dimensional analogue. We use $\mathbb{T}$ to denote a time scale, and $u^{\Delta\Delta}$ stands for the second Δ-derivative of u with respect to the time variable. We assume that the reader is familiar with the time scale calculus (see [3, 8]), which enables us to consider equations with continuous, discrete, or mixed time in a unified way. Eq. (1.1) is obtained from the classical wave equation $\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t)$ by discretizing the space variable (the discretization step is fixed), while the time variable can remain continuous, or can be discretized as well (possibly with variable discretization steps). Without loss of generality, we assume that the spatial discretization step is 1; otherwise, for step size $\Delta x > 0$, a simple change of variables transforms the equation to the form (1.1) with c^2 is replaced by $c^2/(\Delta x)^2$.

Second-order equations such as (1.1) are equivalent to systems of two first-order equations. In Section 2, we do not focus solely on the wave equation, but consider general systems of n linear first-order partial dynamic equations. We obtain some basic results concerning the well-posedness of initial-value problems, formulate a superposition principle for infinite linear combinations of solutions, and discuss symmetric solutions of equations with symmetric right-hand sides. The results and proofs in Section 2 are inspired by the theory of scalar partial dynamic equations developed in [17].

In Section 3, we first show how the previous results apply to the N-dimensional discrete-space wave equation. Then we focus on the one-dimensional case and discuss a method for obtaining fundamental solutions. Our approach is based on generating functions, and is similar to the one demonstrated in [18] for diffusion-type equations. As an example, we find fundamental solutions of the wave equation (1.1) for several particular time scales.

Discrete-space equations are interesting from the viewpoint of numerical mathematics, but also show up directly in applications. For example, the semidiscrete diffusion equation models the flow of a chemical in a system of tanks connected by pipes [19], while the semidiscrete wave equation describes the motion of a chain of masses connected by linear springs [21]. Various diffusion-type equations with continuous, discrete or mixed time appear in the context of stochastic processes [7, 18], signal and image processing [12], or in biology [4].

2 Discrete-space systems of partial dynamic equations

Assume that $N \in \mathbb{N}$, $e_1, \dots, e_N$ is the canonical basis of $\mathbb{R}^N$, $n, r \in \mathbb{N}$, and $A^{(i_1,\dots,i_N)} \in \mathbb{R}^{n\times n}$ for all $i_1, \dots, i_N \in \{-r, \dots, r\}$. We consider the system of n first-order partial dynamic equations written in the vector form

$$u^\Delta(x,t) = \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} A^{(i_1,\dots,i_N)} u\left(x + \textstyle\sum_{k=1}^N i_k e_k, t\right), \quad t \in \mathbb{T},\ x \in \mathbb{Z}^N, \tag{2.1}$$

with the unknown function $u : \mathbb{Z}^N \times \mathbb{T} \to \mathbb{R}^n$. Thus, each component of $u^\Delta(x,t)$ is a linear combination of the values of u lying in the N-dimensional hypercube centered at x and whose side has length $2r+1$.

Note that Eq. (2.1) can be regarded not only as a partial dynamic equation, but also as an infinite system of ordinary dynamic equations indexed by $x \in \mathbb{Z}^N$; equations of a similar type with $N = 1$ have been investigated in [14], but from a different point of view. In the special case $\mathbb{T} = \mathbb{R}$, Eq. (2.1) becomes a countable system of ordinary differential equations; such systems (as well as the related topic of ordinary differential equations in Banach spaces) have received a great deal of attention in the literature (see e.g. [5, 6, 16]).

Let $\ell^\infty(\mathbb{Z}^N)$ denote the space of all bounded N-dimensional arrays of real numbers $\{u_x\}_{x\in\mathbb{Z}^N}$ equipped with the supremum norm

$$\|u\| = \sup_{x\in\mathbb{Z}^N} |u_x|, \quad u \in \ell^\infty(\mathbb{Z}^N).$$

We also need the product space $(\ell^\infty(\mathbb{Z}^N))^n$, whose elements have the form $u = (u^1, \dots, u^n)$ with $u^1, \dots, u^n \in \ell^\infty(\mathbb{Z}^N)$. On this space, we introduce the norm

$$\|u\| = \max\{\|u^1\|, \dots, \|u^n\|\}, \quad u \in (\ell^\infty(\mathbb{Z}^N))^n.$$

For an arbitrary $u \in (\ell^\infty(\mathbb{Z}^N))^n$ and $x \in \mathbb{Z}^N$, we use the symbol u_x to denote the vector $(u^1_x, \dots, u^n_x) \in \mathbb{R}^n$.

The system of partial dynamic equations (2.1) is closely related to the abstract dynamic equation

$$U^\Delta(t) = AU(t), \tag{2.2}$$

where $U : \mathbb{T} \to (\ell^\infty(\mathbb{Z}^N))^n$, and the linear operator $A : (\ell^\infty(\mathbb{Z}^N))^n \to (\ell^\infty(\mathbb{Z}^N))^n$ is given by

$$A(\{u_x\}_{x\in\mathbb{Z}^N}) = \left\{ \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} A^{(i_1,\dots,i_N)} u_{x + \sum_{k=1}^N i_k e_k} \right\}_{x\in\mathbb{Z}^N}. \tag{2.3}$$

Indeed, consider a function $U : [T_1, T_2]_\mathbb{T} \to (\ell^\infty(\mathbb{Z}^N))^n$ which satisfies (2.2), and let $u : \mathbb{Z}^N \times [T_1, T_2]_\mathbb{T} \to \mathbb{R}^n$ be given by $u(x,t) = U(t)_x$. Differentiability of U implies that its components are also differentiable, and we have

$$(U(t)_x)^\Delta = (U^\Delta(t))_x = (AU(t))_x = \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} A^{(i_1,\dots,i_N)} U(t)_{x + \sum_{k=1}^N i_k e_k}.$$

Consequently, u is a solution of (2.1).

We will often need to know the norm of A. If we denote $A^{(i_1,\dots,i_N)}=\{a_{kl}^{(i_1,\dots,i_N)}\}_{k,l=1}^n$ and rewrite (2.3) as

$$A(\{u_x\}_{x\in\mathbb{Z}^N})=\left\{\left(\sum_{i_1,\dots,i_N}\sum_{l=1}^n a_{1l}^{(i_1,\dots,i_N)}u^l_{x+\sum_{k=1}^N i_k e_k},\ \dots,\ \sum_{i_1,\dots,i_N}\sum_{l=1}^n a_{nl}^{(i_1,\dots,i_N)}u^l_{x+\sum_{k=1}^N i_k e_k}\right)\right\}_{x\in\mathbb{Z}^N},$$

it becomes clear that

$$\|A\|=\max_{k\in\{1,\dots,n\}}\sum_{i_1,\dots,i_N\in\{-r,\dots,r\}}\sum_{l=1}^n\left|a_{kl}^{(i_1,\dots,i_N)}\right|. \tag{2.4}$$

Example 2.1. Consider the system of classical partial differential equations

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t)&=d_1\frac{\partial^2 u}{\partial x^2}(x,t)+b_{11}u(x,t)+b_{12}v(x,t),\\ \frac{\partial v}{\partial t}(x,t)&=d_2\frac{\partial^2 v}{\partial x^2}(x,t)+b_{21}u(x,t)+b_{22}v(x,t),\quad x,t\in\mathbb{R}.\end{aligned}$$

After discretizing the space variable and replacing the continuous time domain with a general time scale $\mathbb{T}$, we obtain the discrete-space system of partial dynamic equations

$$\begin{aligned}u^\Delta(x,t)&=d_1(u(x+1,t)-2u(x,t)+u(x-1,t))+b_{11}u(x,t)+b_{12}v(x,t),\\ v^\Delta(x,t)&=d_2(v(x+1,t)-2v(x,t)+v(x-1,t))+b_{21}u(x,t)+b_{22}v(x,t),\quad x\in\mathbb{Z},\ t\in\mathbb{T}.\end{aligned}$$

This system can be rewritten in the vector form (2.1) with $N=1$, $n=2$ and $r=1$; we get

$$\begin{pmatrix}u^\Delta(x,t)\\ v^\Delta(x,t)\end{pmatrix}=\begin{pmatrix}d_1&0\\0&d_2\end{pmatrix}\begin{pmatrix}u(x+1,t)\\ v(x+1,t)\end{pmatrix}+\begin{pmatrix}b_{11}-2d_1&b_{12}\\ b_{21}&b_{22}-2d_2\end{pmatrix}\begin{pmatrix}u(x,t)\\ v(x,t)\end{pmatrix}+\begin{pmatrix}d_1&0\\0&d_2\end{pmatrix}\begin{pmatrix}u(x-1,t)\\ v(x-1,t)\end{pmatrix}.$$

This means that

$$A^{(1)}=A^{(-1)}=\begin{pmatrix}d_1&0\\0&d_2\end{pmatrix},\qquad A^{(0)}=\begin{pmatrix}b_{11}-2d_1&b_{12}\\ b_{21}&b_{22}-2d_2\end{pmatrix}.$$

According to Eq. (2.4), the norm of the corresponding operator $A:(\ell^\infty(\mathbb{Z}))^2\to(\ell^\infty(\mathbb{Z}))^2$ is

$$\|A\|=\max\left(2|d_1|+|b_{11}-2d_1|+|b_{12}|,2|d_2|+|b_{22}-2d_2|+|b_{21}|\right).$$

Our first main result will be concerned with the well-posedness of initial-value problems for Eq. (2.1). To prove it, we need the following lemma, which is a generalization of [17, Lemma 3.4].

Lemma 2.2. *Consider an interval $[\tau_1,\tau_2]_{\mathbb{T}}\subset\mathbb{T}$ and a point $t\in[\tau_1,\tau_2]_{\mathbb{T}}$ such that $|t-\tau_i|<\frac{1}{2\|A\|}$ for $i\in\{1,2\}$. Assume that $u_1,u_2:\mathbb{Z}^N\times[\tau_1,\tau_2]_{\mathbb{T}}\to\mathbb{R}^n$ are bounded solutions of Eq. (2.1). If $u_1(x,t)=u_2(x,t)$ for every $x\in\mathbb{Z}^N$, then u_1 and u_2 coincide on $\mathbb{Z}^N\times[\tau_1,\tau_2]_{\mathbb{T}}$.*

Proof. For every $x\in\mathbb{Z}^N$ and $r\in[\tau_1,\tau_2]_{\mathbb{T}}$, we have

$$u_1(x,r)-u_2(x,r)=u_1(x,t)-u_2(x,t)+\int_t^r(u_1^\Delta(x,s)-u_2^\Delta(x,s))\Delta s=\int_t^r(u_1^\Delta(x,s)-u_2^\Delta(x,s))\Delta s$$

$$=\int_t^r\sum_{i_1,\dots,i_N\in\{-r,\dots,r\}}A^{(i_1,\dots,i_N)}\left(u_1\left(x+\textstyle\sum_{k=1}^N i_ke_k,s\right)-u_2\left(x+\textstyle\sum_{k=1}^N i_ke_k,s\right)\right)\Delta s.$$

Without loss of generality, assume that $\mathbb{R}^n$ is equipped with the supremum norm. Then

$$\left\| \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} A^{(i_1,\dots,i_N)} \left(u_1 \left(x + \sum_{k=1}^N i_k e_k, s \right) - u_2 \left(x + \sum_{k=1}^N i_k e_k, s \right) \right) \right\|$$
$$\le \|A\| \sup_{x\in\mathbb{Z}^N} \|u_1(x,s) - u_2(x,s)\|,$$

and it follows that

$$\|u_1(x,r) - u_2(x,r)\| \le |r-t| \cdot \|A\| \sup_{\substack{s\in[\tau_1,\tau_2]_{\mathbb{T}},\\ x\in\mathbb{Z}^N}} \|u_1(x,s) - u_2(x,s)\| \le \frac{1}{2} \sup_{\substack{s\in[\tau_1,\tau_2]_{\mathbb{T}},\\ x\in\mathbb{Z}^N}} \|u_1(x,s) - u_2(x,s)\|.$$

Passing to the supremum on the left-hand side, we conclude that

$$\sup_{\substack{s\in[\tau_1,\tau_2]_{\mathbb{T}},\\ x\in\mathbb{Z}^N}} \|u_1(x,s) - u_2(x,s)\| \le \frac{1}{2} \sup_{\substack{s\in[\tau_1,\tau_2]_{\mathbb{T}},\\ x\in\mathbb{Z}^N}} \|u_1(x,s) - u_2(x,s)\|.$$

Clearly, this inequality holds only if both suprema vanish, i.e., if u_1 and u_2 coincide. □

We now proceed to the well-posedness of initial-value problems to Eq. (2.1). We consider solutions which can go both forward and backward in time; in the backward direction, a condition involving the graininess of $\mathbb{T}$ is needed to guarantee both existence and uniqueness (the discussion in [17, Section 3] shows that already in the scalar case, the graininess condition cannot be omitted). The next result generalizes Theorems 3.3 and 3.5 from [17].

Theorem 2.3. *Consider an interval $[T_1,T_2]_{\mathbb{T}} \subset \mathbb{T}$ and a point $t_0 \in [T_1,T_2]_{\mathbb{T}}$. Let $u^0 \in (\ell^\infty(\mathbb{Z}^N))^n$. Assume that for every $t \in [T_1,t_0)_{\mathbb{T}}$, the operator $I + A\mu(t)$ is invertible.*

Then, there exists a unique bounded solution $u : \mathbb{Z}^N \times [T_1,T_2]_{\mathbb{T}} \to \mathbb{R}^n$ of Eq. (2.1) such that $u(x,t_0) = u^0_x$ for every $x \in \mathbb{Z}^N$. Moreover, the solution depends continuously on u^0.

Proof. Thanks to the invertibility of $I + A\mu(t)$ for each $t \in [T_1,t_0)_{\mathbb{T}}$, the time scale exponential function $t \mapsto e_A(t,t_0)$ is defined on $[T_1,T_2]_{\mathbb{T}}$. The function $U(t) = e_A(t,t_0)u^0$ is bounded on $[T_1,T_2]_{\mathbb{T}}$ and satisfies Eq. (2.2) with $U(t_0) = u^0$. Consequently, $u(x,t) = U(t)_x$ is a bounded solution of Eq. (2.1) satisfying $u(x,t_0) = u^0_x$.

The prove uniqueness, consider a pair of bounded solutions $u_1, u_2 : \mathbb{Z}^N \times [T_1,T_2]_{\mathbb{T}} \to \mathbb{R}^n$. First, assume that u_1, u_2 do not coincide on $\mathbb{Z}^N \times (t_0,T_2]_{\mathbb{T}}$; let

$$t = \inf\{s \in (t_0,T_2];\ u_1(x,s) \ne u_2(x,s) \text{ for some } x \in \mathbb{Z}^N\}.$$

We claim that $u_1(x,t) = u_2(x,t)$ for every $x \in \mathbb{Z}^N$. If $t = t_0$, the statement is true. If $t > t_0$ and t is left-dense, then the statement follows from continuity. Finally, if $t > t_0$ and t is left-scattered, then $u_1(x,\rho(t)) = u_2(x,\rho(t))$, and the statement follows from the fact that $u_1^\Delta(x,\rho(t)) = u_2^\Delta(x,\rho(t))$. Now, if t is right-scattered, then the relations $u_1(x,t) = u_2(x,t)$ and $u_1^\Delta(x,t) = u_2^\Delta(x,t)$ imply $u_1(x,\sigma(t)) = u_2(x,\sigma(t))$, a contradiction with the definition of t. On the other hand, if t is right-dense, there is a point $\tau \in (t, t + \frac{1}{2\|A\|})_{\mathbb{T}}$, and Lemma 2.2 (with $\tau_1 = t$, $\tau_2 = \tau$) leads to a contradiction again.

Before we proceed to uniqueness in the backward direction, we make the following observation: Denote $U_i(t) = \{u_i(x,t)\}_{x\in\mathbb{Z}}$, $i \in \{1,2\}$. If $t \in [T_1,t_0)_{\mathbb{T}}$ is a right-scattered point, we have

$$U_i(\sigma(t)) = \{u_i(x,\sigma(t))\}_{x\in\mathbb{Z}} = \{u_i(x,t) + \mu(t)u_i^\Delta(x,t)\}_{x\in\mathbb{Z}} = (I + \mu(t)A)U_i(t).$$

Hence, $U_i(t) = (I + \mu(t)A)^{-1}U_i(\sigma(t))$. In other words, the values of the solutions at time $\sigma(t)$ uniquely determine the values at time t.

It remains to discuss the possibility that u_1, u_2 do not coincide on $\mathbb{Z}^N \times [T_1, t_0)_{\mathbb{T}}$; let

$$t = \sup\{s \in [T_1, t_0);\ u_1(x,s) \neq u_2(x,s) \text{ for some } x \in \mathbb{Z}^N\}.$$

We claim that $u_1(x,t) = u_2(x,t)$ for every $x \in \mathbb{Z}^N$. If $t = t_0$, the statement is true. If $t < t_0$ and t is right-dense, then the statement follows from continuity. Finally, if $t < t_0$ and t is right-scattered, it is enough to use our observation. Now, if t is left-scattered, then the relation $u_1(x,t) = u_2(x,t)$ and the observation imply $u_1(x, \rho(t)) = u_2(x, \rho(t))$, a contradiction. On the other hand, if t is left-dense, there is a point $\tau \in (t - \frac{1}{2\|A\|}, t)_{\mathbb{T}}$, and Lemma 2.2 (with $\tau_1 = \tau$, $\tau_2 = t$) leads to a contradiction.

If $u_1^0, u_2^0 \in (\ell^\infty(\mathbb{Z}^N))^n$ are two initial conditions, then the corresponding solutions $U_i(t) = e_A(t, t_0)u_i^0$ satisfy

$$\|U_1(t) - U_2(t)\| \leq \left(\sup_{s \in [T_1, T_2]_{\mathbb{T}}} \|e_A(s, t_0)\| \right) \|u_1^0 - u_2^0\|, \quad t \in [T_1, T_2]_{\mathbb{T}},$$

which proves continuous dependence of solutions on initial values. □

Remark 2.4. In connection with the previous theorem, we point out the following facts:

- If we do not restrict ourselves to bounded solutions, then uniqueness is no longer guaranteed (see [17, Section 3] for a counterexample in the scalar case).
- The condition that $I + A\mu(t)$ is invertible for each $t \in [T_1, t_0)_{\mathbb{T}}$ is known as regressivity. Note that if we are interested only in forward-time solutions, i.e., if $t_0 = T_1$, then the condition vanishes.
- The invertibility of $I + A\mu(t)$ might be difficult to verify. In this case, one can observe that if $\mu(t) < \frac{1}{\|A\|}$ for every $t \in [T_1, t_0)_{\mathbb{T}}$, then $\|I - (I + A\mu(t))\| = \|A\|\mu(t) < 1$, and therefore $I + A\mu(t)$ is invertible.

The next result is the superposition principle for infinite linear combinations of solutions, which generalizes [17, Theorem 3.7].

Theorem 2.5. *Let $u_k : \mathbb{Z}^N \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}^n$, $k \in \mathbb{N}$, be a sequence of bounded solutions of Eq.* (2.1). *Assume there exists a $\beta > 0$ such that $\sum_{k=1}^\infty \|u_k(x, t_0)\| \leq \beta$ for every $x \in \mathbb{Z}^N$. Then, for every bounded sequence $\{c_k\}_{k=1}^\infty$, the function $u(x,t) = \sum_{k=1}^\infty c_k u_k(x,t)$ is a solution of Eq.* (2.1) *on $\mathbb{Z}^N \times [t_0, T]_{\mathbb{T}}$.*

Proof. Find $M > 0$ such that $|c_k| \leq M$, $k \in \mathbb{N}$. Let $\{d_k\}_{k=1}^\infty$ be an arbitrary sequence of numbers such that $|d_k| \leq M$ and consider the functions

$$u^{(m)}(x,t) = \sum_{k=1}^m d_k u_k(x,t), \quad x \in \mathbb{Z}^N, \quad t \in [t_0, T]_{\mathbb{T}}, \quad m \in \mathbb{N}.$$

By linearity, each $u^{(m)}$ is a solution of Eq. (2.1), i.e.,

$$u^{(m)}(x,t) = u^{(m)}(x,t_0) + \int_{t_0}^t \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} A^{(i_1,\dots,i_N)} u^{(m)} \left(x + \textstyle\sum_{l=1}^N i_l e_l, s \right) \Delta s.$$

This leads to the estimate

$$\sup_{x \in \mathbb{Z}^N} \|u^{(m)}(x,t)\| \leq \sup_{x \in \mathbb{Z}^N} \|u^{(m)}(x,t_0)\| + \int_{t_0}^t \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} \|A^{(i_1,\dots,i_N)}\| \sup_{x \in \mathbb{Z}^N} \|u^{(m)}(x,s)\| \Delta s,$$

and Gronwall's inequality [3, Corollary 6.7] gives

$$\sup_{x \in \mathbb{Z}^N} \|u^{(m)}(x,t)\| \leq \sup_{x \in \mathbb{Z}^N} \|u^{(m)}(x,t_0)\| e_{\sum_{i_1,\dots,i_N} \|A^{(i_1,\dots,i_N)}\|}(t, t_0) \leq \beta M e_{\sum_{i_1,\dots,i_N} \|A^{(i_1,\dots,i_N)}\|}(t, t_0)$$

for all $t \in [t_0, T]_{\mathbb{T}}$, $m \in \mathbb{N}$. In particular,

$$|u_j^{(m)}(x,t)| \le \beta M e_{\sum_{i_1,\dots,i_N} \|A^{(i_1,\dots,i_N)}\|}(t,t_0), \quad x \in \mathbb{Z}^N, \quad t \in [t_0,T]_{\mathbb{T}}, \quad m \in \mathbb{N}, \quad j \in \{1,\dots,n\}.$$

For an arbitrary fixed pair (x,t) and $j \in \{1,\dots,n\}$, we can let $d_k = |c_k| \operatorname{sgn}(u_k)_j(x,t)$, $k \in \mathbb{N}$. Then, the previous inequality reduces to

$$\sum_{k=1}^{m} |c_k| \cdot |(u_k)_j(x,t)| \le \beta M e_{\sum_{i_1,\dots,i_N} \|A^{(i_1,\dots,i_N)}\|}(t,t_0),$$

and it follows that the series $\sum_{k=1}^{\infty} c_k u_k(x,t)$ is absolutely convergent.

We claim that $\sum_{k=1}^{\infty} c_k u_k^{\Delta}(x,t)$ is absolutely convergent, too. Indeed, we have

$$\sum_{k=1}^{m} |c_k||(u_k^{\Delta})_j(x,t)| = \sum_{k=1}^{m} |c_k| \left| \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} \sum_{p=1}^{n} a_{jp}^{(i_1,\dots,i_N)} (u_k)_p \left(x + \textstyle\sum_{l=1}^{N} i_l e_l, t\right) \right|$$

$$\le \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} \sum_{p=1}^{n} |a_{jp}^{(i_1,\dots,i_N)}| \sum_{k=1}^{m} |c_k| \left| (u_k)_p \left(x + \textstyle\sum_{l=1}^{N} i_l e_l, t\right) \right|$$

$$\le \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} \sum_{p=1}^{n} |a_{jp}^{(i_1,\dots,i_N)}| \beta M e_{\sum_{i_1,\dots,i_N} \|A^{(i_1,\dots,i_N)}\|}(t,t_0) \le \|A\| \beta M e_{\sum_{i_1,\dots,i_N} \|A^{(i_1,\dots,i_N)}\|}(t,t_0),$$

which proves the assertion. By Lebesgue's dominated convergence theorem, the series $\sum_{k=1}^{\infty} c_k u_k^{\Delta}(x,t)$ can be integrated term by term:

$$\sum_{k=1}^{\infty} c_k u_k(x,t_0) + \int_{t_0}^{t} \left(\sum_{k=1}^{\infty} c_k u_k^{\Delta}(x,s) \right) \Delta s = \sum_{k=1}^{\infty} c_k u_k(x,t_0) + \sum_{k=1}^{\infty} c_k (u_k(x,t) - u_k(x,t_0)) = u(x,t)$$

It follows from this relation that u is continuous with respect to t (since the integral on the left-hand side is a continuous function of its upper bound). Since

$$\sum_{k=1}^{\infty} c_k u_k^{\Delta}(x,t) = \sum_{k=1}^{\infty} c_k \left(\sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} A^{(i_1,\dots,i_N)} u_k \left(x + \textstyle\sum_{k=1}^{N} i_k e_k, t\right) \right)$$

$$= \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} A^{(i_1,\dots,i_N)} u \left(x + \textstyle\sum_{k=1}^{N} i_k e_k, t\right),$$

we see that $\sum_{k=1}^{\infty} c_k u_k^{\Delta}(x,t)$ is continuous with respect to t. Hence, we can differentiate the equality

$$u(x,t) = \sum_{k=1}^{\infty} c_k u_k(x,t_0) + \int_{t_0}^{t} \left(\sum_{k=1}^{\infty} c_k u_k^{\Delta}(x,s) \right) \Delta s$$

with respect to t and obtain

$$u^{\Delta}(x,t) = \sum_{k=1}^{\infty} c_k u_k^{\Delta}(x,t) = \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} A^{(i_1,\dots,i_N)} u \left(x + \textstyle\sum_{k=1}^{N} i_k e_k, t\right),$$

which shows that u is a solution of Eq. (2.1) $\square$

The next result, which generalizes Theorem 5.1 from [17], guarantees that symmetric right-hand sides together with symmetric initial conditions give rise to symmetric solutions. For simplicity, we consider symmetry with respect to the origin, but any other point in $\mathbb{Z}^N$ can serve the same purpose.

Theorem 2.6. *Let $u : \mathbb{Z}^N \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}^n$ be a bounded solution of Eq. (2.1). Assume that the following conditions are satisfied:*

- $A^{(i_1,\dots,i_N)} = A^{-(i_1,\dots,i_N)}$ *for all* $i_1, \dots, i_N \in \{-r, \dots, r\}$.
- *For a certain* $t_0 \in [T_1, T_2]_{\mathbb{T}}$, *we have* $u(x, t_0) = u(-x, t_0)$ *for every* $x \in \mathbb{Z}^N$.
- *For every* $t \in [T_1, t_0)_{\mathbb{T}}$, *the operator* $I + A\mu(t)$ *is invertible.*

Then $u(x, t) = u(-x, t)$ *for every* $t \in [T_1, T_2]_{\mathbb{T}}$ *and* $x \in \mathbb{Z}^N$.

Proof. The function $v : \mathbb{Z}^N \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}^n$ given by $v(x, t) = u(-x, t)$ is a solution of Eq. (2.1), because

$$v^\Delta(x,t) = u^\Delta(-x,t) = \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} A^{(i_1,\dots,i_N)} u\left(-x + \textstyle\sum_{k=1}^N i_k e_k, t\right)$$

$$= \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} A^{-(i_1,\dots,i_N)} v\left(x - \textstyle\sum_{k=1}^N i_k e_k, t\right) = \sum_{i_1,\dots,i_N \in \{-r,\dots,r\}} A^{(i_1,\dots,i_N)} v\left(x + \textstyle\sum_{k=1}^N i_k e_k, t\right).$$

Also, u and v have the same values for $t = t_0$. By the uniqueness of solutions (see Theorem 2.3), we have $u = v$ on $\mathbb{Z}^N \times [T_1, T_2]_{\mathbb{T}}$. □

3 Discrete-space wave equation

The N-dimensional discrete-space wave equation

$$u^{\Delta\Delta}(x,t) = c^2 \left(\sum_{i=1}^N u(x + e_i, t) - 2Nu(x,t) + \sum_{i=1}^N u(x - e_i, t) \right), \quad t \in \mathbb{T},\ x \in \mathbb{Z}^N, \tag{3.1}$$

is equivalent to the first-order system

$$\begin{aligned} u^\Delta(x,t) &= v(x,t), \\ v^\Delta(x,t) &= \sum_{i=1}^N c^2 u(x+e_i,t) - 2Nc^2 u(x,t) + \sum_{i=1}^N c^2 u(x-e_i,t), \end{aligned}$$

which has the form (2.1) with $n = 2$, $r = 1$, $A^{(0,\dots,0)} = \begin{pmatrix} 0 & 1 \\ -2Nc^2 & 0 \end{pmatrix}$, $A^{(i_1,\dots,i_N)} = \begin{pmatrix} 0 & 0 \\ c^2 & 0 \end{pmatrix}$ if exactly one of the $i_1, \dots, i_N$ is nonzero and equals ± 1, and $A^{(i_1,\dots,i_N)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ otherwise.

According to Eq. (2.4), the norm of the operator A given by (2.3) is $\|A\| = \max(1, 4Nc^2)$. The following theorem is now an immediate consequence of Theorems 2.3, 2.6, and Remark 2.4.

Theorem 3.1. *Consider an interval* $[T_1, T_2]_{\mathbb{T}} \subset \mathbb{T}$ *and a point* $t_0 \in [T_1, T_2]_{\mathbb{T}}$. *Let* $u^0, v^0 \in \ell^\infty(\mathbb{Z}^N)$. *Assume that* $\mu(t) < \frac{1}{\max(1,4Nc^2)}$ *for every* $t \in [T_1, t_0)_{\mathbb{T}}$.

Then, there exists a unique bounded solution $u : \mathbb{Z}^N \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}$ *of Eq. (3.1) such that* $u(x, t_0) = u^0_x$ *and* $u^\Delta(x, t_0) = v^0_x$ *for all* $x \in \mathbb{Z}^N$. *Moreover, the solution depends continuously on* u^0 *and* v^0. *Also, if* $u^0_x = u^0_{-x}$ *and* $v^0_x = v^0_{-x}$ *for all* $x \in \mathbb{Z}^N$, *then* $u(x, t) = u(-x, t)$ *for all* $t \in [T_1, T_2]_{\mathbb{T}}$, $x \in \mathbb{Z}^N$.

In the rest of this section, we focus on forward-time solutions of initial-value problems for the one-dimensional wave equation

$$u^{\Delta\Delta}(x,t) = c^2(u(x+1,t) - 2u(x,t) + u(x-1,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{T}, \tag{3.2}$$

$$u(x,t_0) = u_x^0, \quad u^{\Delta}(x,t_0) = v_x^0, \quad x \in \mathbb{Z}, \tag{3.3}$$

where $c > 0$, and u^0, $v^0 \in \ell^\infty(\mathbb{Z})$ are given sequences.

According to the next theorem, which is an immediate consequence of Theorem 2.5, the initial-value problem (3.2)–(3.3) can be solved for an arbitrary choice of u^0, v^0 if we find a pair of fundamental solutions to Eq. (3.2). We use the symbol δ_{ij} to denote the Kronecker delta.

Theorem 3.2. *Let $u_1, u_2 : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ be the unique bounded solutions of Eq.* (3.2) *corresponding to the initial conditions $u_1(x,t_0) = \delta_{x0}$, $u_1^{\Delta}(x,t_0) = 0$, and $u_2(x,t_0) = 0$, $u_2^{\Delta}(x,t_0) = \delta_{x0}$ for all $x \in \mathbb{Z}$.*

Then for arbitrary u^0, $v^0 \in \ell^\infty(\mathbb{Z})$, the function

$$u(x,t) = \sum_{k\in\mathbb{Z}} (u_x^0 \cdot u_1(x-k,t) + v_x^0 \cdot u_2(x-k,t)), \quad x \in \mathbb{Z}, \quad t \in [t_0, T]_{\mathbb{T}},$$

is the unique bounded solution of the initial-value problem (3.2)–(3.3).

To find u_1, we use the generating function method. Instead of defining $F(z,t) = \sum_{x\in\mathbb{Z}} u_1(x,t) z^x$ as in [18], it is more convenient to let

$$F(z,t) = \sum_{x\in\mathbb{Z}} u_1(x,t) z^{2x}.$$

Now, Eq. (3.2) implies

$$F^{\Delta\Delta}(z,t) = c^2 \left(\frac{1}{z^2} - 2 + z^2 \right) F(z,t) = \left(\frac{c}{z} - cz \right)^2 F(z,t). \tag{3.4}$$

Taking into account the initial conditions for u_1, we get

$$F(z,0) = 1, \quad F^{\Delta}(z,0) = 0. \tag{3.5}$$

The solution of the initial-value problem (3.4)–(3.5) can be expressed in terms of the time scale hyperbolic cosine:

$$F(z,t) = \cosh_{c/z-cz}(t,t_0) = \frac{1}{2} \left(e_{c/z-cz}(t,t_0) + e_{-c/z+cz}(t,t_0) \right). \tag{3.6}$$

To get an explicit formula for $u_1(x,t)$, it remains to calculate the Laurent series expansion of F and find the coefficient of z^{2x}.

Once we have u_1, it is a simple observation that the function

$$u_2(x,t) = \int_{t_0}^{t} u_1(x,s)\Delta s, \quad x \in \mathbb{Z}, \quad t \in [t_0, \infty)_{\mathbb{T}},$$

satisfies Eq. (3.2) with $u_2(x,t_0) = 0$ and $u_2^{\Delta}(x,t_0) = u_1(x,t_0) = \delta_{x0}$.

The following examples illustrate the previous method on several particular time scales.

Example 3.3. Consider the semidiscrete one-dimensional wave equation, which is a special case of Eq. (3.2) corresponding to $\mathbb{T} = \mathbb{R}$. We find a pair of fundamental solutions for $t_0 = 0$. In this case, the function F given by Eq. (3.6) becomes

$$F(z,t) = \cosh(ct(1/z - z)) = \frac{1}{2} \left(e^{(1/z-z)ct} + e^{-(1/z-z)ct} \right).$$

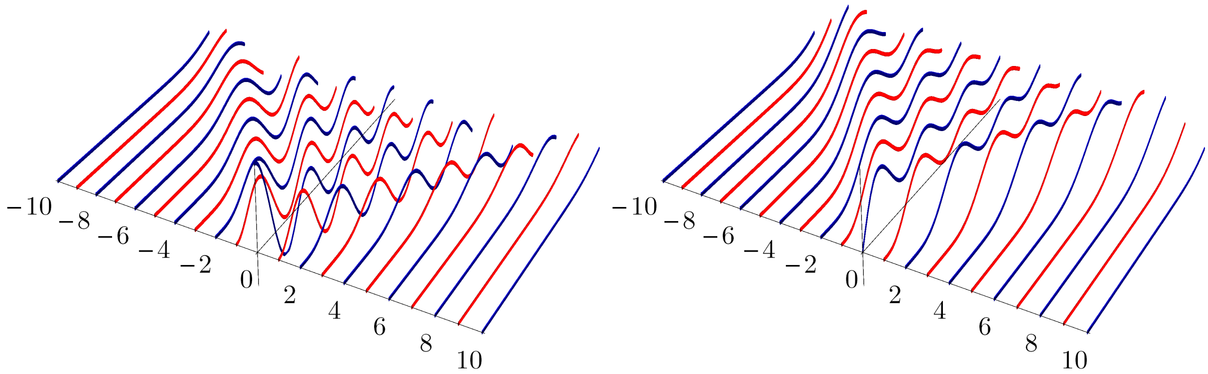

Figure 1: Fundamental solutions u_1 (left) and u_2 (right) for $\mathbb{T} = \mathbb{R}$

We need the identity $e^{\frac{1}{2}t(z-1/z)} = \sum_{x\in\mathbb{Z}} J_x(t)z^x$ (see [15, formula 10.12.1]), where J_x is the Bessel function of the first kind. Consequently,

$$F(z,t) = \frac{1}{2}\sum_{x\in\mathbb{Z}}(J_x(-2ct) + J_x(2ct))z^x.$$

From the definition of the Bessel function (see [15, formula 10.2.2]), it is clear that J_x is even when x is an even integer, and odd if x is an odd integer. Hence,

$$F(z,t) = \sum_{x\in\mathbb{Z}} J_{2x}(2ct)z^{2x},$$

which leads to the result

$$u_1(x,t) = J_{2x}(2ct), \quad u_2(x,t) = \int_0^t J_{2x}(2cs)\,\mathrm{d}s.$$

Our calculations involving the generating function were purely formal; nevertheless, using the identity $J'_x(z) = \frac{1}{2}(J_{x-1}(z) - J_{x+1}(z))$ (see [15, formula 10.6.1]), one can easily verify that u_1 and u_2 are indeed solutions of Eq. (3.2).

The two fundamental solutions are shown in Figure 1. At first sight, the triangles emanating from the origin resemble the causality principle valid for the classical wave equation with continuous space and time. Nevertheless, in the semidiscrete case, the signal propagates with infinite speed: For every fixed $t > 0$, the function $x \mapsto u_1(x,t)$ approaches zero as $x \to \pm\infty$, but does not have a compact support.

In the next two examples, we need the following technical lemma, which is a consequence of [18, Lemma 3.2]. The symbol $\binom{t}{t_1,\dots,t_n}$ stands for the multinomial coefficient, which is equal to $\frac{t!}{t_1!\cdots t_n!}$ when $t, t_1, \dots, t_n$ are nonnegative integers, and otherwise is zero.

Lemma 3.4. *For every $t \in \mathbb{N}_0$ and $x \in \mathbb{Z}$, the coefficient of z^x in $(a/z + b + cz)^t$ is*

$$\sum_{j=0}^{t}\binom{t}{j,\, t-2j-x,\, j+x} a^j b^{t-2j-x} c^{j+x}.$$

0	7	−7	−14	29	−14	−7	7	0
0	1	9	−30	41	−30	9	1	0
0	0	5	−10	11	−10	5	0	0
0	0	1	2	−5	2	1	0	0
0	0	0	3	−5	3	0	0	0
0	0	0	1	−1	1	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0	0

0	1	15	−34	43	−34	15	1	0
0	0	6	−4	2	−4	6	0	0
0	0	1	6	−9	6	1	0	0
0	0	0	4	−4	4	0	0	0
0	0	0	1	1	1	0	0	0
0	0	0	0	2	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0

Table 1: Fundamental solutions u_1 (left) and u_2 (right) for $\mathbb{T} = \mathbb{Z}$

Example 3.5. Consider the purely discrete one-dimensional wave equation, which is a special case of Eq. (3.2) corresponding to $\mathbb{T} = \mathbb{Z}$. Again, let $t_0 = 0$. In this case, we have $e_\alpha(t, 0) = (1 + \alpha)^t$, and therefore

$$F(z,t) = \frac{1}{2}\left(\left(1 + \frac{c}{z} - cz\right)^t + \left(1 - \frac{c}{z} + cz\right)^t\right).$$

Recalling that $u_1(x, t)$ is the coefficient of z^{2x} in $F(z, t)$ and using Lemma 3.4, we obtain

$$\begin{aligned} u_1(x,t) &= \frac{1}{2}\left(\sum_{j=0}^{t}\binom{t}{j, t-2j-2x, j+2x}(c^j(-c)^{j+2x} + (-c)^j c^{j+2x})\right) \\ &= \sum_{j=0}^{t}\binom{t}{j, t-2j-2x, j+2x}(-1)^j c^{2j+2x}, \\ u_2(x,t) &= \sum_{s=0}^{t-1} u_1(x,s) = \sum_{s=0}^{t-1}\sum_{j=0}^{s}\binom{s}{j, s-2j-2x, j+2x}(-1)^j c^{2j+2x} \\ &= \sum_{j=0}^{t-1}(-1)^j c^{2j+2x}\sum_{s=j}^{t-1}\binom{s}{j, s-2j-2x, j+2x} = \sum_{j=0}^{t-1}(-1)^j c^{2j+2x}\binom{2j+2x}{j}\sum_{s=j}^{t-1}\binom{s}{2j+2x}. \end{aligned}$$

Finally, using the well-known identity $\sum_{k=0}^{m}\binom{k}{n} = \binom{m+1}{n+1}$, the last formula simplifies to

$$u_2(x,t) = \sum_{j=0}^{t-1}(-1)^j c^{2j+2x}\binom{2j+2x}{j}\left(\binom{t}{2j+2x+1} - \binom{j}{2j+2x+1}\right).$$

Table 1 shows the values of $u_1(x, t)$ and $u_2(x, t)$ for $c = 1$ and $t \in \{0, \dots, 7\}$ (horizontal direction corresponds to spatial location, and the upward direction corresponds to increasing values of time, starting with $t = 0$).

Example 3.6. Consider the time scale $\mathbb{T} = \{H_n, n \in \mathbb{N}_0\}$, where $H_0 = 0$ and $H_n = \sum_{k=1}^{n}\frac{1}{k}$ are the harmonic numbers. Assume that $t_0 = 0$. It is known (see [3, Example 2.53]) that the values of the time scale exponential function are the binomial coefficients: $e_\alpha(H_n, 0) = \binom{n+\alpha}{n}$. Therefore,

$$F(z, H_n) = \frac{1}{2}\left(\binom{n + c/z - cz}{n} + \binom{n - c/z + cz}{n}\right), \quad n \in \mathbb{N}_0.$$

Using the identity (see [15, formula 26.8.7])

$$x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} s(n,l)x^l, \quad x \in \mathbb{R}, n \in \mathbb{N}_0,$$

where $s(n,l)$ are the Stirling numbers of the first kind, we obtain

$$F(z,H_n) = \frac{1}{2n!}\sum_{l=0}^{n} s(n,l)\left((n+c/z-cz)^l + (n-c/z+cz)^l\right).$$

Recalling that $u_1(x,H_n)$ is the coefficient of z^{2x} in $F(z,H_n)$ and using Lemma 3.4, we find that

$$\begin{aligned}
u_1(x,H_n) &= \frac{1}{2n!}\sum_{l=0}^{n}\sum_{j=0}^{l} s(n,l)\binom{l}{j,\,l-2j-2x,\,j+2x}\left(c^j n^{l-2j-2x}(-c)^{j+2x} + (-c)^j n^{l-2j-2x} c^{j+2x}\right)\\
&= \frac{1}{n!}\sum_{l=0}^{n}\sum_{j=0}^{l} s(n,l)\binom{l}{j,\,l-2j-2x,\,j+2x} n^{l-2j-2x}(-1)^j c^{2j+2x},\\
u_2(x,H_n) &= \sum_{k=0}^{n-1} u_1(x,H_k)\mu(H_k) = \sum_{k=0}^{n-1} u_1(x,H_k)\frac{1}{k+1}.
\end{aligned}$$

Example 3.7. Consider the time scale $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$. Assume that $t_0 = 1$. It is known (see [3, Example 2.55]) that the time scale exponential function is given by the formula

$$e_\alpha(q^n,1) = \prod_{s\in[1,q^n)_{\mathbb{T}}} (1+(q-1)\alpha s) = \prod_{k=0}^{n-1}(1+(q-1)\alpha q^k), \quad n\in\mathbb{N}_0.$$

Recalling the definition of the q-Pochhammer symbol (also known as the q-shifted factorial)

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),$$

we see that

$$e_\alpha(q^n,1) = ((1-q)\alpha;q)_n.$$

Using the identity (see [11, Eq. (0.3.5)])

$$(a;q)_n = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n\\ k\end{bmatrix}_q (-a)^k$$

where $\begin{bmatrix} n\\ k\end{bmatrix}_q$ is the q-binomial coefficient, we obtain

$$e_\alpha(q^n,1) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n\\ k\end{bmatrix}_q ((q-1)\alpha)^k.$$

Therefore,

$$\begin{aligned}
F(z,q^n) &= \frac{1}{2}\left(e_{c/z-cz}(q^n,1) + e_{-c/z+cz}(q^n,1)\right) = \frac{1}{2}\sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n\\ k\end{bmatrix}_q (q-1)^k\left((c/z-cz)^k + (-c/z+cz)^k\right)\\
&= \frac{1}{2}\sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n\\ k\end{bmatrix}_q (q-1)^k c^k\left((1/z-z)^k + (-1/z+z)^k\right).
\end{aligned}$$

Note that $(1/z - z)^k + (-1/z + z)^k$ equals $2(1/z - z)^k$ if k is even, and zero if k is odd. Hence,

$$F(z, q^n) = \sum_{l=0}^{\lfloor n/2 \rfloor} q^{\binom{2l}{2}} \begin{bmatrix} n \\ 2l \end{bmatrix}_q (q-1)^{2l} c^{2l} (1/z - z)^{2l} = \sum_{l=0}^{\lfloor n/2 \rfloor} q^{\binom{2l}{2}} \begin{bmatrix} n \\ 2l \end{bmatrix}_q (q-1)^{2l} c^{2l} \left(\sum_{m=0}^{2l} \binom{2l}{m} (-1)^m z^{2m-2l} \right).$$

Since $u_1(x, q^n)$ is the coefficient of z^{2x} in $F(z, q^n)$, we get

$$\begin{aligned} u_1(x, q^n) &= \sum_{l=0}^{\lfloor n/2 \rfloor} q^{\binom{2l}{2}} \begin{bmatrix} n \\ 2l \end{bmatrix}_q (q-1)^{2l} c^{2l} \binom{2l}{l+x} (-1)^{l+x}, \\ u_2(x, q^n) &= \sum_{k=0}^{n-1} u_1(x, q^k) \mu(q^k) = \sum_{k=0}^{n-1} \sum_{l=0}^{\lfloor k/2 \rfloor} q^{\binom{2l}{2}} \begin{bmatrix} k \\ 2l \end{bmatrix}_q q^k (q-1)^{2l+1} c^{2l} \binom{2l}{l+x} (-1)^{l+x}. \end{aligned}$$

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