Toeplitzova transformace

$\{c_{n,k}\}_{n=1,\;k=1}^{\infty,\;n}$ ... reálná čísla splňující

(i) $\lim\limits_{n\to\infty} c_{n,k} = 0$ pro každé $k \in \mathbb{N}$

(ii) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} c_{n,k} = 1$

(iii) existuje $C>0$ takové, že pro všechna $n \in \mathbb{N}$

$$\sum_{k=1}^{n} |c_{n,k}| \le C .$$

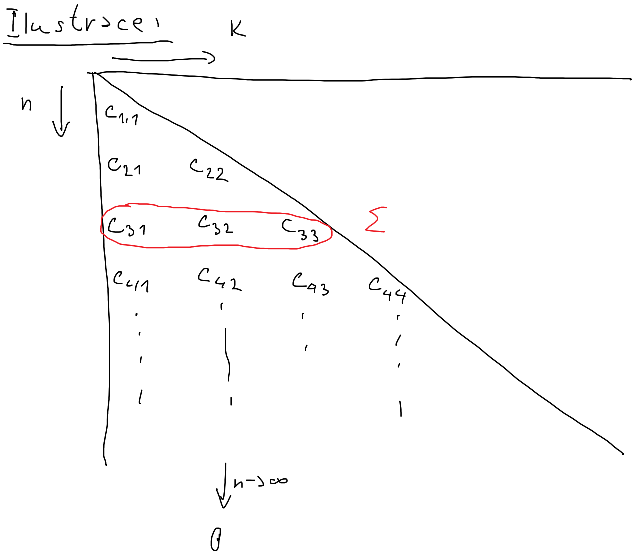
Nechť $\{a_n\}$ je konvergentní posloupnost. Zadefinujme novou posloupnost $\{b_n\}$ vztahem

$$b_n = \sum_{k=1}^{n} c_{n,k}\, a_k , \qquad n \in \mathbb{N} .$$

Pak $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} a_n$.

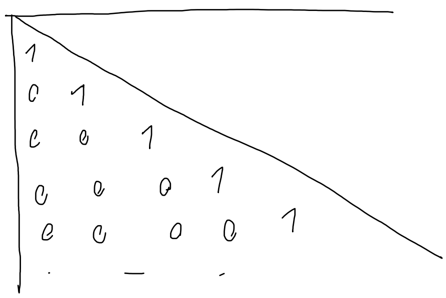
Ilustrace:

$c_{n,k}$



Příklady:

1)



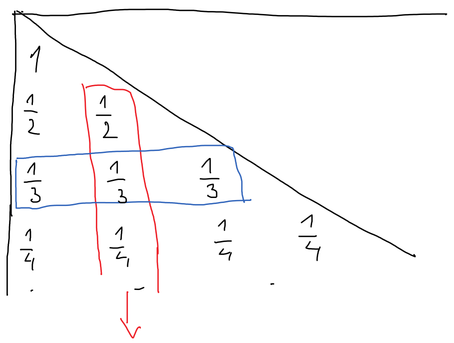
$c_{nn} = 1$, $n \in \mathbb{N}$

$c_{n,k} = 0$, $n \in \mathbb{N}$, $k \in \mathbb{N}$, $k < n$

Pak

$$b_n = \sum_{k=1}^{n} c_{n,k}\, a_k = c_{nn}\, a_n = a_n$$

2)



$$c_{n,k} = \frac{1}{n}, \quad n \in \mathbb{N}, \ k \in \mathbb{N}, \ k \le n$$

Pak

$$b_n = \sum_{k=1}^{n} c_{n,k}\, a_k = \frac{1}{n} \sum_{k=1}^{n} a_k = \frac{a_1 + \dots + a_n}{n}$$

Pak $\lim\limits_{n\to\infty} a_n = a \Rightarrow \lim\limits_{n\to\infty} \frac{a_1 + \dots + a_n}{n} = a$

Důkaz Toeplitzovy věty:

1. krok: Nechť $a_n = a$, $n \in \mathbb{N}$. Pak

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sum_{k=1}^{n} c_{n,k}\, a_k = a \underbrace{\lim_{n\to\infty} \sum_{k=1}^{n} c_{n,k}}_{} \overset{(ii)}{=} a \quad ✓$$

2. krok: Nechť $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$. Lze psát

$a_n = \underbrace{(a_n - a)} + \underbrace{a}$ víme, že pro tuto posloupnost tvrzení platí

Stačí tedy dokázat tvrzení pro posloupnost $a_n - a$, která má limitu 0.

3. krok: Nechť $\lim\limits_{n\to\infty} a_n = 0$. Pak $\forall m > 1$, $n \ge m$ platí

$$|b_n| = |b_n - 0| = \left| \sum_{k=1}^{n} c_{n,k}\, a_k \right| \le \sum_{k=1}^{m-1} |c_{n,k}|\,|a_k| + \sum_{k=m}^{n} |c_{n,k}|\,|a_k| \qquad (*)$$

Nechť $\varepsilon > 0$. Protože $a_n \to 0$, pak $\exists n_1 \in \mathbb{N}$ $\forall n \ge n_1$: $|a_n| < \frac{\varepsilon}{2C}$,

kde C je konstanta z (iii)

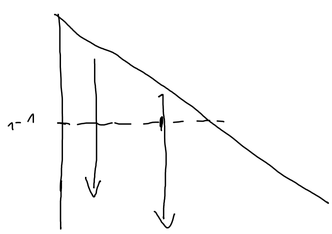
$$\sum_{k=1}^{n} |c_{n,k}| \le C, \quad n \in \mathbb{N}$$

$\{a_n\}$ je konvergentní $\Rightarrow$ $\{a_n\}$ je omezená $\Rightarrow$ $|a_n| \le D$ $\forall n \in \mathbb{N}$.

Z podmínky (i) plyne, že $\exists n_2 \in \mathbb{N}$ $\forall n \ge n_2$

$$\sum_{k=1}^{n_1 - 1} |c_{n,k}| < \frac{\varepsilon}{2D}.$$

$$\left(|c_{n,k}| < \frac{\varepsilon}{2D\, n_1} \right)$$



Použijeme $(*)$ s $m = n_1$, pro $n \ge \max\{n_1, n_2\}$

$$|b_n| \le \sum_{k=1}^{n_1-1} |c_{n,k}|\,|a_k| + \sum_{k=n_1}^{n} |c_{n,k}|\,|a_k|$$

$$\le D \sum_{k=1}^{n_1-1} |c_{n,k}| + \frac{\varepsilon}{2C} \sum_{k=n_1}^{n} |c_{n,k}|$$

$$< D \cdot \frac{\varepsilon}{2D} + \frac{\varepsilon}{2C} \cdot C = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \, .$$

Tedy $\lim_{n\to\infty} b_n = 0$.