(1)

Invariant functions and covariants

$K \ldots$ an infinite field ($= \mathbb{Q}, \mathbb{R}, \mathbb{C} \ldots$)

W/K, $\dim_K W < \infty$; $f: W \to K$ is polynomial (regular) if given by a polyn. in the coordinates w.r.t. a basis of W (basis independent).

$K[W]$ .. K-algebra of polyn. fctions on W (coordinate ring of W $=$ ring of regular fctions)

$\{w_1, \ldots, w_n\}$ basis of W
$\{x_1, \ldots, x_n\}$ basis of W^* (coordinate fctions) $\Big\}$ $K[W] = K[x_1, \ldots, x_n]$.
(K is infinite field $\Rightarrow$ $K[W]$ is polyn. ring.)

$f \in K[W]$ is homogeneous of degree d if $f(tw) = t^d f(w)$ $\forall t \in K, w \in W$.

$\Rightarrow K[W] = \bigoplus_{d \in \mathbb{N}_0} K[W]_d$ graded K-algebra,
$K[W]_d \ldots$ degree d (or d-homogeneous) pol.

($K[W]_{d_1} \cdot K[W]_{d_2} \subseteq K[W]_{d_1+d_2}$, so the monomials $x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n}$, $\sum d_i = d$, form a basis of $K[W]_d$)

$K[W]_1 = W^*$, so $K[W] = \bigoplus_{d \in \mathbb{N}_0} K[W]_d = \bigoplus_d S^d(W^*) = \underbrace{S(W^*)}_{\text{symmetric algebra of } W^*}$.

$GL(W)$ .. the general linear group, assume $G < GL(W)$ (or $\rho: G \to GL(W)$ in general)

Def 1: $f \in K[W]$ is G-invariant (or, just invariant) if $f(g \cdot w) = f(w)$ $\forall g \in G$, $w \in W$. The invariants form a subalgebra of $K[W]$ (called) invariant ring, $K[W]^G$.

Orbit of $w \in W$: $W \supseteq G \cdot w := \{g \cdot w \mid g \in G\}$
Stabilizer —"— : $G \supseteq G_w := \{g \in G \mid g \cdot w = w\}$
$f \in K[W]$ is G-invariant $\Leftrightarrow$ f is constant on $\forall$ orbits

A subset $X \subseteq W$ is G-stable if it is a union of orbits of G in W, i.e. $g \cdot x \in X$ for all $x \in X$, $g \in G$.

<u>Example 2</u>: $SL(2,K)$ (or, $GL(2,K)$) acting on K^2 has two orbits. The stabilizer of $e_1 = (1,0) \in K^2$ is $U := \{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in K \}$, and for any other $(x,y) \in K^2$ is $G_{(x,y)}$ conjugate (in $G = SL(2,K)$) to U;
$G_{(0,0)} = SL(2,K)$.

Another way to describe the ring of invariants: the linear action of G on $K[W]$: $(g, f) \longmapsto g \cdot f$ $\quad (g f)(w) := f(g^{-1} \cdot w)$
$g \in G$, $f \in K[W]$, $w \in W$.

$f \in K[W]^G$ iff f is G-fixed point w.r. to this action.
$K[W]_d$, $d \in \mathbb{N}$, is G-stable subspace $\Rightarrow K[W]^G = \bigoplus_{d \in \mathbb{N}} K[W]_d^G$
(and is a graded algebra, too.)

<u>~~Example~~ Exercise 3</u>: The linear action of $G = GL(2,K)$ on $K[x,y]$:

a/ Show that $K[x,y]^{GL(2,K)} = K[x,y]^{SL(2,K)} = K$

b/ Show that $K[x,y]^{U} = K[y]$ with U introduced in Example 2.

<u>Example 4</u>: The multiplicative group $K^* = GL(1,K)$ has 2-dimensional representation on $W = K^2$, $t \longmapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.
Then the invariant ring is generated by $x \cdot y$:
$K[W]^{K^*} = K[xy]$. The subspaces $\langle x^a y^b \rangle \subseteq K[W]$ a K^*-stable, and $t \cdot (x^a y^b) = t^{b-a} x^a y^b$.

Example 5: $G = SL(n,K) \leq GL(n,K)$, its representation on $M_n(K)$ by left multiplication: $(g,A) \mapsto g\cdot A$, $g \in SL(n,$ $A \in M_n(K)$. The determinant function $A \mapsto \det(A)$ is invariant. We claim that invariant ring is

$$K[M_n]^{SL(n,K)} = K[\det]$$

(outline of) Proof: Assume f is an invariant, define the polynomial $p \in K[t]$ by $p(t) := f\left(\begin{pmatrix} t & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix}\right)$. Then $f(A) = p(\det A)$ for all invertible matrices A, because A can be written in the form

$$A = g\cdot\begin{pmatrix} \det A & & & \emptyset \\ & 1 & & \\ & & 1 \ddots & \\ \emptyset & & & 1\end{pmatrix} \quad \text{with } g \in SL(n,K)$$

and f is $G = SL(n,K)$-invariant. As we know (and recall again), $GL(n,K)$ is Zariski dense in $M_n(K)$. This implies $f(A) = p(\det A)$ for $\forall A \in M_n(K)$, thus $f \in K[\det]$. ■

Exercise 6: Determine the invariant rings $K[M_2(K)]^{U}$ and $K[M_2(K)]^{T}$, with U as in Example 2 and $T := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \mid t \in K^* \right\}$.

Basics of algebraic geometry:

Def 7: A subset $X \subseteq W$ is called Zariski dense if every fuion $f \in K[W]$ vanishing on X is the zero function. In general, $X \subseteq Y (\subseteq W)$ is called Zariski dense in Y if $\forall f \in K[W]$ vanishing on X also vanishes on Y.

(4) This means that $\forall f \in K[W]$ is completely detemined by its restriction to a Zariski dense subset $X \subseteq W$.

$I(X) := \{f \in K[W] \mid f(a) = 0 \ \forall a \in X\}$ is the ideal in $K[W]$ of functions vanishing on X

$m_a := I(\{a\})$ ... ideal of functions vanishing in $a \in W$, i.e.

$$\underbrace{m_a}_{\text{maximal ideal of } a\in W} = \operatorname{Ker}(\underbrace{ev_a}_{\text{evaluation homomorphism}} = \epsilon_a : K[W] \to K)$$
$$f \mapsto f(a)$$

so that $I(X) = \bigcap_{a \in X} m_a$; in coordinates $x_1, \dots, x_n$

$m_a = (x_1 - a_1, \dots, x_n - a_n)$
with $a = (a_1, \dots, a_n)$.

By definition, $X \subseteq Y\ (\subset W)$ is Zariski dense iff $I(X) = I(Y)$.

<u>Remark 8</u>: $X \subseteq Y \subseteq W$, $W = W/K$. $\forall L/K$ field extension, we have $W \subseteq W/L := L \otimes_K W$. If X is Zariski dense in Y, then the same conclusion holds for subsets of W/L. This follows from $L \otimes_K I_K(X) = I_L(X)$, which is implied by the previous description of I.

A subset $X \subseteq W$ defined as the zero locus of polynomial equations is called Zariski closed, and its complement in W Zariski open.

<u>Lemma 9</u>: Let $h \in K[W]$ be a non-zero function, and define $W_h := \{w \in W \mid h(w) \neq 0\}$. Then W_h is Zariski dense in W.

<u>Pf</u>: If f vanishes on W_h then fh vanishes on W, and since $K[W]$ is UFD, $fh = 0$ implies $(h \neq 0)$ $f = 0$. ▨

An example of Zariski dense subset is $GL_n(K) = M_n(K)_{\det}$ in $M_n(K)$.

Example 10: Let W be a G-module, and assume G has Zariski dense orbit in W. Then every invariant function is constant: $K[W]^G = K$. A typical example is the natural representation of $SL(V)$ on

$$V^r := \underbrace{V \oplus V \oplus \dots \oplus V}_{r\times} \quad \text{with } r < \dim V = n.$$

In coordinates, the action of $SL(V)$ is on $n\times r$-matrices $M_{n\times r}(K)$.

In Example 5, we have seen that for $r=n$ the invariants are generated by the determinant function det. As we justify later on, there are no non-constant invariants for $r<n$ because the restriction of det to $n\times r$ matrices $M_{n\times r}(K)$ vanishes for $r<n$ ($\forall$ invariant of $r<n$ copies is a restriction of an invariant on n-copies.)

The general problem (for arbitrary r) is solved by the First Fundamental Theorem for $SL(n,K)$.

Exercise 11 a/ $X \subseteq Y \subseteq W$, assume X is Zariski dense in Y. Then the linear spans $\langle X \rangle$ and $\langle Y \rangle$ are equal.

b/ Show ~~that~~ $K[V\oplus V^*]^{GL(V)} = K[q]$, where the bilinear form q is defined by $q(v,\xi) := \xi(v)$, $v\in V$, $\xi \in V^*$.

(The subset $Z := \{(v,\xi) \mid \xi(v) \neq 0\}$ of $V\oplus V^*$ is Zariski dense. Fix a pair (v_0,ξ_0) s.t. $\xi_0(v_0)=1$. Then for $\forall$ $(v,\xi)\in Z$ there is a $g\in GL(V)$ s.t. $g(v,\xi) = (v_0, \lambda\xi_0)$ with $\lambda = \xi(v)$.)